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On Integrally Closed Noetherian Rings

By

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Introduction. An interesting characterization of integrally closed Noetherian integral domains by the notion of symbolic powers was obtained by Prof. S. Mori and T. Dodo in the following form.

THEOREM (M). Let $\mathfrak{o}$ be a Noetherian integral domain with a unit element. If $\mathfrak{o}$ is integrally closed, then it follows that

i) Every prime divisor of any principal ideal $(a)(\neq(0), \neq\mathfrak{o})$ is minimal, and

ii) Every primary ideal which belongs to any minimal prime ideal $\mathfrak{p}$, is a symbolic powor of $\mathfrak{p}$.

Conversely, if the following condition iii) is satisfied, $\mathfrak{o}$ is integrally closed.

iii) If $\mathfrak{p}$ is any prime divisor of any principal ideal then there exists no primary ideal between $\mathfrak{p}$ and $\mathfrak{p}^{(2)}$.[1]

In this note we extend this theorem to the case where $\mathfrak{o}$ is not free from zero divisors. The main purpose of this paper is to prove the following

THEOREM 1. *Let $\mathfrak{o}$ be a Noetherian ring with a unit element, and let K be its total quotient ring. Assume first $\mathfrak{o}$ is integrally closed in K. Then*

i) *Let $\mathfrak{p}$ be any prime divisor of any regular principal ideal $(a)(\neq\mathfrak{o})$. Then $\mathfrak{p}$ contains properly only one prime ideal and this prime ideal is a primary component of the zero ideal.*

ii) *Let $\mathfrak{p}$ be any minimal regular prime ideal in $\mathfrak{o}$, then every primary ideal which belongs to $\mathfrak{p}$ is a symbolic power of $\mathfrak{p}$.*

Conversely, if $\mathfrak{o}$ satisfies the following condition iii), *$\mathfrak{o}$ is integrally closed.*

iii) *If $\mathfrak{p}$ is any prime divisor of any regular principal ideal, then there exists no primary ideal between $\mathfrak{p}$ and $\mathfrak{p}^{(2)}$.*

Our proof is entirely based on the so-called primary ideal theorem and device of forming quotient rings.

Conventions of terminology. Let $\mathfrak{o}$ be a Noetherian ring and let K be its total quotient ring. If $\mathfrak{a}$ is an ideal in $\mathfrak{o}$, we call prime divisors of $\mathfrak{a}$ the prime ideals which occur as associated prime ideals of the primary ideals in a shortest representation of $\mathfrak{a}$ as an intersection of primary ideals.

Non zero divisor of $\mathfrak{o}$ shall be called regular. We shall call an ideal $\mathfrak{a}$ in $\mathfrak{o}$

1) S. Mori und T. Dodo, *Bedingungen für ganze Abgeschlossenheit in Integritätsbereichen,* This Journal 7 (1937) 15-28.

regular if $\mathfrak{a}$ contains some regular elements, and call a regular prime ideal $\mathfrak{p}$ minimal regular if $\mathfrak{p}$ is minimal in the set of all regular prime ideals. If an ideal $\mathfrak{a}$ in $\mathfrak{o}$ is not regular, we shall call $\mathfrak{a}$ also a zero divisor ideal.

$\mathfrak{p}^{-1}$ means as usual the set of such element x in K as $x\mathfrak{p}\subseteq\mathfrak{o}$.

1. Symbolic powers of prime ideals. In this section, we shall deal with some consequences of the so-called primary ideal theorem, due to Krull.

