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On the Non-Commutative Solutions of the Exponential Equation $e^x e^y = e^{x+y}$

By

Kakutaro MORINAGA and Takayuki NÔNO

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1. Introduction. In our previous paper,[1] it was shown that if x and y are matrices of order n whose all characteristic roots μ have the imaginary parts $I(\mu)$ such that $-\pi \leqq I(\mu) < \pi$, then $e^x e^y = e^y e^x$ implies $xy = yx$. And if x and y are simultaneously transformed to the hermitian (symmetric or pure imaginary) matrices, then $e^x e^y = e^{x+y}$ implies $e^x e^y = e^y e^x$. For, from $e^x e^y = e^{x+y}$, it follows $e^{{}^t\bar{y}} e^{{}^t\bar{x}} = e^{{}^t\bar{x} + {}^t\bar{y}}$, where ${}^t x$ denotes the transposed matrix of x, and $\bar{x}$ the complex conjugate matrix of x; since x and y are considered to be hermitian, i.e., ${}^t\bar{x} = x$ and ${}^t\bar{y} = y$, we have $e^y e^x = e^{y+x}$, hence $e^y e^x = e^x e^y$. (For the other cases, this fact is proved by the similar argument). In the case where x and y are simultaneously transformed to the hermitian matrices, the imaginary parts of the characteristic roots of x and y are zero, hence we see that if x and y are simultaneously transformed to the hermitian matrices, then $e^x e^y = e^{x+y}$ implies $xy = yx$ i.e., there exists no non-commutative solution of $e^x e^y = e^{x+y}$. In theoretical physics, the characteristic roots of matrices must be real, so that the matrices treated there are usually hermitian; therefore, for the matrices representing the opetators in theoretical physics, $e^x e^y = e^{x+y}$ implies $xy = yx$.

As for the arbitrary matrices, the question concerning the non-commutative solutions of $e^x e^y = e^{x+y}$ seems, at least for us, to be an open one.

Recently M. Fréchet[2] studied the matrices of order two such that $e^x e^y = e^{x+y} = e^y e^x$ but $xy \neq yx$, and proved that the only non-commutative solutions $x = a$, $y = b$ of order two of this equation satisfy: $e^a = \nu \cdot 1$, $e^b = \nu' \cdot 1$, and a, b and 1 are linearly independent. And, in his review of Fréchet's paper, W. Givens[3] wrote that if α is any non-zero root of $(1-\xi)e^\xi = 1$, then $x = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfy: $e^x e^y = e^{x+y}$ but $e^x e^y \neq e^y e^x$.

1) K. Morinaga and T. Nôno, On the logarithmic functions of matrices I, J. Sci. Hiroshima Univ. (A), Vol. 14, No. 2 (1950), p. 112.

2) M. Fréchet, *Solutions non commutables de l'équation matricielle $e^{A+B} = e^A e^B$.* Compt. rend. acad. sci. (Paris), n^o 22 (1951), pp. 1339-1340. *Les solutions non commutables de l'équation matricielle $e^X e^Y = e^{X+Y}$.* Rend. Circ. Mat. Palermo. (2) 1 (1952), pp. 11-27.

We have not yet read the latter, the content of which, however, we knew in the review of Math. Rev., vol. 14, no. 3 (1953), p. 237.

3) Math. Rev., loc. cit.

In this paper, we shall first consider the non-commutative solutions of $e^x e^y = e^{x+y}$ for the complex algebras of degree two, and then, as the special cases, for the ternary algebras of degree two, the quaternion algebra and the total matric algebra of order two.

2. Preliminary for a complex algebra of degree two. Let $\mathfrak{A}$ be a complex algebra with the unit 1 and of degree two; and let $\mathfrak{C}$ be the field of complex numbers. In this section we shall prove some lemmas for $\mathfrak{A}$. Greek letters $\alpha, \beta, \cdots, \lambda, \mu, \cdots$ denote the elements of $\mathfrak{C}$ and $\alpha \cdot 1$ will be written as α simply. Moreover $\mathfrak{C} \cdot 1$ will be identified with $\mathfrak{C}$.

LEMMA 1. *If $xy \neq yx$, $x, y \in \mathfrak{A}$, then* 1, *x and y are linearly independent, and then x and y are uniquely expressed as*

$$x = \alpha_1 + x_0, \ x_0^2 = \alpha \qquad \textit{and} \qquad y = \beta_1 + y_0, \ y_0^2 = \beta,$$

respectively. And then it holds

$$(x_0 + y_0)^2 = \gamma \qquad \textit{and} \qquad x_0 y_0 + y_0 x_0 = \gamma - \alpha - \beta.$$

PROOF. If 1, x and y are linearly dependent, then $xy = yx$, that is, if $xy \neq yx$, then 1, x and y are linearly independent. Since the degree of $\mathfrak{A}$ is two, x and y satisfy

$$x^2 - 2\alpha_1 x + \alpha_2 = 0, \qquad \alpha_1, \alpha_2 \in \mathfrak{C}$$

and

$$y^2 - 2\beta_1 y + \beta_2 = 0, \qquad \beta_1, \beta_2 \in \mathfrak{C}$$

respectively. If we write $x_0 = x - \alpha_1$, $y_0 = y - \beta_1$, $\alpha = \alpha_1^2 - \alpha_2$ and $\beta = \beta_1^2 - \beta_2$, then we have

$$(1) \qquad x_0^2 = \alpha, \qquad y_0^2 = \beta, \qquad \alpha, \beta \in \mathfrak{C}.$$

Since 1, x and y are linearly independent, these expressions are unique. And then $x_0 + y_0$, $x_0 - y_0 \in \mathfrak{A}$, so it holds that

$$(x_0 + y_0)^2 - 2\sigma_1(x_0 + y_0) + \sigma_2 = 0, \qquad \sigma_1, \sigma_2 \in \mathfrak{C}$$

and

$$(x_0 - y_0)^2 - 2\tau_1(x_0 - y_0) + \tau_2 = 0, \qquad \tau_1, \tau_2 \in \mathfrak{C}.$$

