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Orthocomplemented Lattices Satisfying the Exchange Axiom

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The set of all closed linear manifolds of a (not necessarily separable) Hilbet space is a (I) *complete*, (II) *relatively atomic*, and (IV) *orthocomplemented* lattice satisfying (III) the *exchange axiom* of MacLane [1][1], together with the following condition:

(V) *If* b, c *are orthogonal elements, then it holds* $(b,c)M$, i.e., $a \leqq c$ *implies* $(a \smile b) \frown c = a \smile (b \frown c)$.

The purpose of this paper is to study an abstract lattice L satisfying these five conditions (I)-(V). The main results are as follows[2]:

(1) Any element a of L has an orthogonal basis, that is, there exists an orthogonal system of points whose join is equals to a.

(2) If P, Q are both orthogonal bases of an element, then P and Q have the same cardinal numbers provided that L satisfies furthermore an "counrability condition of dependence".[3]

(3) Any quotient lattice of L has the same properties as L.

(4) L is a direct sum of irreducible sublattices.

(5) Projections and permutability of elements are defined and their interrelation is investigated.

§1. The lattice of closed linear manifolds of a Hibert space.

Let $\mathfrak{H}$ be a (not necessarily separable) Hilbert space, and let the set of all closed linear manifolds of $\mathfrak{H}$ be denoted by L. It is well known that L is a complete, relatively atomic, and orthocomplemented lattice, partially ordered by set-inclusion. Croisot [1] has shown that L satisfies the exchange axiom of MacLane[4]. Hence we have the following:

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1) The numbers in square brackets refer to the list of references at the end of the paper.

2) An *exchange lattice* of MacLane [1], which is equivalent to a "*matroid lattice*" in the terminology of F. Maeda [3], is a complete, relatively atomic lattice satisfying the exchange axiom together with the "finiteness condition of dependence". MacLane [1] has shown that analogous theorems to (1)-(3) above are valid in any exchange lattice (cf. ibid. 458, Theorem 3, 4 and 6); (4) has been shown for an exchange lattice by U. Sasaki and S. Fujiwara [1] 188, Theorem 4.

3) Cf. the condition (VI) in Theorem 2.2 below.

4) Cf. Croisot [1] 259, Lemma 1 and 261, Lemma 3 which are respectively (η'') and (ξ'') in Remark (2) below.

THEOREM 1.1. *The lattice L of all closed linear manifolds of a (not necessarily separable) Hilbert space $\mathfrak{H}$ satisfies the following conditions:*

(I) *It is complete.*

(II) *It is relatively atomic: $a < b$ implies $a < a \smile p \leqq b$ for some point p.*

(III) *It satisfies the exchange axiom of MacLane:*

$$(\eta') \qquad p \leqq q \smile a, \quad p \frown a = 0 \quad \text{imply} \quad q \leqq p \smile a.$$

(IV) *It is orthocomplemented: There exists a one-to-one mapping $a \rightarrow a^{\perp}$ of L into L such that* (1) *$a \leqq b$ implies $a^{\perp} \geqq b^{\perp}$,* (2) *$(a^{\perp})^{\perp} = a$, and* (3) *$a \frown a^{\perp} = 0$*[5].

(V) *$b \leqq c^{\perp}$ implies $(b, c)M$*[6].

PROOF. It remains merely to show the condition (V). As is well known, L is isomorphic to the lattice of all projections in $\mathfrak{H}$, partially ordered by the relation $\leqq$ which is defined as follows: $E \leqq F$ if and only if $EF = E$. Now let F, G be orthogonal projections in $\mathfrak{H}$, and let E be any projection with $E \leqq G$, then it is obvious that $(E+F)G = E$, completing the proof.

REMARK. The following propositions will be used frequently in the sequel without further reference:

(1) *In a relatively atomic lattice, any element a is the join of all points contained in a*[7].

(2) *In a relatively atomic lattice, the following conditions are equivalent to each other*[8]*:*

(η') $\quad p \leqq q \smile a, \quad p \frown a = 0 \quad$ *imply* $\quad q \leqq p \smile a$.

(η'') $\quad$ *If* $p \nleqq a$ *then* $p \smile a$ *covers* a.

(ξ'') $\quad$ *If* a *covers* $a \frown b$ *then* $a \smile b$ *covers* b.

(3) *An orthocomplemented lattice is self-dual, whence any proposition yields its dual.* For example, in a lattice satisfying the conditions (II), (III), (IV), it holds the dual of (ξ''):

$(\bar{\xi}'')$ $\quad$ *If* $a \smile b$ *covers* a *then* b *covers* $a \frown b$.