THEOREM (K). *Let $\mathfrak{p}(\neq\mathfrak{o})$ be a prime ideal in a Noetherian ring $\mathfrak{o}$ and let*

$$(0)=\bigcap_{i=1}^{r}\mathfrak{q}_i$$

be an irredundant representation of the zero ideal as an intersection of primary ideals. Let $\mathfrak{p}_i$ be the prime divisor associated to $\mathfrak{q}_i$. Then we have

$$\bigcap_{\nu=1}^{\infty}\mathfrak{p}^{(\nu)}=\bigcap_{\mathfrak{p}_i\subseteq\mathfrak{p}}\mathfrak{q}_i,$$

where $\mathfrak{p}^{(\nu)}$ means the ν-th symbolic power of $\mathfrak{p}$. In particular, if $\mathfrak{o}$ is an integral domain, we have

$$\bigcap_{\nu=1}^{\infty}\mathfrak{p}^{(\nu)}=(0).$$ [2]

LEMMA 1. *Let $\mathfrak{p}$ be the prime ideal associated to a primary ideal $\mathfrak{q}$ in a commutative ring $\mathfrak{o}$, and let $\mathfrak{a}$ be an ideal in $\mathfrak{o}$ such that $\mathfrak{a}\not\subseteq\mathfrak{q}$. Then $\mathfrak{q}:\mathfrak{a}$ is a primary ideal and belongs to $\mathfrak{p}$. Moreover if $\mathfrak{p}$ is a strongly primary ideal with the exponent $\rho\geqq 2$, then $\mathfrak{q}:\mathfrak{p}$ has the exponent $\rho-1$.*

LEMMA 2. *Let $\mathfrak{a}(\neq(0))$ be an ideal in a commutative ring $\mathfrak{o}$, such that $\mathfrak{a}^2=\mathfrak{a}$ and $\mathfrak{a}$ has a finite set of generators. Then $\mathfrak{a}$ has a unit element.*[3]

From Theorem (K), Lemmas 1 and 2, we deduce the following theorem on the exponents of symbolic powers.

THEOREM 2. *Let $\mathfrak{p}$ be a prime ideal in a Noetherian ring $\mathfrak{o}$. Then $\mathfrak{p}^{(i)}$ has the exponent i for $i=1,2,\cdots$, if $\mathfrak{p}$ is not among the exceptional ones enumerated below in (*), (**).*

(*) *$\mathfrak{p}$ is minimal. Let $\mathfrak{q}$ be the primary component of the zero ideal belonging to $\mathfrak{p}$, and let ρ be its exponent. Then*

$$\mathfrak{p}\supset\mathfrak{p}^{(2)}\supset\cdots\cdots\supset\mathfrak{p}^{(\rho)}=\mathfrak{q}=\mathfrak{p}^{(\rho+1)}=\cdots\cdots.$$

(**) *$\mathfrak{p}=\mathfrak{o}$ and $\mathfrak{o}$ is a direct sum of a ring with a unit element and a nilpotent ring.*

PROOF. Let ρ_i be the exponent of $\mathfrak{p}^{(i)}$. If the set $\{\rho_i\}$ is not bounded, we see

2) W. Krull, *Primidealketten in allgemeinen Ringbereichen,* Sitzungsber. d. Heidelberger Akademie, Math.-Naturw. Klasse 1928 3 Abhandl.

C. Chevalley, *On the notion of the ring of quotients of a prime ideal,* Bull. Amer. Math. Soc. 50 (1944) 93–97.

3) S. Mori, *Ueber Ringe, in denen die grössten Primärkomponenten jedes Ideals eindeutig bestimmt sind,* This Journal 1 (1931) Satz 8, P. 174.

easily $\rho_i = i$ by Lemma 1. If the set $\{\rho_i\}$ is bounded, let ρ be its maximum. Assume first that $\mathfrak{p} \neq \mathfrak{o}$, then we have $\bigcap_{\mathfrak{p}_i \subseteq \mathfrak{p}} \mathfrak{q}_i \geqq \mathfrak{p}^\rho$ (with the same notations as in Theorem (K)), whence $\mathfrak{p}$ is minimal. If $\mathfrak{p} = \mathfrak{o}$, then our assumption says

$$\mathfrak{o} \supset \mathfrak{o}^2 \supset \cdots\cdots \supset \mathfrak{o}^\rho = \mathfrak{o}^{\rho+1} = \cdots\cdots .$$

Set $\mathfrak{o}_1 = \mathfrak{o}^\rho$, then $\mathfrak{o}_1^2 = \mathfrak{o}_1$. Hence if $\mathfrak{o}_1 \neq (0)$, $\mathfrak{o}_1$ has a unit element by Lemma 2. Let $\mathfrak{o}_2$ be the annihilator of $\mathfrak{o}_1$, then we have $\mathfrak{o} = \mathfrak{o}_1 + \mathfrak{o}_2$.