Adding these two equations and taking account of $x_0^2 = \alpha$ and $y_0^2 = \beta$, we have

$$-2(\sigma_1 + \tau_1)x_0 - 2(\sigma_1 - \tau_1)y_0 + 2(\alpha + \beta) + \sigma_2 + \tau_2 = 0.$$

Since 1, x_0 and y_0 are linearly independent, we have

$$\sigma_1 + \tau_1 = 0, \ \sigma_1 - \tau_1 = 0, \qquad \text{i.e.,} \qquad \sigma_1 = \tau_1 = 0;$$

from which it follows that

$$(2) \qquad (x_0 + y_0)^2 = \gamma, \qquad \gamma \in \mathfrak{C},$$

and consequently

$$x_0 y_0 + y_0 x_0 = \gamma - \alpha - \beta . \tag{3}$$

In the following we shall use the notations in Lemma 1.

LEMMA 2. *For x and y such that $xy \neq yx$, if $x_0 y_0 = \mu_2 x_0 + \mu_1 y_0 + \mu_3$, then $x_0^2 = \mu_1^2$, $y_0^2 = \mu_2^2$, $(x_0 + y_0)^2 = (\mu_1 - \mu_2)^2$ and $\mu_1 \mu_2 + \mu_3 = 0$. And $x_0 y_0 - y_0 x_0 = 2\mu_2 x_0 + 2\mu_1 y_0$, if and only if $x_0 y_0 = \mu_2 x + \mu_1 y_0 + \mu_3$.*

PROOF. By means of (3), from

$$x_0 y_0 = \mu_2 x_0 + \mu_1 y_0 + \mu_3 , \tag{4}$$

we have

$$y_0 x_0 = -\mu_2 x_0 - \mu_1 y_0 + \nu_3 , \tag{5}$$

where $\mu_3 + \nu_3 = \gamma - \alpha - \beta$. Since 1, x_0 and y_0 are linearly independent, by substituting (1), (4) and (5) into

$$x_0^2 y_0 = x_0 (x_0 y_0), \quad x_0 y_0^2 = (x_0 y_0) y_0, \quad (x_0 y_0) x_0 = x_0 (y_0 x_0), \quad (y_0 x_0) y_0 = y_0 (x_0 y_0),$$

we have

$$\mu_1^2 = \alpha, \ \mu_2^2 = \beta, \ \mu_1 \mu_2 + \mu_3 = 0 \quad \text{and} \quad \mu_3 = \nu_3 = \frac{1}{2}(\gamma - \alpha - \beta), \tag{6}$$

from which $\gamma = (\mu_1 - \mu_2)^2$. By $\mu_3 = \nu_3$, it follows that

$$x_0 y_0 - y_0 x_0 = 2\mu_2 x_0 + 2\mu_1 y_0 . \tag{7}$$

By means of (3), it is easily seen that (4) follows from (7). This lemma is proved.

REMARK. $xy - yx = 2\lambda_2 x + 2\lambda_1 y + \lambda_3$, if and only if $xy = \kappa_2 x + \kappa_1 y + \kappa_3$. And then $\kappa_1 \kappa_2 + \kappa_3 = 0$; only for x_0 and y_0, we have $\lambda_1 = \kappa_1$ and $\lambda_2 = \kappa_2$.

Next we shall define, as usual, e^x by $e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$, then for $x \in \mathfrak{A}$ it is easily verified that

$$e^x = e^{\alpha_1} e^{x_0}, \qquad e^{x_0} = p(\alpha) x_0 + q(\alpha), \tag{8}$$

where

$$\begin{aligned} p(\alpha) &= \frac{sh(\sqrt{\alpha})}{\sqrt{\alpha}} && (\alpha \neq 0) \\ &= 1 && (\alpha = 0), \\ q(\alpha) &= ch(\sqrt{\alpha}). \end{aligned} \tag{9}$$

LEMMA 3. *For x and y such that $xy \neq yx$, it is necessary and sufficient for $e^x e^y = e^y e^x$ that $x_0^2 + l^2 \pi^2 = 0$ or $y_0^2 + m^2 \pi^2 = 0$ where l and m are non-zero integers, in other words, that $e^x = \nu$ or $e^y = \nu'$, $\nu, \nu' \in \mathfrak{C}$.*

PROOF. $xy \neq yx$ is equivalent to $x_0 y_0 \neq y_0 x_0$, and also $e^x e^y = e^y e^x$ is equivalent to $e^{x_0} e^{y_0} = e^{y_0} e^{x_0}$; by means of (8) we have

$$(10) \qquad e^{x_0}e^{y_0}-e^{y_0}e^{x_0}=p(\alpha)p(\beta)(x_0y_0-y_0x_0),$$

hence, for x and y such that $xy \neq yx$, we have $p(\alpha)p(\beta)=0$, if and only if $e^xe^y=e^ye^x$. From (9), if $p(\alpha)=0$, then $\sqrt{\alpha}=\sqrt{-1}\,l\pi$, i.e., $\alpha=-l^2\pi^2$ where l is a non-zero integer; and then, by (8), $e^x=\nu$, $\nu \in \mathfrak{C}$. If $p(\beta)=0$, then, similarly, $e^y=\nu'$, $\nu' \in \mathfrak{C}$. This lemma is proved.