If L satisfies (IV), (V), then the dual of (V) is valid:

$(\bar{\mathrm{V}})$ *If $b \leqq c^{\perp}$, then it holds $(b, c)\bar{M}$,* i.e. *$a \geqq c$ implies $(a \frown b) \smile c = a \frown (b \smile c)$.*

§ 2. Orthogonal bases.

DEFINITION 2.1. In an orthocomplemented lattice, an element a is said *orthogonal* to b, denoted by $a \perp b$, if $a \leqq b^{\perp}$. It is obvious that $a \perp b$ implies $b \perp a$. By an *orthogonal system,* we mean the the set of elements $\{a_{\alpha};\ \alpha \in I\}$ such that if

5) Cf. Halmos [1] 24, Theorem 1, 3, 5.

6) By $(b, c)M$, it is meant that $a \leqq c$ implies $(a \smile b) \frown c = a \smile (b \frown c)$.

7) Cf. F. Maeda [2] 88, Lemma 1.1.

8) Cf. F. Maeda [3] 180, Theorem 2.

$\alpha \neq \beta$ then $a_\alpha \perp a_\beta$. $\vee(a_\alpha;\ \alpha \in I)$ is denoted by $\vee(\oplus a_\alpha;\ \alpha \in I)$ provided that $\{a_\alpha;\ \alpha \in I\}$ is an orthogonal system. In particular, if P is an orthogonal system of points, $\vee(p;\ p \in P)$ is denoted by $\vee(\oplus P)$, and P is called a ***orthogonal basis*** of a, if $a = \vee(\oplus P)$.

LEMMA 2.1. *In a lattice satisfying the the conditions* (II) *and* (IV) *in Theorem* 1.1, *the condition* (V) *is equivalent to*

(V′) $\quad p \leqq a \oplus b$ *and* $p \leqq b^\perp$ *imply* $p \leqq a$.

PROOF. First we shall show that (V) implies (V′). Let $p \leqq a \smile b$, and $p, a \leqq b^\perp$. Since it holds $(b, b^\perp)M$ by (V), we have $(a \smile b) \frown b^\perp = a$, and it follows $p \leqq a$. Next we shall prove that (V′) implies (V). Let $a \leqq c$, then it holds $(a \smile b) \frown c \geqq a \smile (b \frown c)$ in general. Thus it is sufficient to prove that $p \leqq (a \smile b) \frown c$ implies $p \leqq a \smile (b \frown c)$. Now $p \leqq (a \smile b) \frown c$ implies $p \leqq a \smile b$ and $p \leqq c$. Since $b \perp c$, it follows $a \perp b$ and $p \leqq c \leqq b^\perp$, whence we have $p \leqq a$ by (V′).

Unless otherwise stated, in the remainder of this section we shall denote by L the lattice satisfying the conditions (I)–(V) in Theorem 1.1.

LEMMA 2.2. *If* $a > b$ *in* L, *then there exists a point* p *such that* $a \geqq b \oplus p$.

PROOF. There exists a point q such that $a \geqq b \smile q > b$ by (II). In view of the fact $(b^\perp, b)\bar{M}$, we have $\{(b \smile q) \frown b^\perp\} \smile b = b \smile q$. Since L satisfies (η''), $b \smile q$ covers b, whence $(b \smile q) \frown b^\perp$ covers $\{(b \smile q) \frown b^\perp\} \frown b = 0$ by $(\bar{\xi}'')$. The point $p = (b \smile q) \frown b^\perp$ is the desired one.

THEOREM 2.1. *Let L be a lattice satisfying the conditions* (I)–(V) *in Theorem* 1.1. *If P is an orthogonal system of points such that* $\vee(\oplus P) \leqq a$, *then there exists an orthogonal basis of a containing P. Consequently every element* $a (\neq 0)$ *has an orthogonal basis.*

PROOF. Let S be the set of points contained in a, then there exists by Zorn's lemma, a maximal orthogonal system of points Q such that $P \subseteqq Q \subset S$. Then Q is a basis of a, since otherwise we have $\vee(\oplus Q) < a$, which implies the existence of a point p such that $p \oplus \vee(\oplus Q) \leqq a$, whence $\{p, Q\}$ is an orthogonal system containing Q, contrary to the maximality of Q.