Corollary. *Let $\mathfrak{p}_i$ $(i=1, \cdots, r)$ be the prime divisors of an ideal $\mathfrak{a}$ in a Noetherian ring $\mathfrak{o}$, and let $\mathfrak{a} = [\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$ be an irredundant representation of $\mathfrak{a}$, where $\mathfrak{q}_i$ belongs to $\mathfrak{p}_i$. Then $\mathfrak{q}_i$ is uniquely determined if $\mathfrak{p}_i$ satisfies one of the conditions:* i) *$\mathfrak{p}_i$ is isolated,* ii) *$\mathfrak{p}_i = \mathfrak{o}$ and $\mathfrak{o}/\mathfrak{a}$ is a direct sum of a ring with unit element and a nilpotent ring. In the other case, $\mathfrak{q}_i$ can be chosen in infinitely many ways.*[4]

Proof. The only point to be verified is that if $\mathfrak{p}_i$ satisfies the condition ii) $\mathfrak{q}_i$ is uniquely determined. We may assume $\mathfrak{a} = (0)$, so let $\mathfrak{o} = \mathfrak{o}_1 + \mathfrak{o}_2$ (direct sum), where $\mathfrak{o}_1$, $\mathfrak{o}_2$ are ideals in $\mathfrak{o}$, $\mathfrak{o}_1$ has a unit element and $\mathfrak{o}_2$ is nilpotent. Let $\mathfrak{p}_i$ $(i=1, \cdots, s)$ be the prime divisors of $\mathfrak{o}_2$ and let $\mathfrak{o}_2 = [\mathfrak{q}_1, \cdots, \mathfrak{q}_s]$ be an irredundant representation of $\mathfrak{o}_2$. Obviously $\mathfrak{p}_i \neq \mathfrak{o}$ $(i=1, \cdots, s)$. Then it is easily seen that $(0) = \mathfrak{o}_1 \cap \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ is an irredundant representation of the zero ideal. Therefore the prime divisors of the zero ideal are $\mathfrak{o}$ and $\mathfrak{p}_i$ $(i=1, \cdots, s)$. Let $(0) = \mathfrak{q} \cap \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_s$ be an irredundant representation of (0), where $\mathfrak{q}$, $\mathfrak{q}'_i$ belong to $\mathfrak{o}$ and $\mathfrak{p}_i$ respectively. Then since $\mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_s$ is an isolated component of (0), $\mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_s = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s = \mathfrak{o}_2$. Hence we have $(0) = \mathfrak{q} \cap \mathfrak{o}_2$ and $\mathfrak{q} \geqq \mathfrak{o}_1$, whence $\mathfrak{q} = \mathfrak{o}_1$.

2. Proof of the Theorem 1.

Lemma 3. *Let $\mathfrak{o}$ be an integrally closed Noetherian ring with a unit element and let $\mathfrak{p}$ be any prime divisor of any regular principal ideal $(a)(\neq \mathfrak{o})$. Then we have*

$$\mathfrak{p}\mathfrak{p}^{-1} \supset \mathfrak{p}.$$

This lemma is due to van der Waerden. We shall note here its proof for the sake of completeness.

Proof, Let $(a) = [\mathfrak{q}, \mathfrak{q}_1, \cdots, \mathfrak{q}_r]$ be an irredundant representation with $\mathfrak{q}$ belonging to $\mathfrak{p}$. Set $\mathfrak{b} = [\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$, then $\mathfrak{p}^{\rho-1}\mathfrak{b} \not\leqq (a)$, $\mathfrak{p}^\rho \mathfrak{b} \leqq (a)$ for some integer $\rho \geqq 1$. For any element b such that $b \in \mathfrak{p}^{\rho-1}\mathfrak{b}$, $b \notin (a)$, we have $b/a \in \mathfrak{p}^{-1}$, $b/a \notin \mathfrak{o}$; hence $\mathfrak{p}^{-1} \supset \mathfrak{o}$. Assume that $\mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}$, then we have $\mathfrak{p} = \mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}\mathfrak{p}^{-2} = \cdots$, whence $\mathfrak{p}^{-1} \leqq \mathfrak{o}$ since $\mathfrak{o}$ is integrally closed. This is a contradiction.