3. Non-commutative solutions of $e^xe^y=e^{x+y}$. It is evident that $e^xe^y=e^{x+y}$ is equivalent to $e^{x_0}e^{y_0}=e^{x_0+y_0}$. And from (1), (2) and (8) we see that $e^{x_0}e^{y_0}=e^{x_0+y_0}$ is equivalent to

$$(11) \qquad \left\{ \begin{aligned} p(\alpha)p(\beta)x_0y_0+p(\alpha)q(\beta)x_0+p(\beta)q(\alpha)y_0+q(\alpha)q(\beta) \\ =p(\gamma)(x_0+y_0)+q(\gamma). \end{aligned} \right.$$

Here if $p(\alpha)p(\beta)=0$, since 1, x_0 and y_0 are linearly independent for non-commutative x and y (by Lemma 1), then from (11) it follows

$$(12) \qquad p(\alpha)p(\beta)=0, \quad p(\alpha)q(\beta)=p(\beta)q(\alpha)=p(\gamma). \quad q(\alpha)q(\beta)=q(\gamma).$$

From (12) it follows

$$p(\gamma)^2=p(\alpha)p(\beta)q(\alpha)q(\beta)=0, \qquad \text{i.e.,} \qquad p(\gamma)=0.$$

Since $q(\alpha)^2-\alpha p(\alpha)^2=1$, $p(\alpha)$ and $q(\alpha)$ are not both zero, therefore from (12) we have

$$(13) \qquad p(\alpha)=p(\beta)=p(\gamma)=0,$$

that is,

$$(14) \qquad \alpha=-l^2\pi^2, \qquad \beta=-m^2\pi^2, \qquad \gamma=-n^2\pi^2,$$

where l, m and n are non-zero integers. And then we have

$$(15) \qquad q(\alpha)=(-1)^l, \qquad q(\beta)=(-1)^m, \qquad q(\gamma)=(-1)^n,$$

where, by $q(\alpha)q(\beta)=q(\gamma)$, it must be satisfied that $l+m+n \equiv 0$ (*mod.* 2).

Thus we have the

THEOREM 1. *All the non-commutative solutions of $e^xe^y=e^{x+y}=e^ye^x$ are given by $x=\alpha_1+x_0$, $y=\beta_1+y_0$ such that*

$$x_0^2=-l^2\pi^2, \; y_0^2=-m^2\pi^2 \quad \textit{and} \quad (x_0+y_0)^2=-n^2\pi^2,$$

where α_1 and β_1 are arbitrary complex numbers, and l, m and n are non-zero integers such that $l+m+n \equiv 0$ (mod. 2).

These solutions are also given by the other condition (Fréchet's result). (see Lemma 3):

$$e^x=\nu, \qquad e^y=\nu' \qquad \textit{and} \qquad e^{x+y}=\nu\nu', \qquad \nu, \nu' \in \mathfrak{C}.$$

Next if $p(\alpha)p(\beta) \neq 0$, then from (11) 1, x_0, y_0 and $x_0 y_0$ are linearly dependent; we shall write this linear relation as follows:

$$x_0 y_0 = \mu_2 x_0 + \mu_1 y_0 + \mu_3, \tag{16}$$

which is equivalent to (by Lemma 2)

$$x_0 y_0 - y_0 x_0 = 2\mu_2 x_0 + 2\mu_1 y_0 .$$

And moreover this is equivalent to

$$xy - yx = 2\mu_2 x + 2\mu_1 y - 2\mu_2 \alpha_1 - 2\mu_1 \beta_1 .$$

Then, by means of Lemma 2, it holds that

$$\alpha = \mu_1^2,\ \beta = \mu_2^2,\ \gamma = (\mu_1 - \mu_2)^2 \quad \text{and} \quad \mu_3 = -\mu_1 \mu_2 . \tag{17}$$

By substituting (16) into (11) and by using (17) and the linear independency of 1, x_0 and y_0, we have

$$\begin{cases} p(\mu_1^2)(p(\mu_2^2)\mu_2 + q(\mu_2^2)) = p((\mu_1 - \mu_2)^2), \\ p(\mu_2^2)(p(\mu_1^2)\mu_1 + q(\mu_1^2)) = p((\mu_1 - \mu_2)^2), \\ -p(\mu_1^2)p(\mu_2^2)\mu_1\mu_2 + q(\mu_1^2)q(\mu_2^2) = q((\mu_1 - \mu_2)^2). \end{cases} \tag{18}$$

The last relation is satisfied identically; and (18) can be written as follows:

$$\begin{cases} e(-2\mu_1) = e(-2\mu_2) & \text{for} \quad \mu_1 \neq \mu_2, \\ e(-2\mu_1) = e(-2\mu_2) & \text{for} \quad \mu_1 = \mu_2, \end{cases} \tag{19}$$

where $e(\mu) = \dfrac{e^{\mu} - 1}{\mu}$ for $\mu \neq 0$, $=1$ for $\mu = 0$.

Moreover $p(\mu^2) \neq 0$ is equivalent to $e(-2\mu) \neq 0$; and from $xy \neq yx$ it follows that μ_1 and μ_2 are not both zero.

Thus we have the

THEOREM 2. *All the non-commutative solutions of* $e^x e^y = e^{x+y} \neq e^y e^x$ *are given by* $x = \alpha_1 + x_0$, $y = \beta_1 + y_0$ *such that*

$$x_0^2 = -\mu_1^2, \quad y_0^2 = -\mu_2^2, \quad (x_0 + y_0)^2 = (\mu_1 - \mu_2)^2$$

and

$$x_0 y_0 = \mu_2 x_0 + \mu_1 y_0 - \mu_1 \mu_2,$$

(which may be replaced by $x_0 y_0 - y_0 x_0 = 2\mu_2 x_0 + 2\mu_1 y_0$*), where* α_1 *and* β_1 *are arbitrary complex numbers, and* μ_1 *and* μ_2 *satisfy* $e(-2\mu_1) = e(-2\mu_2) \neq 0$ *for* $\mu_1 \neq \mu_2$ *and* $e(2\mu_1) = 1$ *for* $\mu_1 = \mu_2 \neq 0$.

REMARK 1. If $x_0 y_0 = \mu_2 x_0 + \mu_1 y_0 + \mu_3$ and $x_0^2, y_0^2 \in \mathfrak{C}$, then $x_0^2 = \mu_1^2$, $y_0^2 = \mu_2^2$, $(x_0 + y_0)^2 = (\mu_1 - \mu_2)^2$ and $\mu_3 = -\mu_1 \mu_2$. (See Lemma 2).