LEMMA 2.3. *Let S be an orthogonal system of points of L. If* $p \leqq \vee(\oplus S)$, *then there exists a unique subset T of S such that* $p \leqq \vee(\oplus T)$, *but* $p \nleqq \vee(\oplus T')$ *for any proper subset T' of T.*

PROOF. (i) Put $T = \{t \in S;\ t \nleqq p^\perp\}$ and $R = \{r \in S;\ r \leqq p^\perp\}$, then we have $p \leqq \vee(\oplus T) \oplus \vee(\oplus R)$, and $p \leqq \{\vee(\oplus R)\}^\perp$. It follows from (V′) that $p \leqq \vee(\oplus T)$.

(ii) For any $t \in T$, put $T' = T - t$. If $p \leqq \vee(\oplus T')$, then we have $p \leqq t^\perp$ contradicting the definition of T. Thus $p \nleqq V(\oplus T')$ for any proper subset T' of T.

(iii) Let U be a subset of S such that $p \leqq \vee(\oplus U)$, and $p \nleqq \vee(\oplus U')$ for any proper subset U' of U. If $r \in S - U$, then it holds $r^\perp \geqq \vee(\oplus U) \geqq p$ and so $r \in R = S - T$, whence we have $U \geqq T$. Hence it holds that $U = T$, completing the proof.

It is known that all bases of an element have the same cardinal number in any exchange lattice[9]. However, in a lattice satisfying the conditions (I)–(V), the uniqueness of the cardinal number of orthogonal bases of an element is secured under an additional condition stated below.

THEOREM 2.2. *Let L be a lattice satisfying the conditions* (I)–(V) *in Theorem* 1.1. *and in addition*

(VI)[10] *$p \leqq \vee(P)$ implies $p \leqq \vee(p_i;\ i=1,2\cdots)$ for some $p_i \in P (i=1,2,\cdots)$, where P is a set of points of L.*

If both P and Q are orthogonal bases of an element a, then P and Q have the same cardinal number.

PROOF. *Case* 1. Either P or Q is a finite set.

By symmetry we can assume that P is a finite set $\{p_i;\ i=1,2,\cdots,n\}$. The cardinal number of Q can not exceed n. For otherwise there exist points $q_1, q_2, \cdots \cdots, q_{n+1} \in Q$, and it follows from U. Sasaki and S. Fujiwara [1] Lemma 2.2 that $\bigvee_{i=1}^{n}(\oplus q_i)=\bigvee_{i=1}^{n}(\oplus p_i)=a \geqq q_{n+1}$, contrary to the orthogonality of Q. Similarly $\overline{\overline{P}}$, the cardinal number of P, can not exceed $\overline{\overline{Q}}$, and so we have $\overline{\overline{P}}=\overline{\overline{Q}}$.

Case 2. Both P and Q are infinite sets.

Since $a=\vee(\oplus P)=\vee(\oplus Q)$, it follows from the cendition (VI) and Lemma 2.2 that for any $q \in Q$ there exists a countable subset T_q of P such that $q \leqq \vee(\oplus T_q)$ and $q \not\leqq \vee(\oplus T')$ for any proper subset T' of T_q. Since any point of P belongs to some $T_q (q \in Q)$[11], we have $P=\bigcup(T_q;\ q \in Q)$. On the other hand the cardinal number of $\bigcup(T_q;\ q \in Q)$ can not exceed $\overline{\overline{Q}} \cdot \aleph_0=\overline{\overline{Q}}$. Thus we have $\overline{\overline{P}} \leqq \overline{\overline{Q}}$. Similarly we have $\overline{\overline{Q}} \leqq \overline{\overline{P}}$, completing the proof.

§ 3. Quotient lattices.

DEFINITION 3.1. Let $a > b$ in an orthocomplemented lattice, we shall denote by a/b the lattice of all elements c such that $a \geqq c \geqq b$, and by $a \ominus b$ we shall mean the element $a \frown b^{\perp}$.

LEMMA 3.1. *In a lattice satisfying the conditious* (IV) *and* (V), *the following rules of computation hold:*

1. $a \geqq b$ *implies* $a=b \oplus (a \ominus b)$.
2. $c \leqq a$ *implies* $a \ominus (a \ominus c)=c$.
3. $b \oplus c \leqq a$ *implies* $a \ominus (b \oplus c)=(a \ominus b) \ominus c$.
4. $a \oplus x=a \oplus y$ *implies* $x=y$.
5. $x, y \leqq a$ *and* $a \ominus x=a \ominus y$ *imply* $x=y$.

9) Cf. MacLane [1] 458, Theorem 6.

10) Compare with the condition (F_1) of MacLane [1] 456. Remark also that (VI) is satisfied in the lattice of all closed linear manifolds of a Hilbert space, cf. Halmos [1], 22 Theorem 1.

