JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 17, NO. 3, MAR., 1954

The Intersection Theorem on Noetherian Rings

By

Michio YOSHIDA and Motoyoshi SAKUMA

(Received November 7, 1953)

Introduction. The intersection theorem, due to Krull, Chevalley and Zariski, is of paramount importance in the theory of Noetherian rings. But it being stated only for rings with unit elements, we tried to formulate and prove the theorem for rings not assumed to have unit elements. Our proof is based on an investigation of relations between ideals in a ring and those in a ring obtained by adjoining a unit element to the original one. This may be blamed too long and too compliciated in order to obtain only the theorem in view. However Lemmas I* and III, which state relations between two rings, will be of some interest in themselves.

1. Theorems known. For the sake of convenience, we state the theorems due to Krull, Chevalley and Zariski, and sketch their proofs briefly.

Let $\mathfrak{o}$ be a commutative ring and form the set $\mathfrak{o}^*$ which consists of all pairs $[a, m]$ where $a \in \mathfrak{o}$ and m is an integer. $\mathfrak{o}^*$ becomes a ring if we define addition and multiplication in $\mathfrak{o}^*$ as follows:

$$[a, m]+[b, n]=[a+b, m+n],$$
$$[a, m]\cdot[b, n]=[ab+na+mb, mn].$$

We may identify $[a, 0]$, $[0, m]$ with a, m respectively in obvious reasons. Then $\mathfrak{o}^*$ is a ring with the unit element 1, and contains $\mathfrak{o}$ and the ring of integers Z as an ideal and a subring respectively. Every ideal in $\mathfrak{o}$ is an ideal in $\mathfrak{o}^*$. $\mathfrak{o}$ is a prime ideal in $\mathfrak{o}^*$, and every prime ideal in $\mathfrak{o}^*$, which contains $\mathfrak{o}$ strictly and is not equal to $\mathfrak{o}^*$, is a maximal ideal. If $\mathfrak{o}$ is Noetherian, $\mathfrak{o}^*$ is too.

(L_1) *Let* $\mathfrak{a}$ *be an ideal in a Noetherian ring and put* $\mathfrak{c}=\bigcap_{n=1}^{\infty}\mathfrak{a}^n$ *then we have* $\mathfrak{c}\mathfrak{a}=\mathfrak{c}$.

Let $\mathfrak{c}\mathfrak{a}=[\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$ be a representation of $\mathfrak{c}\mathfrak{a}$ as an intersection of primary ideals and let $\mathfrak{p}_i$ be the prime divisor of $\mathfrak{q}_i$. If $\mathfrak{a} \not\leq \mathfrak{p}_i$, then $\mathfrak{c} \leq \mathfrak{q}_i$ follows from $\mathfrak{c}\mathfrak{a} \leq \mathfrak{q}_i$. If $\mathfrak{a} \leq \mathfrak{q}_i$, then $\mathfrak{c} \leq \mathfrak{a}^{\rho_i} \leq \mathfrak{p}_i^{\rho_i} \leq \mathfrak{q}_i$, ρ_i being the exponent of $\mathfrak{q}_i$. After all we have always $\mathfrak{c} \leq \mathfrak{q}_i$.

(L_2) *Let* $\mathfrak{c}$, $\mathfrak{a}$ *be ideals in a commutative ring* $\mathfrak{o}$ *such that* $\mathfrak{c}\mathfrak{a}=\mathfrak{c}$ *and suppose* $\mathfrak{c}$ *has a finite base. Then there exists an element* a *in* $\mathfrak{c}$ *such that* $c \cdot a=c$ *for any element* c *in* $\mathfrak{c}$.[1)]

1) S. Mori, *Ueber Productzerlegung der Ideals.* This Journal 2 (1932) Satz 1, p. 1.

Put $\mathfrak{c}=(c_1, \cdots, c_n)$, then we have $c_i=\Sigma a_{ij}c_j$, $a_{ij}\in\mathfrak{a}$. Forming $\mathfrak{o}^*$ if necessary, we have $c_i\varDelta=0$, where $\varDelta=\det(\delta_{ij}-a_{ij})=1-a$, $a\in\mathfrak{a}$, then $ca=c$ for any element c in $\mathfrak{c}$.