REMARK 2. Since $p(\mu^2)\mu + q(\mu^2) = e^{\mu}$ and $-p(\mu_1^2)p(\mu_2^2)\mu_1\mu_2 + q(\mu_1^2)q(\mu_2^2) = $

$q((\mu_1-\mu_2)^2)$, by regarding as $\alpha\equiv\mu_1^2$ and $\beta\equiv\mu_2^2$, (12) and (18) are unified into

$$\begin{cases} p(\mu_1^2)e^{\mu_2}=p(\tau) \\ p(\mu_2^2)e^{\mu_1}=p(\tau) \\ q((\mu_1-\mu_2)^2)=q(\tau)\,. \end{cases}$$

We can derive from this that either $p(\tau)=0$ or $\tau=(\mu_1-\mu_2)^2$, corresponding to the preceding two cases respectively. For the case $\mu_1\neq\mu_2$, it is easily verified that $p(\mu_1^2)e^{\mu_2}=p(\mu_2^2)e^{\mu_1}$ implies $p(\mu_2^2)e^{\mu_1}=p((\mu_1-\mu_2)^2)$, hence $p((\mu_1-\mu_2)^2)=p(\tau)$. On the other hand $q((\mu_1-\mu_2)^2)=q(\tau)$, therefore, taking account of $q(\tau)^2-\tau p(\tau)^2=1$ and $q((\mu_1-\mu_2)^2-(\mu_1-\mu_2)^2p((\mu_1-\mu_2)^2)=1$, we have

$$p(\tau)^2(\tau-(\mu_1-\mu_2)^2)=0\,.$$

That is, $p(\tau)=0$ or $\tau=(\mu_1-\mu_2)^2$. For the case $\mu_1=\mu_2$, we have $q(\tau)=q(0)=1$, from $q(\tau)^2-\tau p(\tau)^2=1$ we have $\tau p(\tau)^2=0$. That is, $p(\tau)=0$ or $\tau=0(=(\mu_1-\mu_2)^2)$.

REMARK 3. These exist no real numbers u_1 and u_2 such that

$$e(u_1)=e(u_2) \quad \text{and} \quad u_1\neq u_2\,.$$

By the definition of $e(u)$,

$$\begin{aligned} e(u)&=\frac{e^u-1}{u} && (u\neq 0) \\ &=1 && (u=0)\,. \end{aligned}$$

Since we have $\dfrac{de(u)}{du}=\dfrac{(u-1)e^u-1}{u^2}>0$ for $u\neq 0$, $e(u)$ is a monotone increasing function. Therefore, if $u_1\neq u_2$, then $e(u_1)\neq e(u_2)$. This proves our assertion.

4. Complex ternary algebras of degree two. There are two types of complex ternary algebras of degree two; one of which is commutative, and the other is as follows:[1]

The basic elements 1, e_1, e_2 satisfy the relations:

$$1e_i=e_i1=e_i\ (i=1,2),\ e_1^2=1,\ e_1e_2=e_2,\ e_2e_1=-e_2,\ e_2^2=0\,.$$

Any elements x and y of this algebra are expressed as

$$x=\xi_0+\xi_1e_1+\xi_2e_2\,, \qquad y=\eta_0+\eta_1e_1+\eta_2e_2\,,$$

and then $(x-\xi_0)^2=\xi_1^2$, $(y-\eta_0)^2=\eta_1^2$. The condition in Theorem 1 is written as

$$\text{(20)} \qquad \begin{cases} \xi_1=\sqrt{-1}\,l\pi\,, \\ \eta_1=\sqrt{-1}\,m\pi\,, \end{cases}$$

1) L. E. Dickson, Linear algebras, Cambridge tracts in Mathematics and Mathematical physics, no. 16 (1914), pp. 23-26.

where l, m are non-zero integers such that $l+m \neq 0$. And it is easily seen that $xy=yx$ if and only if $\xi_1:\xi_2=\eta_1:\eta_2$. The condition in Theorem 2 is written as

$$
(21) \qquad \begin{cases} \xi_1\eta_2-\xi_2\eta_1=\mu_2\xi_2+\mu_1\eta_2, \\ \qquad\qquad 0=\mu_2\xi_1+\mu_1\eta_1, \end{cases}
$$

where $\xi_1^2=\mu_2^2$, $\eta_1^2=\mu_2^2$, $\xi_1\eta_1=-\mu_1\mu_2$ and $\xi_1:\xi_2 \neq \eta_1:\eta_2$; from this we have $\mu_1=\xi_1$ and $\mu_2=-\eta_1$. Therefore, $\mu_1=\xi_1$ and $\mu_2=-\eta_1$ must satisfy:

$$
(22) \qquad \begin{cases} e(-2\xi_1)=e(2\eta_1) \neq 0 & \text{for} \quad \xi_1+\eta_1 \neq 0, \\ e(2\xi_1)=1 & \text{for} \quad \xi_1=-\eta_1 \neq 0. \end{cases}
$$

Moreover it is easily seen that there exists no real number satisfying (22). (See Remark 3, p. 350).

Thus summarizing these results, we have the corollary.

Corollary 1. *In the non-commutative complex ternary algebra of degree two, all the non-commutative solutions of $e^x e^y = e^{x+y}$ are given by*

$$
\begin{cases} x=\xi_0+\xi_1 e_1+\xi_2 e_2, \\ y=\eta_0+\eta_1 e_1+\eta_2 e_2, \end{cases}
$$

where $e(-2\xi_1)=e(2\eta_1)$ for $\xi_1+\eta_1 \neq 0$ and $e(2\xi_1)=1$ for $\xi_1=-\eta_1 \neq 0$, $\xi_1:\xi_2 \neq \eta_1:\eta_2$ and ξ_0, η_0 are arbitrary complex numbers. In the real ternary algebra of degree two, $e^x e^y = e^{x+y}$ implies $xy=yx$.