11) In fact, suppose that a point $p (\in P)$ belongs to none of $T_q (q \in Q)$, then we have $p \leqq \vee(\oplus Q) \leqq \vee(P-p) \leqq p^{\perp}$, which is a contradiction.

PROOF. (1)-(3) are obvious.

(4) $a\oplus x=a\oplus y$ implies $(a\smile x)\frown a^{\perp}=(a\smile y)\frown a^{\perp}$, whence it holds $x=y$ in view of $(a, a^{\perp})M$, and $x, y\leqq a^{\perp}$.

(5) It follows from (1) that $x\oplus(a\ominus x)=y\oplus(a\ominus y)$, whence we have $x=y$ by (4).

THEOREM 3.1. *Let L be a lattice satisfying the conditions* (I)-(V) *in Theorem* 1.1. *If $a>b$ in L, then the quotient lattice a/b satisfies also* (I)-(V), *and its points are all the elements of L of the form $b\oplus p$, where p is a point of L such that $p\oplus b\leqq a$.*

PROOF. Such an arbitrary element $b\oplus p$ is a point of a/b, since $b\oplus p$ obviously covers b which is the zero element of a/b. Conversely if d is a point of a/b, then $a\geqq d$ and d covers b, it follows from Lemma 2.1 that there exists a point p with $d=b\oplus p\leqq a$.

Now it is sufficient to prove (II)-(V), since a/b is obviously complete.

(II) If $c, d\in a/b$ and $c>d$, then we have $c\geqq d\oplus p>d$ for some point p of L by Lemma 2.1, whence it holds $c\geqq d\smile(b\oplus p)>d$. Thus a/b is relatively atomic.

(III) Let $b\oplus p$ be a point of a/b which is not contained in $c\in a/b$, then $c\smile(c\oplus p)=c\smile p$ covers c since $p\nleqq c$. Thus a/b satisfies (η'') and so does (η').

(IV) Put $c'=(a\ominus c)\oplus b$ for any $c\in a/b$. We shall show that c' is the orthocomplement of c in a/b. Let $c, d\in a/b$ and $c'=d'$, then the applications of (4) and (5) of Lemma 3.1 yield $c=d$, which shows that the mapping $c\rightarrow c'$ is one-to-one. Obviously $c\leqq d$ implies $c'\geqq d'$. The applications of (3), (2), and (1) of Lemma 3.1 yield $c''=c$. Since it holds $(c^{\perp}, c)M$, we have $b\leqq c\frown c'=c\frown\{(a\ominus c)\smile b\}\leqq c\frown(c^{\perp}\smile b)=b$. whence $c\frown c'=b$. Hence a/b is an orthocomplemented lattice.

(V) Let $y, z\in a/b$ and $y\leqq z'$, then $x\leqq z$ implies

$$(x\smile y)\frown z\leqq(x\smile z')\frown z\leqq(x\smile z^{\perp})\frown z=x,$$

whence we have $(y, z)M$.

This completes the proof.

If L satisfies in addition the condition (VI) in Theorem 2.2, then it is easily seen that the quotient lattice a/b satisfies also (VI).

§ 4. Direct decomposition.

DEFINITION 4.1. In a lattice with 0, if p, q are points such that $p\leqq q\smile a$, $p\frown a=0$ for some element $a\in L$, then p is said *perspective* to q and denoted by $p\sim q$.

Remark that $p\sim q$ is equivalent to $q\sim p$ in a lattice satisfying the condition (η').

LEMMA 4.1. *Let L be a lattice satisfying the conditions* (II), (III), (IV). *If $p\leqq q\smile a$; p, q being points and a being an element $(\neq 0)$ of L, then there exists a point r such that $p\leqq q\smile r$ and $r\leqq a$.*

PROOF. We can assume without loss of generality that $p\neq q$, and $p\nleqq a$. Since $q\nleqq a$, we have by (η'') that $(p\smile q)\smile a=q\smile a$ covers a. It follows from $(\bar{\xi}'')$ that $(p\smile q)\frown a$ is covered bp $p\smile q$, whence $r=(p\smile q)\frown a$ is a point of L with $r\leqq a$

and $r \leqq p \smile q$. By using (η') we have $p \leqq q \smile r$.

An immediate consequence of this lemma is the followihg corollary which will be used frequently without further reference.

COROLLARY. *In a lattice satisfying the conditions* (II), (III), (IV), *two points* p, q *are perspective to each other if and only if* $p \leqq q \smile r$ *for some point* $r(\neq p)$.