THEOREM (K). *Let $\mathfrak{o}$ be a Noetherian integral domain, and let $\mathfrak{o}$ be an ideal in $\mathfrak{o}$, but not equal to $\mathfrak{o}$ if $\mathfrak{o}$ has a unit element. Then we have $\bigcap\mathfrak{a}^n=(0)$.*[2]

THEOREM (C). *Let $\mathfrak{a}$ be an ideal in a Noetherian ring with unit element 1, and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be the prime divisors of the zero ideal. Then the following statements are equivalents;* i) $\bigcap\mathfrak{a}^n=(0)$ ii) *no element which is $\equiv 1\,(mod.\,\mathfrak{a})$ is a zero divisor,* iii) $\mathfrak{p}_i+\mathfrak{a}\neq(1)\ (i=1, \cdots, r)$.[3]

This follows from L_1, L_2 and the fact that the set of all zero divisors is $\bigcup_{i=1}^{r}\mathfrak{p}_i$.

(L_3) *Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals in a Noetherian ring $\mathfrak{o}$ with unit element and let $\mathfrak{p}_i\ (i=1, \cdots, r)$ be the prime divisors of $\mathfrak{b}$, and suppose $\mathfrak{b}\neq\mathfrak{o}$. Then the following statements are equivalent:* i) $\bigcap(\mathfrak{b}+\mathfrak{a}^n)=\mathfrak{b}$, ii) $\mathfrak{p}_i+\mathfrak{a}\neq(1)\ (i=1, \cdots, r)$.

This is easily seen if we apply the Theorem (C) to the residue ring $\mathfrak{o}/\mathfrak{b}$.

THEOREM (Z). *Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals in a Noetherian ring $\mathfrak{o}$ with a unit element 1, and let $\mathfrak{b}=[\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$ be an irredundant representation of $\mathfrak{b}$ as an intersection of primary ideals, and let $\mathfrak{p}_i$ be the prime divisor of $\mathfrak{q}_i$. If $\mathfrak{p}_i+\mathfrak{a}\neq(1)$ for $i=1, \cdots, s$ and $\mathfrak{p}_j+\mathfrak{a}=(1)$ for $j=s+1, \cdots, r$, then*

$$\bigcap(\mathfrak{b}+\mathfrak{a}^n)=[\mathfrak{q}_1, \cdots, \mathfrak{q}_s].$$[4]

We may assume $s<r$. Then $[\mathfrak{q}_{s+1}, \cdots, \mathfrak{q}_r]+\mathfrak{a}^n=(1)$ for any positive integer n, whence $[\mathfrak{q}_1, \cdots, \mathfrak{q}_s]\subseteq(\mathfrak{b}+\mathfrak{a}^n)$. On the other hand, we have by L_3, $[\mathfrak{q}_1, \cdots, \mathfrak{q}_s]=\bigcap([\mathfrak{q}_1, \cdots, \mathfrak{q}_s]+\mathfrak{a}^n)\supseteq\bigcap(\mathfrak{b}+\mathfrak{a}^n)$.

2. Lemmas. Let $\mathfrak{o}$ be a commutative ring and let $\mathfrak{o}^*$ be the ring obtained by adjoining a unit element in the usual manner described in the preceding section.

LEMMA I. *Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ and assume that $\mathfrak{p}$ is not prime in $\mathfrak{o}^*$. Then there exists a only one prime ideal $\mathfrak{p}^*$ in $\mathfrak{o}^*$ such that*

$$\mathfrak{p}=\mathfrak{p}^*\cap\mathfrak{o}.$$

$\mathfrak{p}^*$ consists of elements such that

$$c+m(b_0-r_0) \qquad \textit{where } c\in\mathfrak{p} \textit{ and } m \textit{ is an integer.}$$

r_0 is uniquely determined positive integer and b_0 is an element in $\mathfrak{o}$ uniquely determined mod. $\mathfrak{p}$ such that $ab_0\equiv ar_0\,(mod.\,\mathfrak{p})$ for any a in $\mathfrak{o}$. The following statements are equivalent:

i) $r_0=1$, ii) $\mathfrak{o}/\mathfrak{p}$ *has a unit element,* iii) $\mathfrak{o}+\mathfrak{p}^*=\mathfrak{o}^*$.

When $r_0=1$, b_0 is a unit element mod. $\mathfrak{p}$ in $\mathfrak{o}$.