5. Quaternion algebra. Now let $\mathfrak{A}$ be the complex quternion algebra, then any elements x and y of $\mathfrak{A}$ are written as

$$
\begin{cases} x=\xi_0+x_0=\xi_0+\xi_1 i+\xi_2 j+\xi_3 k, & \xi_0, \xi_1, \xi_2, \xi_3 \in \mathfrak{C} \\ y=\eta_0+y_0=\eta_0+\eta_1 i+\eta_2 j+\eta_3 k, & \eta_0, \eta_1, \eta_2, \eta_3 \in \mathfrak{C} \end{cases}
$$

where $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$. And if we write $N(x_0) = \sum_{i=1}^{3} \xi_i^2$ and $N(y_0)=\sum_{i=1}^{3} \eta_i^2$, then we have

$$
(x-\xi_0)^2=-N(x_0),
$$

$$
(y-\eta_0)^2=-N(y_0).
$$

Therefore, in this case, Theorem 1 and 2 are stated as follows.

Corollary 2. *In the complex quaternion algebra, all the non-commutative solutions of $e^x e^y = e^{x+y} = e^y e^x$ are given by*

$$
\begin{cases} x=\xi_0+\xi_1 i+\xi_2 j+\xi_3 k \\ y=\eta_0+\eta_1 i+\eta_2 j+\eta_3 k \end{cases}
$$

such that ξ_0 and η_0 are arbitrary complex numbers and

$$N(x_0)\equiv\sum_{i=1}^{3}\xi_i^2=l^2\pi^2,\qquad N(y_0)\equiv\sum_{i=1}^{3}\eta_i^2=m^2\pi^2,$$

$$\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3=-\frac{1}{2}(l^2+m^2-n^2)\pi^2,$$

(the last relation may be replaced by $N(x_0+y_0)\equiv\sum_{i=1}^{3}(\xi_i+\eta_i)^2=n^2\pi^2$*),*
and

$$\xi_1:\xi_2:\xi_3\neq\eta_1:\eta_2:\eta_3,$$

where l, m and n are non-zero integers such that $l+m+n\equiv 0$ (mod 2).

COROLLARY 3. *In the complex quaternion algebra, all the non-commutative solutions of $e^xe^y=e^{x+y}\neq e^ye^x$ are given by*

$$\begin{cases} x=\xi_0+\xi_1 i+\xi_2 j+\xi_3 k, \\ y=\eta_0+\eta_1 i+\eta_2 j+\eta_3 k \end{cases}$$

such that

$$\text{(i)}\qquad \begin{cases} \xi_2\eta_3-\xi_3\eta_2=\mu_2\xi_1+\mu_1\eta_1, \\ \xi_3\eta_1-\xi_1\eta_3=\mu_2\xi_2+\mu_1\eta_2, \\ \xi_1\eta_2-\xi_2\eta_1=\mu_2\xi_3+\mu_1\eta_3, \end{cases}$$

$$\xi_1:\xi_2:\xi_3\neq\eta_1:\eta_2:\eta_3,$$

where $e(-2\mu_2)=e(-2\mu_1)\neq 0$ for $\mu_1\neq\mu_2$ and $e(2\mu_1)=1$ for $\mu_1=\mu_2\neq 0$.

And then it holds (see Remark 1, p. 349) that

$$\text{(ii)}\qquad N(x_0)\equiv\sum_{i=1}^{3}\xi_i^2=-\mu_1^2,\qquad N(y_0)\equiv\sum_{i=1}^{3}\eta_i^2=-\mu_2^2.$$

$$\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3=\mu_1\mu_2,\qquad \xi_1:\xi_2:\xi_3\neq\eta_1:\eta_2:\eta_3,$$

and consequently

$$N(x_0y_0)\equiv(\xi_2\eta_3-\xi_3\eta_2)^2+(\xi_3\eta_1-\xi_1\eta_3)^2+(\xi_1\eta_2-\xi_2\eta_1)^2=0,$$

which is the necessary and sufficient condition for that 1, *x, y and xy are linearly dependent. (Now we are in the case where* 1, *x, y and xy are linearly dependent.)*

REMARK. (i) is written in the tensor notation as follows:

$$\varepsilon^{ij}{}_{k}\xi_i\eta_j=\mu_2\xi_k+\mu_1\eta_k\quad\text{and}\quad \xi_i\neq\kappa\eta_i\ (i,j,k=1,2,3),$$

where

$$\varepsilon^{ij}{}_{k}=\begin{cases} 1, \text{ if } (ijk) \text{ is an even permutation of } 123, \\ -1, \text{ if } (ijk) \text{ is an odd permutation of } 123, \\ 0, \text{ for the other cases.} \end{cases}$$

(ii) is written as

$$N(x_0)=-\mu_1^2,\ N(y_0)=-\mu_2^2,\ N(x_0+y_0)=-(\mu_1-\mu_2)^2\ \text{ and }\ x_0\neq\kappa y_0.$$

It was mentioned above that (i) implies (ii). Now let us suppose (ii), and if we put $\overset{\circ}{\varepsilon}{}^{ij}{}_{k}\xi_i\eta_j=\mu_2\xi_k+\mu_1\eta_k+\zeta_k$, then from (ii) we have

$$\xi^i\zeta_i=0 \quad\text{and}\quad \eta^i\zeta_i=0,\quad\text{hence}\quad \zeta_k=\rho\,\overset{\circ}{\varepsilon}{}^{ij}{}_{k}\xi_i\eta_j.$$

Therefore we have

(iii) $$(1-\rho)\,\varepsilon^{ij}{}_{k}\xi_i\eta_j=\mu_2\xi_k+\mu_1\xi_k.$$

We may suppose that $\mu_1\neq0$ and $\xi_1\eta_2-\xi_2\eta_1\neq0$, then from (iii) we have

$$(1-\rho)(\xi_2(\xi_2\eta_3-\xi_3\eta_2)-\xi_1(\xi_3\eta_1-\xi_1\eta_3))=\mu_1(\xi_2\eta_1-\xi_1\eta_2).$$

And furthermore, by using (ii) and (iii) we have

$$(1-(1-\rho)^2)\mu_1(\xi_1\eta_2-\xi_2\eta_1)=0,\qquad\text{that is}\qquad (1-\rho)^2=1.$$

Thus we have

(i′) $$\overset{\circ}{\varepsilon}{}^{ij}{}_{k}\xi_i\eta_j=\pm(\mu_2\xi_k+\mu_1\eta_k).$$

As before it is obvious that (i)′ implies (ii).