Now we shall show that the perspectivity is transitive.

LEMMA 4.2. *In a lattice satisfying the conditions* (II), (III) *and* (IV),

$$p \sim q \quad \textit{and} \quad q \sim r \quad \textit{imply} \quad p \sim r.$$

PROOF. We may assume without loss of generality that p, q, r are distinct and $p \nleqq q \smile r$. Since $p \leqq q \smile s$ and $q \leqq r \smile t$ for some points $s(\neq p)$ and $t(\neq q)$, it holds that $p \leqq r \smile s \smile t$. Now suppose that $p \leqq s \smile t$, then we have $s \smile t = p \smile s \geqq q$ and $s \smile t = q \smile t \geqq r$ by (η'), and it follows from U. Sasaki and S. Fujiwara [1] Lemma 2 (2°) that $q \smile r = s \smile t$ and so $q \smile r \geqq p$, contrary to the assumption. Thus we have $p \frown (s \smile t) = 0$, and it follows that $p \sim r$.

DEFINITION 4.2. In a lattice with 0, by $a \nabla b$, we mean that $(a \smile x) \frown b = x \frown b$ for every $x \in L$. If S is any subset of L, we denote by S^{∇} the set of a such that $a \nabla b$ for all $b \in S$.

Unless otherwise stated, we shall denote by L in the remainder of this section the lattice satisfying the conditions (I)–(V).

LEMMA 4.3. *The following conditions are equivalent in* L:

(α) $\quad a \nabla b$.

(β) $\quad$ *There exist no points* p, q *such that* $p \leqq a$, $q \leqq b$, *and* $p \sim q$.

PROOF. The implication $(\alpha) \to (\beta)$ may be proved by a verbatim repetition of F. Maede [2] Lemma 1.6, while to prove the converse we need a slight modification.

Let us assume that (α) is false, then there exists $x \neq 0$ such that $(a \smile x) \frown b > x \frown b$. It follows from Lemma 2.2 that there exists a point q such that $(a \smile x) \frown b \geqq (x \frown b) \oplus q$. Then $x \frown b \frown q = 0$, $q \leqq b$ and so we have $q \frown x = 0$. Now since $q \leqq a \smile x = x \oplus \{(a \smile x) \ominus x\}$, the element $(a \smile x) \ominus x$ contains at least one point r which is not orthogonal to q, for for otherwise we have $q \leqq \{(a \smile x) \ominus x\}^{\perp}$ and it follows from the condition (V′) that $q \leqq x$, contrary to the preceding statement. Then we have $r \leqq a \smile x$ and $r \leqq x^{\perp}$, whence r is not orthogonal to some point $p \leqq a$, since otherwise we have $r \leqq a^{\perp}$ and so $r \leqq (a \smile x) \frown a^{\perp} \frown x^{\perp} = 0$, which is a contradiction. Here we remark that *two points which are not orthogonal are perspective to each other.* Consequently we have $q \sim r$ and $r \sim p$ and so it holds $q \sim p$ and $p \leqq a$, contrary to (β).

DEFINITION 4.3. Let $\{S_{\alpha}; \alpha \in I\}$ be a family of subsets of L. If

12) Proof: If q is not orthogonal to r, then we have $q \nleqq r^{\perp}$. Since $q \leqq r \smile r^{\perp}$, it holds $q \sim r$ by Definition 4.1.

(1) every $a \in L$ is expressible in the form

$$a = \bigvee(a_\alpha;\ \alpha \in I), \qquad a_\alpha \in S_\alpha(\alpha \in I), \qquad \text{and}$$

(2) $\alpha \neq \beta$ implies $S_\beta \leqq S_\alpha^\nabla$,

then L is called a *direct sum* of $S_\alpha(\alpha \in I)$, and is denoted by $L = \Sigma(\oplus S_\alpha;\ \alpha \in I)$.

LEMMA 4.4. *If $b \nabla a_\alpha(\alpha \in I)$ in L, then it holds $b \nabla \bigvee(a_\alpha;\ \alpha \in I)$.*

PROOF. Suppose that $b \nabla \bigvee(a_\alpha;\ \alpha \in I)$ is false, then it follows from Lemma 4.3 that there exist points p, q such that $p \leqq \bigvee(a_\alpha;\ \alpha \in I)$, $q \leqq b$ and $p \sim q$. Now let $\{p_\beta^\alpha;\ \beta \in I_\alpha\}$ be the set of points contained in a_α, then we have $p \leqq \bigvee(p_\beta^\alpha;\ \beta \in I_\alpha,\ \alpha \in I)$. It follows that there exists a point p_β^α which is not orthogonal to p, whence $p_\beta^\alpha \sim p$ and so $p_\beta^\alpha \sim q$ for some $\alpha \in I$, $\beta \in I_\alpha$, contrary to $b \nabla a_\alpha$.