---

2) W. Krull, *Dimensionstheorie in Stellenringen,* J. Reine Angew. Math., 179 (1938) Satz 1, p. 206.

3) C. Chevalley, *On the theory of local rings,* Ann. of Math., 44 (1943) Lemma 2, p. 692.

4) O. Zariski, *Generalized semi-local rings,* Sum. Bras. Math., 1 (1946) Lemma 3, p. 178 and Theorem 3, p. 180.

PROOF. Set $\mathfrak{p}^*=\mathfrak{p}:\mathfrak{o}$ (ideal quotient in $\mathfrak{o}^*$). Let α, β be two elements in $\mathfrak{o}^*$ such that $\alpha\cdot\beta\in\mathfrak{p}^*$. Then $(\alpha\cdot\beta)\mathfrak{o}\leqq\mathfrak{p}$, hence $(\alpha\mathfrak{o})\cdot(\beta\mathfrak{o})\leqq\mathfrak{p}$. Since $\mathfrak{p}$ is prime in $\mathfrak{o}$, we have $\alpha\mathfrak{o}\leqq\mathfrak{p}$ or $\beta\mathfrak{o}\leqq\mathfrak{p}$, that is $\alpha\in\mathfrak{p}^*$ or $\beta\in\mathfrak{p}^*$. This shows $\mathfrak{p}^*$ is prime in $\mathfrak{o}^*$. Since $\mathfrak{o}\neq\mathfrak{p}$ and $(\mathfrak{p}^*\cap\mathfrak{o})\mathfrak{o}\leqq\mathfrak{p}$, we have $\mathfrak{p}^*\cap\mathfrak{o}\leqq\mathfrak{p}$, hence $\mathfrak{p}^*\cap\mathfrak{o}=\mathfrak{p}$. Furthermore since $\mathfrak{p}$ is not prime in $\mathfrak{o}^*$, this representation of $\mathfrak{p}$ is irredundant. We next consider the set of integers n such that $b+n\in\mathfrak{p}^*$ for some b in $\mathfrak{o}$. This set forms a principal ideal (r_0) in a ring of integer Z and we can find an element b_0 in $\mathfrak{o}$ such that $b_0-r_0\in\mathfrak{p}$. Then b_0 is uniquely determined mod. $\mathfrak{p}$. The other assertions in our Lemma will be easily seen. q. e. d.

LEMMA II. (With the same assumptions and notations.) *If* $\mathfrak{a}$ *is an ideal in* $\mathfrak{o}$, *then the following two statements are equivalent*:

i) $\mathfrak{a}+\mathfrak{p}^*=\mathfrak{o}^*$, ii) $\mathfrak{o}/\mathfrak{p}$ *has a unit element, and* $\mathfrak{a}+\mathfrak{p}=\mathfrak{o}$.

PROOF. If $\mathfrak{a}+\mathfrak{p}^*=\mathfrak{o}^*$, then $\mathfrak{o}/\mathfrak{p}$ has a unit element (by Lemma I) and moreover $\mathfrak{o}=(\mathfrak{a}+\mathfrak{p}^*)\cap\mathfrak{o}=\mathfrak{a}+\mathfrak{p}^*\cap\mathfrak{o}=\mathfrak{a}+\mathfrak{p}$. Conversely if $\mathfrak{o}/\mathfrak{p}$ has a unit element, then $\mathfrak{o}+\mathfrak{p}^*=\mathfrak{o}^*$ (by Lemma I). Therefore if we have $\mathfrak{a}+\mathfrak{p}=\mathfrak{o}$ in addition, then follows $\mathfrak{a}+\mathfrak{p}^*=\mathfrak{o}^*$.

q. e. d.

If $\mathfrak{a}^*$ and $\mathfrak{a}$ are ideals in $\mathfrak{o}^*$ and $\mathfrak{o}$ respectively such that $\mathfrak{a}^*\cap\mathfrak{o}=\mathfrak{a}$, then we shall say that $\mathfrak{a}^*$ lies over $\mathfrak{a}$. The following Lemma will be an easy consequence of Lemma I.

LEMMA I*. *If* $\mathfrak{p}$ *is a prime ideal in* $\mathfrak{o}$ *other than* $\mathfrak{o}$, *there lies over* $\mathfrak{p}$ *one and only one prime ideal in* $\mathfrak{o}^*$.