This means that

$$x_0y_0=\pm(\mu_2x_0+\mu_1y_0)+\mu_3,$$

which is equivalent to (see (4) and (5) in the proof of Lemma 2)

$$x_0y_0=\mu_2x_0+\mu_1y_0+\mu_3\qquad\text{or}\qquad y_0x_0=\mu_2x_0+\mu_1y_0+\mu_3.$$

By Remark 1 after Theorem 2, this is equivalent to that

$$e^xe^y=e^{x+y}\neq e^ye^x\qquad\text{or}\qquad e^ye^x=e^{y+x}\neq e^xe^y.$$

Thus (ii) is the necessary and sufficient condition that

$$e^xe^y=e^{x+y}\neq e^ye^x\qquad\text{or}\qquad e^ye^x=e^{y+x}\neq e^xe^y.$$

Next for the real quaternions, by means of the fact that $\sum_{i=1}^{3}\xi_i^2\sum_{i=1}^{3}\eta_i^2-\left(\sum_{i=1}^{3}\xi_i\eta^i\right)^2$ $=(\xi_2\eta_3-\xi_3\eta_2)^2+(\xi_3\eta_1-\xi_1\eta_3)^2+(\xi_1\eta_2-\xi_2\eta_1)^2>0$ (since $\xi_i\neq k\eta_i$), we have

$$D\equiv(l+m+n)(l-m+n)(l+m-n)(l-m-n)<0.$$

And from

$$N(x_0y_0)\equiv(\xi_2\eta_3-\xi_3\eta_2)^2+(\xi_3\eta_1-\xi_1\eta_3)^2+(\xi_1\eta_2-\xi_2\eta_1)^2=0$$

in Corollary 3, we have $\xi_i=k\eta_i$, hence there exist no real quaternions satisfying $e^xe^y=e^{x+y}\neq e^ye^x$. From Corollaries 2 and 3, we have

COROLLARY 4. *All the non-commutative real quaternions x and y satisfying $e^xe^y=e^{x+y}=e^ye^x$ are given by*

$$\begin{cases} x=\xi_0+x_0=\xi_0+\xi_1i+\xi_2j+\xi_3k,\\ y=\eta_0+y_0=\eta_0+\eta_1i+\eta_2j+\eta_3k,\end{cases}$$

such that ξ_0 and η_0 are arbitrary real numbers and

$$N(x_0)\equiv\sum_{i=1}^{3}\xi_i^2=l^2\pi^2,\ N(y_0)\equiv\sum_{i=1}^{3}\eta_i^2=m^2\pi^2 \quad \textit{and} \quad N(x_0+y_0)\equiv\sum_{i=1}^{3}(\xi_i+\eta_i)^2=n^2\pi^2,$$

where l, m and n are non-zero integers such that

$$D\equiv(l+m+n)(l-m+n)(l+m-n)(l-m-n)<0, \qquad l+m+n\equiv 0 \ (mod.\ 2).$$

And there exist no real quaternions satisfying

$$e^x e^y=e^{x+y}\neq e^x e^y.$$

6. Total matric algebra of order two. As the complex total matric algebra is isomorphic to the complex quaternion algebra, so the condition to be found for the complex total matric algebra is obtained from the results of **5.** But now we shall determine the canonical form of non-commutative matrices satisfying $e^x e^y =e^{x+y}$.

Since, by means of Theorem 1, $x_0^2=-l^2\pi^2$, x_0 is written as

$$x_0=p^{-1}\begin{pmatrix}\sqrt{-1}\,l\pi & 0\\ 0 & -\sqrt{-1}\,l\pi\end{pmatrix}p, \quad (p \text{ is a non-singular matrix}). \tag{23}$$

And then, from $y_0^2=-m^2\pi^2$, it follows that

$$y_0=p^{-1}y_1p,\quad y_1=\begin{pmatrix}\lambda & \mu\\ \nu & -\lambda\end{pmatrix},\quad \lambda^2+\mu\nu=-m^2\pi^2,$$

furthermore, from $(x_0+y_0)^2=-n^2\pi^2$, we have

$$(\lambda+\sqrt{-1}\,l\pi)^2+\mu\nu=-n^2\pi^2,$$

therefore we have

$$\lambda=\frac{-\sqrt{-1}\,\pi}{2l}(l^2+m^2-n^2),$$

$$\mu\nu=\frac{\pi^2}{4l^2}D=\frac{\pi^2}{4l^2}(l+m+n)(l+m-n)(l-m+n)(l-m-n).$$

Thus we get

$$y_0=p^{-1}y_1p,\quad y_1=\begin{pmatrix}\dfrac{-\sqrt{-1}\,\pi}{2l}(l^2+m^2-n^2) & \mu\\ \nu & \dfrac{-\sqrt{-1}\,\pi}{2l}(l^2+m^2-n^2)\end{pmatrix}, \tag{24}$$

where $\mu\nu=\dfrac{\pi^2}{4l^2}D=\dfrac{\pi^2}{4l^2}(l+m+n)(l-m+n)(l+m-n)(l-m-n)$. By transforming y_1 by a suitable matrix $\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}$ leaving invariant $\begin{pmatrix}\sqrt{-1}\,l\pi & 0\\ 0 & -\sqrt{-1}\,l\pi\end{pmatrix}$, we have

$$\text{(25)}\qquad \begin{cases} x_0=q^{-1}\begin{pmatrix}\sqrt{-1}\,l\pi & 0\\ 0 & -\sqrt{-1}\,l\pi\end{pmatrix}q, \\[2ex] y_0=q^{-1}\begin{pmatrix}\dfrac{-\sqrt{-1}\,\pi}{2l}(l^2+m^2-n^2) & \dfrac{\pi}{2l}((l-m^2)-n^2)\\[1ex] \dfrac{\pi}{2l}((l+m)^2-n^2) & \dfrac{\sqrt{-1}\,\pi}{2l}(l^2+m^2-n^2)\end{pmatrix}q, \end{cases}$$

where l, m and n are non-zero integers such that $l+m+n\equiv 0 \pmod{2}$. Moreover $D=0$, if and only if 1, x, y and xy are linearly dependent; and then we have, taking account of the transformation by a matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$,

$$\begin{cases} x_0=q^{-1}\begin{pmatrix}\sqrt{-1}\,l\pi & 0\\ 0 & -\sqrt{-1}\,l\pi\end{pmatrix}q, \\[1ex] y_0=q^{-1}\begin{pmatrix}\sqrt{-1}\,m\pi & 1\\ 0 & -\sqrt{-1}\,m\pi\end{pmatrix}q, \quad (l+m\neq 0). \end{cases}$$