By making use of this lemma, we can easily verify the following:

LEMMA 4.5. *If L is a direct sum of sublattices $S_\alpha(\alpha \in I)$, then any element $a \in L$ is expressible uniquely as*

$$a = \bigvee(a_\alpha;\ \alpha \in I), \qquad a_\alpha \in S_\alpha(\alpha \in I),$$

and L is isomorphic to the direct product $\Pi(S_\alpha;\ \alpha \in I)$.

PROOF. Cf. F. Maeda [1] Lemma 2.2 and Theorem 2.1.

By a similar argument as in F. Maeda [2] Theorem 1.1, we obtain the following[13].

LEMMA 4.6. *L is a direct sum of sublattices S_α of L, that is, $L = \Sigma(\oplus S_\alpha;\ \alpha \in I)$. And any two points of the same S_α are perspective to each other, and two points which are contained in distinct S_α and S_β are not perspective.*

Thus we obtain the following theorems by verbatim repetitions of the arguments in U. Sasaki and S. Fujiwara [1], Theorem 3 and 4.

THEOREM 4.1. *A lattice satisfying the conditions* (I)–(V) *in Theorem* 1.1 *is irreducible if and only if any two points are perspective to each other.*

THEOREM 4.2. *A lattice L satisfying the conditions* (I)–(V) *in Theorem* 1.1 *is a direct sum of irreducible sublattlces which satisfy the same conditions as L.*

§ 5. Projection.

In this section, we shall denote by L a lattice satisfying the conditions (I)–(V) in Theorem 1.1.

LEMMA 5.1. *If $p \not\leqq b$, $p \leqq a \oplus b$ in L, then $(p \smile b) \frown a$ is a point.*

PROOF. By virtue of $(a, b)M$, we have $\{(p \smile b) \frown a\} \smile b = p \smile b$. If follows from (η'') that $\{(p \smile b) \frown a\} \smile b$ covers b, whence it holds by (ξ'') that $\{(p \smile b) \frown a\} \frown b = 0$ is covered by $(p \smile b) \frown a$, completing the proof.

LEMMA 5.2. *If $p \leqq a \oplus b$, $a, b \neq 0$ in L, then $p \leqq q \oplus r$ for some points $q \leqq a$, $r \leqq b$; in particular if $p \not\leqq a, b$, then q, r are determined uniquely dependent on p, a, b.*

13) Remark that if $p \leqq \bigvee(P)$, where P is a set of points which is perspective to a fixed point p_0, then it holds $p \sim p_0$, in fact p is not orthogonal to some point $p' \in P$, whence $p \sim p' \sim p_0$.

PROOF. We can assume without loss of generality that $p \nleqq a$ and $p \nleqq b$. Put $q=(p \smile b) \frown a$, then by Lemma 5.1, q is a point such that $q \leqq p \smile b$ and $q \leqq a$. Hence by (η') we have $p \leqq q \oplus b$. Then by a similar argument, $r=(p \smile q) \frown b$ is a point and $r \leqq b$, $r \leqq p \smile q$, whence $p \leqq q \oplus r$ by (η').

Next suppose that $p \leqq q' \oplus r'$, $q' \leqq a$, and $r' \leqq b$. In view of (η') it holds that $q' \leqq p \smile r' \leqq p \smile b$, whence $q' \leqq (p \smile b) \frown a = q$. Consequently we have $q'=q$. By symmetry it holds $r'=(p \smile a) \frown b$.

REMARK. If a is an element $(\neq 0, 1)$ of L, and if p is a point such that $p \nleqq a$, $p \nleqq a^{\perp}$, then there exist uniquely determined points q, r such as $p \leqq q \oplus r$, $q \leqq a$, $r \leqq a^{\perp}$; q, r being $(p \smile a^{\perp}) \frown a$, $(p \smile a) \frown a^{\perp}$, respectively.

We may consider that this fact corresponds to the one in a Hilbert space $\mathfrak{H}$ that if f is an element of $\mathfrak{H}$ and $\mathfrak{M}$ is a closed linear manifold, then there exist elements g, h such that $f=g+h$, $g \in \mathfrak{M}$, $h \in \mathfrak{M}^{\perp}$, motivating the following:

DEFINITION 5.1. Let a and p be respectively an element and a point of L. We shall define $\varphi_a p$ by $(p \smile a^{\perp}) \frown a$ and call it a *projection* of p on a. In particular we shall define $\varphi_a 0=0$.