Lemma 4. *Let $\mathfrak{p}$ be a regular prime ideal in a Noetherian ring $\mathfrak{o}$ such that $\mathfrak{p}\mathfrak{p}^{-1} \supset \mathfrak{p}$. Then*

i) *$\mathfrak{p}$ contains only one prime ideal properly and this prime ideal is a primary component of the zero ideal.*[5]

4) S. Mori, *Minimale Primärideale eines Ideals,* This Journal **2** (1932) Satz 7, P. 32.

5) W. Krull, *Ueber den Aufbau des Nullideals in ganz abgeschlossenen Ringen mit Teilerkettensatz,* Math. Ann. **102** (1930) Satz 1, P. 365.

ii) *Any primary ideal which belongs to* $\mathfrak{p}$, *is a symbolic power of* $\mathfrak{p}$.[6]

i) and ii) are due to W. Krull and S. Mori respectively. Here we shall give a shorter proof.

Proof. Let $(0)=[\mathfrak{q}_1, \cdots, \mathfrak{q}_r]$ be an irredundant representation of (0) and let $\mathfrak{p}_i$ be the prime divisor of $\mathfrak{q}_i$. We set $\mathfrak{a}=\cap\mathfrak{p}^{(\nu)}=\bigcap_{\mathfrak{p}_i\subseteq\mathfrak{p}}\mathfrak{q}_i$. If we denote by K a total quotient ring of $\mathfrak{o}$, and set $\mathfrak{o}'=\mathfrak{o}/\mathfrak{a}$ and $K'=K/\mathfrak{a}K$, then $\mathfrak{o}'$ may be considered as a subring of K' since $\mathfrak{a}K\cap\mathfrak{o}=\mathfrak{a}$. Then our assumption implies

$$\mathfrak{o}'\supseteq\mathfrak{p}'(\mathfrak{p}^{-1})'\supset\mathfrak{p}'$$

where $\mathfrak{p}'=\mathfrak{p}/\mathfrak{a}$ and $(\mathfrak{p}^{-1})'=(\mathfrak{p}^{-1}+\mathfrak{a}K)/\mathfrak{a}K$. Since $\mathfrak{o}'$ has no zero divisor outside of $\mathfrak{p}'$, we can form the quotient ring of $\mathfrak{p}'$ with respect to $\mathfrak{o}'$, which shall be denoted by $\mathfrak{o}^*$. Then we may assume $\mathfrak{o}^*$ and K' are subrings of the total quotient ring of $\mathfrak{o}'$ and we have

$$\mathfrak{o}^*\supseteq\mathfrak{p}^*(\mathfrak{p}^{-1})'\mathfrak{o}^*\supset\mathfrak{p}^*,$$

where $\mathfrak{p}^*=\mathfrak{p}'\mathfrak{o}^*$. Therefore we have $\mathfrak{p}^*(\mathfrak{p}^*)^{-1}=\mathfrak{o}^*$, whence $\mathfrak{p}^*=(\pi)$, where π is any element of $\mathfrak{o}^*$ such that $\pi\in\mathfrak{p}^*$ and $\pi\notin\mathfrak{p}^{*2}$. That is, $\mathfrak{o}^*$ is a discrete valuation ring. From this, assertions i) and ii) will be easily seen.

The direct part of the Theorem is thus proved by Lemmas 3 and 4, so we shall prove the converse part.