Since, by means of Theorem 2, $x_0^2=\mu_1^2$, x_0 is written as:

$$\text{(26)}\qquad \begin{cases} x_0=p^{-1}\begin{pmatrix}\mu_1 & 0\\ 0 & -\mu_1\end{pmatrix}p \qquad (\mu_1\neq 0), \\[1ex] \text{or} \quad\ \ =p^{-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}p \qquad (\mu_1=0). \end{cases}$$

And then, from $y_0^2=\mu_2^2$, it follows that

$$y_0=p^{-1}y_1p,\quad y_1=\begin{pmatrix}\lambda & \mu\\ \nu & -\lambda\end{pmatrix},\quad \lambda^2+\mu\nu=\mu_2^2,$$

next, from $x_0y_0=\mu_2x_0+\mu_1y_0-\mu_1\mu_2$, we have

$$\lambda=-\mu_2 \qquad \text{and} \qquad \nu=0,$$

that is,

$$y_1=\begin{pmatrix}-\mu_2 & \mu\\ 0 & \mu_2\end{pmatrix}.$$

By transforming by a suitable matrix $\begin{pmatrix}1&0\\0&t\end{pmatrix}$ and $\begin{pmatrix}1&s\\0&1\end{pmatrix}$ leaving invariant $\begin{pmatrix}\mu_1&0\\0&-\mu_1\end{pmatrix}$ $(\mu_1\neq 0)$ and $\begin{pmatrix}0&1\\0&0\end{pmatrix}$ respectively, we have

$$\text{(27)}\qquad \begin{cases} x_0=q^{-1}\begin{pmatrix}\mu_1 & 0\\ 0 & -\mu_1\end{pmatrix}q \qquad (\mu_1\neq 0), \\[1ex] y_0=q^{-1}\begin{pmatrix}-\mu_2 & 1\\ 0 & \mu_2\end{pmatrix}q, \end{cases}$$

where $e(-2\mu_1)=e(-2\mu_2)\neq 0$ for $\mu_1\neq\mu_2$, and $e(2\mu_1)=1$ for $\mu_1=\mu_2\neq 0$; and

$$(28)\qquad \begin{cases} x_0=q^{-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}q, \\ y_0=q^{-1}\begin{pmatrix}-\mu_2 & 0\\ 0 & \mu_2\end{pmatrix}q, \end{cases}$$

where $e(-2\mu_2)=1$ and $\mu_2 \neq 0$.

Thus we have the

COROLLARY 5. *All the non-commutative complex matrices x and y of order two satisfying $e^x e^y = e^{x+y}$ are given by*

$$x=\alpha_1+q^{-1}\mathring{x}q, \qquad y=\beta_1+q^{-1}\mathring{y}q,$$

where q is an arbitrary non-singular matrix, α_1 and β_1 are arbitrary complex numbers, and $\mathring{x}$ and $\mathring{y}$ are given as follows:

(I): *The case where $e^x e^y = e^y e^x$ is satisfied.*

$$\begin{cases} \mathring{x}=\begin{pmatrix}\sqrt{-1}\,l\pi & 0\\ 0 & -\sqrt{-1}\,l\pi\end{pmatrix}, \\ \mathring{y}=\dfrac{\pi}{2l}\begin{pmatrix}-\sqrt{-1}\,(l^2+m^2-n^2) & (l-m)^2-n^2\\ (l+m)^2-n^2 & \sqrt{-1}\,(l^2+m^2-n^2)\end{pmatrix}, \end{cases}$$

where l, m and n are non-zero integers such that $l+m+n\equiv 0$ (mod. 2). And $D\equiv(l+m+n)(l-m+n)(l+m-n)(l-m-n)=0$ is the condition for the linear dependency of 1, x, y and xy, and then

$$\begin{cases} \mathring{x}=\begin{pmatrix}\sqrt{-1}\,l\pi & 0\\ 0 & -\sqrt{-1}\,l\pi\end{pmatrix}, \\ \mathring{y}=\begin{pmatrix}\sqrt{-1}\,m\pi & 0\\ 0 & -\sqrt{-1}\,m\pi\end{pmatrix}, \end{cases}$$

where l and m are arbitrary non-zero integers such that $l+m\neq 0$.

(II): *The case where $e^x e^y \neq e^y e^x$ is satisfied.*

$$(\mathrm{II}_1)\qquad \begin{cases} \mathring{x}=\begin{pmatrix}\mu_1 & 0\\ 0 & -\mu_1\end{pmatrix} \quad (\mu_1\neq 0), \\ \mathring{y}=\begin{pmatrix}-\mu_2 & 1\\ 0 & \mu_2\end{pmatrix}, \end{cases}$$

where $e(-2\mu_1)=e(-2\mu_2)\neq 0$ for $\mu_1\neq\mu_2$ and $e(2\mu_1)=1$ for $\mu_1=\mu_2\neq 0$.

$$(\mathrm{II}_2)\qquad \begin{cases} \mathring{x}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \\ \mathring{y}=\begin{pmatrix}-\mu_2 & 0\\ 0 & \mu_2\end{pmatrix}, \end{cases}$$

where $e(-2\mu_2)=1$ and $\mu_2\neq 0$.