REMARK. *For any point p and any element a of L, we have* $p \leqq \varphi_a p \oplus \varphi_{a^{\perp}} p$ in view of the preceding remark.

LEMMA 5.3. *The following propositions are valid in L:*

$$(1) \qquad \varphi_a p = p \quad \textit{if and only if} \quad p \leqq a.$$

$$(2) \qquad \varphi_a p = 0 \quad \textit{if and only if} \quad p \leqq a^{\perp}.$$

PROOF. (1) The necessity is trivial and the sufficiency follows from the fact $(a^{\perp}, a)M$. (2) The sufficiency is obvious and the necessity is immediate from Lemma 5.1.

LEMMA 5.4. *If $p \leqq b \oplus c$, $b \leqq a$, $c \leqq a^{\perp}$ for some element a in L, then it holds $\varphi_a p = \varphi_b p = (p \smile c) \frown b$.*

PROOF. It follows from $c \leqq b^{\perp}$ that $(p \smile c) \frown b \leqq (p \smile b^{\perp}) \frown b$, whence we can infer $(p \smile c) \frown b = \varphi_b p$. For otherwise we have $0=(p \smile c) \frown b < \varphi_b p$, it follows from Lemma 5.1 that $p \leqq c$, whence $p \leqq b^{\perp}$, contrary to $0 < \varphi_b p$ by Lemma 5.3. Similarly $(p \smile c) \frown b = \varphi_a p$, whence we have $\varphi_a p = \varphi_b p$.

Now we shall characterize the projections.

THEOREM 5.1. *Let L be a lattice satisfying the conditions* (I)–(V) *Theorem* 1.1, *and let P be the set of all points of L together with* 0. *A mapping φ of P into P is identical to a projection on some element a of L if and only if it satisfies the following conditions:*

$$(1) \qquad p \leqq \bigvee(\varphi p_\alpha;\ \alpha \in I) \quad \textit{implies} \quad \varphi p = p.$$

$$(2) \qquad p \perp \varphi q \quad \textit{implies} \quad \varphi p \perp q.$$

(3) $\quad p \leqq q \smile r$ *implies* $\varphi p \leqq \varphi q \smile \varphi r$.

PROOF. *Necessity.* (1) is obvious, since $\varphi_a p_\alpha \leqq a (\alpha \in I)$. (2) Suppose $p \perp \varphi_a q$ then it holds $(q \smile a^\perp) \frown a \leqq p^\perp$, whence $p \leqq (q^\perp \frown a) \oplus a^\perp$. It follows that $p \smile a^\perp \leqq (q^\perp \frown a) \oplus a^\perp$, and so $(p \smile a^\perp) \frown a \leqq \{(q^\perp \frown a) \oplus a^\perp\} \frown a = q^\perp \frown a \leqq q^\perp$. That is, $\varphi_a p \perp q$. (3) Since $q \leqq \varphi_a q \oplus \varphi_{a^\perp} q$, and $r \leqq \varphi_a r \oplus \varphi_{a^\perp} r$, we have

$$p \leqq (\varphi_a q \smile \varphi_a r) \oplus (\varphi_{a^\perp} q \smile \varphi_{a^\perp} r).$$

It follows from Lemma 5.4 that $\varphi_a p$ is equal to the projection of p on $\varphi_a q \smile \varphi_a r$, whence we have $\varphi_a p \leqq \varphi_a q \smile \varphi_a r$.

Sufficiency. Put $a = \vee(\varphi q;\ q \in P)$. Then since $\varphi_a p \leqq a = \vee(\varphi q;\ q \in P)$, it follows from the condition (1) that $\varphi(\varphi_a p) = \varphi_a p$. Next we have $\varphi \varphi_{a^\perp} p = 0$, in fact, it holds $\varphi_{a^\perp} p \perp \varphi q$ for every $q \in P$, and so $\varphi \varphi_{a^\perp} p$ is orthogonal to every $q \in P$ by the condition (2), whence $\varphi \varphi_{a^\perp} p = 0$. Now $p \leqq \varphi_a p \oplus \varphi_{a^\perp} p$ for any $p \in P$. It follows from the condition (3) that $\varphi p \leqq \varphi \varphi_a p \oplus \varphi \varphi_{a^\perp} p$. Hence by the above results we have $\varphi p \leqq \varphi_a p$ which yields $\varphi p = \varphi_a p$ for every $p \in P$. For otherwise $0 = \varphi p < \varphi_a p$, and it holds $\varphi p \perp q$ for every $q \in P$, and so p is orthogonal to every $\varphi q (q \in P)$, whence $p \perp a$, contrary to the fact $\varphi_a p > 0$. Now it is clear that the condition (1) yields $\varphi 0 = 0$. Hence φ is identical with φ_a.