LEMMA III. *Let* $\mathfrak{q}$ *be a strongly primary ideal in* $\mathfrak{o}$ *which belongs to a prime ideal* $\mathfrak{p}(\neq\mathfrak{o})$. *Then there exists an only one primary ideal* $\mathfrak{q}^*$ *in* $\mathfrak{o}^*$ *which lies over* $\mathfrak{q}$. *Moreover* $\mathfrak{q}^*$ *belongs to the prime ideal* $\mathfrak{p}^*$ *which lies over* $\mathfrak{p}$.

PROOF. Set $\mathfrak{q}^*=\mathfrak{q}:\mathfrak{o}$, $\mathfrak{p}^*=\mathfrak{p}:\mathfrak{o}$. We have obviously $\mathfrak{q}^*\leqq\mathfrak{p}^*$. Let α, β be two elements such that $\alpha\cdot\beta\in\mathfrak{q}^*$ and $\alpha\notin\mathfrak{p}^*$, then we have $(\alpha\mathfrak{o})(\beta\mathfrak{o})\leqq\mathfrak{q}$ and $\alpha\mathfrak{o}\not\leqq\mathfrak{p}$, whence $\beta\mathfrak{o}\leqq\mathfrak{q}$ i.e. $\beta\in\mathfrak{q}^*$. If $\alpha\in\mathfrak{p}^*$, we have $\alpha^\rho\mathfrak{o}^\rho\leqq\mathfrak{p}^\rho\leqq\mathfrak{q}$ where ρ is the exponent of $\mathfrak{q}$. Hence we have $(\alpha^\rho\mathfrak{o})\mathfrak{c}^{\rho-1}\leqq\mathfrak{q}$, which implies $\alpha^\rho\mathfrak{o}\leqq\mathfrak{q}$, $\alpha^\rho\in\mathfrak{q}^*$. Therefore $\mathfrak{q}^*$ is a primary ideal in $\mathfrak{o}^*$ and belongs to $\mathfrak{p}^*$. The relation $\mathfrak{q}=\mathfrak{q}^*\cap\mathfrak{o}$ can be proved similarly as in Lemma I. If $\mathfrak{p}$ is prime in $\mathfrak{o}^*$ (in this case evidently $\mathfrak{p}:\mathfrak{o}=\mathfrak{p}$), $\mathfrak{q}^*=\mathfrak{q}$. q. e. d.

LEMMA IV. *If* $\mathfrak{q}$ *is a primary ideal in a Noetherian ring* $\mathfrak{o}$ *and has* $\mathfrak{o}$ *as its radical, then its representations have such a form as* $\mathfrak{q}=[\mathfrak{q}_0, \mathfrak{q}_1^*, \cdots, \mathfrak{q}_r^*]$, *where* $\mathfrak{q}_0$ *belongs to* $\mathfrak{o}$, *and* $\mathfrak{q}_i^*$ *to a maximal ideal which contains* $\mathfrak{o}$.

3. Intersection theorem

(Z^*) *Let* $\mathfrak{a}$, $\mathfrak{b}$ *be any ideals in a Noetherian ring and let* $\mathfrak{b}=[\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$ *be an irredundant representation of* $\mathfrak{b}$. *Let* $\mathfrak{p}_i$ *denotes the prime divisor of* $\mathfrak{q}_i$. *If* $\mathfrak{p}_i$ *satisfies the three conditions "*$\mathfrak{p}_i\neq\mathfrak{o}$, $\mathfrak{o}/\mathfrak{p}_i$ *has a unit element,* $\mathfrak{a}+\mathfrak{p}_i=\mathfrak{o}$*" for* $s+1\leqq i\leqq r$, *and fails to satisfy the conditions for* $1\leqq i\leqq s$, *then*

$$\bigcap_{n=1}^{\infty}(\mathfrak{b}+\mathfrak{a}^n)=[\mathfrak{q}_1, \cdots, \mathfrak{q}_s].$$

From Lemmas II, III and IV, we can write down an irredundant representation of an ideal $\mathfrak{b}$ as an intersection of primary ideals in $\mathfrak{o}^*$ from the one in $\mathfrak{o}$. Therefore this theorem is an immediate consequence of Theorem (Z).

Department of Mathematics,
Hiroshima University