The four types (even the matrices of the same type for distinct values of l, m, n, μ_1 and μ_2) are not transformable to each other by any transformation $x'=\xi+p^{-1}xp$, $y'=\eta+p^{-1}yp$.

Next a real matrix x_0 such that $x_0^2=-l^2\pi^2$ $(l\neq 0)$ is written as

$$x_0=p^{-1}\begin{pmatrix}0 & -l^2\pi^2\\ 1 & 0\end{pmatrix}p, \qquad (p \text{ is a real non-singular matrix});$$

furthermore, since

$$\begin{pmatrix}0 & -l^2\pi^2\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -l\pi\end{pmatrix}^{-1}\begin{pmatrix}0 & l\pi\\ -l\pi & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -l\pi\end{pmatrix},$$

we have

$$x_0=q^{-1}\begin{pmatrix}0 & l\pi\\ -l\pi & 0\end{pmatrix}q, \qquad (q \text{ is a real non-singular matrix}).$$

And then a real matrix y_0 such that $y_0^2=-m^2\pi^2$ is written as

$$y_0=q^{-1}y_1q, \qquad y_1=\begin{pmatrix}\lambda & \mu\\ \nu & -\lambda\end{pmatrix}, \qquad \lambda^2+\mu\nu=-m^2\pi^2.$$

From $(x_0+y_0)^2=-n^2\pi^2$, it follows that

$$\nu-\mu=\frac{\pi}{l}(l^2+m^2-n^2).$$

Therefore, it is easily seen that $\lambda^2+\mu\nu$ and $\nu-\mu$ are invariant for the transformations by matrices leaving invariant $\begin{pmatrix}0 & l\pi\\ -l\pi & 0\end{pmatrix}$. By a matrix $\begin{pmatrix}1 & t\\ -t & 1\end{pmatrix}$ leaving invariant $\begin{pmatrix}0 & l\pi\\ -l\pi & 0\end{pmatrix}$, we have

$$y_1'=\begin{pmatrix}\lambda' & \mu'\\ \nu' & -\lambda'\end{pmatrix}=\begin{pmatrix}1 & t\\ -t & 1\end{pmatrix}^{-1}\begin{pmatrix}\lambda & \mu\\ \nu & -\lambda\end{pmatrix}\begin{pmatrix}1 & t\\ -t & 1\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{1+t^2}(\lambda-(\mu+\nu)t-\lambda t^2) & \frac{1}{1+t^2}(\mu+2\lambda t-\nu t^2)\\ \frac{1}{1+t^2}(\nu+2\lambda t-\mu t^2) & -\frac{1}{1+t^2}(\lambda-(\mu+\nu)t-\lambda t^2)\end{pmatrix}.$$

So if we take $t=t_0$ (real number) such that

$$\lambda-(\mu+\nu)t_0-\lambda t_0^2=0,$$

then we have

$$y_1''=\begin{pmatrix}0 & \mu''\\ \nu'' & 0\end{pmatrix},$$

where $\nu''-\mu''=\frac{\pi}{l}(l^2+m^2+n^2)$ and $\mu''\nu''=-m^2\pi^2$, i.e.,

$$\mu'' = \frac{\pi}{2l}\left(-(l^2+m^2-n^2)+\sqrt{D}\right) \quad \text{and} \quad \nu'' = \frac{\pi}{2l}\left(l^2+m^2-n^2+\sqrt{D}\right),$$

$D \equiv (l+m+n)(l-m+n)(l+m-n)(l-m-n) > 0$. (If $D=0$, then $x_0y_0=y_0x_0$).
Thus we have

$$\begin{cases} x_0 = r^{-1}\begin{pmatrix} 0 & l\pi \\ -l\pi & 0 \end{pmatrix} r, \\ y_0 = r^{-1}\begin{pmatrix} 0 & \frac{\pi}{2l}\left(-(l^2+m^2-n^2)+\sqrt{D}\right) \\ \frac{\pi}{2l}\left(l^2+m^2-n^2+\sqrt{D}\right) & 0 \end{pmatrix} r. \end{cases}$$

For real matrices x_0 and y_0, if 1, x_0 and y_0 are linearly independent and $x_0y_0 = \mu_2x_0+\mu_1y_0-\mu_1\mu_2$, then μ_1 and μ_2 must be real. There are no real numbers such that $e(-2\mu_1)=e(-2\mu_2)\neq 0$ for $\mu_1\neq\mu_2$ and $e(2\mu_1)=1$ for $\mu_1=\mu_2\neq 0$. (See Remark 3, p. 350). That is, there exist no real matrices of order two such that $e^xe^y=e^{x+y} \neq e^ye^x$.

Thus we have

COROLLARY 6. *All the non-commutative real matrices x and y of order two satisfying $e^xe^y=e^{x+y}$ are given by*

$$\begin{cases} x=\alpha_1+q^{-1}\mathring{x}q, \\ y=\beta_1+q^{-1}\mathring{y}q, \end{cases}$$

where q is an arbitrary non-singular real matrix, α_1 and β_1 are arbitrary real numbers, and $\mathring{x}$ and $\mathring{y}$ are given as follows:

$$\text{(I):} \quad \begin{cases} \mathring{x} = \begin{pmatrix} 0 & l\pi \\ -l\pi & 0 \end{pmatrix}, \\ \mathring{y} = \begin{pmatrix} 0 & \frac{\pi}{2l}\left(-(l^2+m^2-n^2)+\sqrt{D}\right) \\ \frac{\pi}{2l}\left(l^2+m^2-n^2+\sqrt{D}\right) & 0 \end{pmatrix}, \end{cases}$$

where l, m and n are non-zero integers such that $l+m+n\equiv 0 \pmod 2$ and $D\equiv(l+m+n)(l-m+n)(l+m-n)(l-m-n)>0$.

(II): *There exist no real matrices of order two such that $e^xe^y=e^{x+y}\neq e^ye^x$; that is, for real matrices, $e^xe^y=e^{x+y}$ implies $e^xe^y=e^ye^x$.*

Mathematical Institute,
Hiroshima University