COROLLARY. *If $a \leqq b$ in L, then $\varphi_a \varphi_b p = \varphi_a p$ for every point $p \in L$.*

PROOF. For any point $p \in L$, it holds $p \leqq \varphi_b p \oplus \varphi_{b^\perp} p$. It follows Theorem 5.1 (3) that $\varphi_a p \leqq \varphi_a \varphi_b p \smile \varphi_a \varphi_{b^\perp} p$. Since $a \leqq b$, we have $\varphi_a \varphi_{b^\perp} p = 0$ by Lemma 5.3 (2). Hence $\varphi_a p \leqq \varphi_a \varphi_b p$, whence we can infer without difficulty that $\varphi_a p = \varphi_a \varphi_b p$.

DEFINITION 5.2. Let a, b be elements of L. If $a = (a \frown b) \oplus (a \frown b^\perp)$, then a is called *permutable* with b.

LEMMA 5.5. *If a is permutable with b, then b is permutable with a.*[14]

PROOF. Since $a = (a \frown b) \oplus (a \frown b^\perp)$, we have $a^\perp = (a \frown b)^\perp \frown (a^\perp \smile b)$, whence it holds $b \frown a^\perp = (a \frown b)^\perp \frown (a^\perp \smile b) \frown b = (a \frown b)^\perp \frown b$. It follows from the fact $((a \frown b)^\perp, a \frown b)\bar{M}$, that

$$(b \frown a) \oplus (b \frown a^\perp) = \{(a \frown b)^\perp \frown b\} \smile (a \frown b) = b.$$

Interrelation between the permutability of the elements a, b and the commutativity of the projections φ_a, φ_b is verified by the following:

THEOREM 5.2. *Let a, b be elements of a lattice satisfying the condition* I–V *in Theorem* 1.1. *Then the following propositions are equivalent to each other:*

(α) $\quad$ *a is permutable with b.*

(β) $\quad \varphi_a \varphi_b p = \varphi_b \varphi_a p = \varphi_{a \frown b} p$ *for any point $p \in L$.*

(γ) $\quad \varphi_b p \leqq a$ *for any point $p \leqq a$.*

14) Cf. F. Maeda [4] 207, Lemma 1.2.

PROOF. $(\alpha)\rightarrow(\beta)$. Since $a=(a\frown b)\oplus(a\frown b^{\perp})$, we have $\varphi_a p\leqq(a\frown b)\oplus(a\frown b^{\perp})$ for every point $p\in L$. By virtue of the fact $a\frown b\leqq b$, $a\frown b^{\perp}\leqq b^{\perp}$, it follows from Lemma 5.4 that $\varphi_b\varphi_a p=\varphi_{a\frown b}\varphi_a p$. An application of Corollary of Theorem 5.1 yields $\varphi_{a\frown b}\varphi_a p=\varphi_{a\frown b}p$, whence we have $\varphi_b\varphi_a p=\varphi_{a\frown b}p$. In view of Lemma 5.5, it holds similarly $\varphi_a\varphi_b p=\varphi_{a\frown b}p$.

$(\beta)\rightarrow(\gamma)$. Since $\varphi_a p=p$ for any point $p\leqq a$, we have

$$\varphi_b p=\varphi_b\varphi_a p=\varphi_{a\frown b}p\leqq a\,.$$

$(\gamma)\rightarrow(\alpha)$. It is sufficient to prove that $a\leqq(a\frown b)\oplus(a\frown b^{\perp})$, since the converse inequality is valid in general. Now let $p\leqq a$, then it holds by hypothesis, $\varphi_b p\leqq a\frown b$. We have in general $p\leqq\varphi_b p\oplus\varphi_{b^{\perp}}p$, where we can assume $p\neq\varphi_b p$, since otherwise the result is obvious. It follows from (η') that $\varphi_{b^{\perp}}p\leqq p\smile\varphi_b p\leqq a$, whence $\varphi_{b^{\perp}}p\leqq a\frown b^{\perp}$. Hence $p\leqq(a\frown b)\oplus(a\frown b^{\perp})$. Thus we have in view of the condition (II),

$$a\leqq(a\frown b)\oplus(a\frown b^{\perp})\,.$$

This completes the proof.

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