Let K be a total quotient ring of $\mathfrak{o}$. We shall show that if $b/a\in K$ (a is a non zero divisor in $\mathfrak{o}$) is integral over $\mathfrak{o}$, then $b/a\in\mathfrak{o}$. If a is a unit in $\mathfrak{o}$, obviously $b/a\in\mathfrak{o}$. Therefore we may assume a is not a unit. Let

$$(a)=[\mathfrak{q}_1, \cdots\cdots, \mathfrak{q}_r]$$

be an irredundant representation of (a), and let $\mathfrak{p}_i$ be the prime divisor of $\mathfrak{q}_i$. We set $\mathfrak{p}=\mathfrak{p}_i$, $\mathfrak{q}=\mathfrak{q}_i$ (where i is any one of $1, \cdots, r$) $\mathfrak{a}=\cap\mathfrak{p}^{(\nu)}$, $\bar{\mathfrak{o}}=\mathfrak{o}/\mathfrak{a}$ and $\bar{\mathfrak{p}}=\mathfrak{p}/\mathfrak{a}$. Since zero divisors in $\bar{\mathfrak{o}}$ are all contained in $\bar{\mathfrak{p}}$, we can form the quotient ring of $\bar{\mathfrak{p}}$ with respect to $\bar{\mathfrak{o}}$, which we shall denote by $\mathfrak{o}^*$. Set $\mathfrak{p}^*=\bar{\mathfrak{p}}\mathfrak{o}^*$. Then it follows from our assumption that there exists no primary ideal between $\bar{\mathfrak{p}}$ and $\bar{\mathfrak{p}}^{(2)}$. Hence there exists no ideal between $\mathfrak{p}^*$ and $\mathfrak{p}^{*2}$, and obviously $\mathfrak{p}^*\neq\mathfrak{p}^{*2}$. Let π be any element in $\mathfrak{o}^*$ such that $\pi\in\mathfrak{p}^*$ and $\pi\notin\mathfrak{p}^{*2}$; we have $\mathfrak{p}^*=(\pi)+\mathfrak{p}^{*2}$. This implies

$$\mathfrak{p}^*=(\pi)+\mathfrak{p}^{*\nu}\qquad\text{for any positive integer } \nu.$$

Thus we have

$$\mathfrak{p}^*=\cap_{\nu=1}^{\infty}((\pi)+\mathfrak{p}^{*\nu})=(\pi),$$ [7]

hence $\mathfrak{o}^*$ is a discrete valuation ring.

On the other hand since b/a is integral over $\mathfrak{o}$, there exists a non zero divisor λ in $\mathfrak{o}$ such that

6) S. Mori, *Ueber ganz abgeschlossene Ringe*, This Journal 3 (1933) Satz 1. P. 165.

7) W. Krull, *Dimensionstheorie in Stellenringen*, J. Reine Angew. Math., 179 (1938) Satz 2, P. 207.

$$\lambda\left(\frac{b}{a}\right)^{\nu}\in\mathfrak{o} \qquad \text{for any positive integer } \nu.$$

Let us denote by $\bar{a}$, $\bar{b}$ and $\bar{\lambda}$ the residue classes modulo $\mathfrak{a}$ which contain the elements a, b and λ respectively. Then we have

$$\bar{\lambda}\bar{b}^{\nu}\in\bar{a}^{\nu}\bar{\mathfrak{o}} \qquad (\bar{\lambda}\neq\bar{0}) \qquad \text{for any positive integer } \nu,$$

whence

$$\bar{b}\in\bar{a}\,\mathfrak{o}^{*},$$

since $\mathfrak{o}^{*}$ is a discrete valuation ring. It follows

$$\bar{b}=\bar{a}\,\frac{\bar{d}}{\bar{c}}\,(\bar{c},\ \bar{d}\in\bar{\mathfrak{o}} \text{ and } \bar{c}\notin\bar{\mathfrak{p}}).$$

Let c and d be any element in $\mathfrak{o}$ whose residue classes modulo $\mathfrak{a}$ are $\bar{c}$ and $\bar{d}$ in $\bar{\mathfrak{o}}$ respectively. Then we have

$$bc\equiv ad \ (\mathrm{mod.}\ \mathfrak{a}) \qquad (c\notin\mathfrak{p}).$$

Since $\mathfrak{a}\subseteq\mathfrak{q}$, we have $bc\in\mathfrak{q}$ which implies $b\in\mathfrak{q}$, whence $b\in(a)$.

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