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A regularity criterion for a density-dependent incompressible liquid crystals model with vacuum

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ABSTRACT. This paper proves a new regularity criterion for a density-dependent incompressible liquid crystals model with vacuum but without any compatibility condition.

1. Introduction

In this paper, we consider the regularity criterion for the density-dependent incompressible nematic liquid crystals model [2, 3, 14, 18]:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \pi - \Delta u = -\nabla \cdot (\nabla d \odot \nabla d), \tag{1.2}$$

$$\partial_t d + u \cdot \nabla d - \Delta d = d|\nabla d|^2, \qquad |d| = 1, \tag{1.3}$$

$$\operatorname{div} u = 0, \tag{1.4}$$

$$(\rho, u, d)(\cdot, 0) = (\rho_0, u_0, d_0), \qquad |d_0| = 1 \qquad \text{in } \mathbb{R}^3, \tag{1.5}$$

$$\lim_{|x|\to\infty} (\rho, u, d) = (\tilde{\rho}_0, 0, \tilde{d}_0), \qquad \tilde{\rho}_0 = \lim_{|x|\to\infty} \rho_0, \qquad \tilde{d}_0 = \lim_{|x|\to\infty} d_0, \tag{1.6}$$

where ρ denotes the density, u the velocity, π the pressure, and d represents the macroscopic molecular orientations. $(\nabla d \odot \nabla d)_{i,j} := \sum_k \partial_i d_k \partial_j d_k$, and hence

$$\nabla \cdot (\nabla d \odot \nabla d) = \sum_k \Delta d_k \nabla d_k + \frac{1}{2} \nabla |\nabla d|^2.$$

When d is a constant vector with $|d| = 1$, then (1.1), (1.2) and (1.4) represent the well-known density-dependent Navier-Stokes system, which has been an object of many studies [7, 1, 12, 15]. Fan-Ozawa [7] proved the

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following regularity criterion

$$u \in L^{2/(1-r)}(0, T; \dot{B}^{-r}_{\infty,\infty}) \qquad \text{with } 0 < r < 1. \tag{1.7}$$

Very recently, Li [15] shows the local well-posedness of strong solutions to the problem (1.1)–(1.6) with vacuum but without the following compatibility condition

$$\nabla \pi_0 - \Delta u_0 + \nabla \cdot (\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0} g$$

for some $(\pi_0, g) \in H^1 \times L^2$, which was used in [1, 12].

When $u = 0$, (1.3) is the well-known harmonic heat flow.

Li-Wang [16] have proved that the problem (1.1)–(1.6) has a unique local strong solution. Fan-Gao-Guo [5] and Fan-Li [6] showed some blow-up criteria. The aim of this paper is to prove a new regularity criterion

$$u, \nabla d \in L^{2/(1-r)}(0, T; \dot{B}^{-r}_{\infty,\infty}) \qquad \text{with } 0 < r < 1 \tag{1.8}$$

for the problem (1.1)–(1.6) in the framework of [15]. We will prove

Theorem 1.1. *Let* $0 \le \rho_0 \le \sup \rho_0 < \infty$, $\nabla \rho_0 \in L^2 \cap L^p$ $(2 < p \le 6)$, $u_0 \in H^1$, $\nabla d_0 \in H^1$, $\operatorname{div} u_0 = 0$ *and* $|d_0| = 1$ *in* $\mathbb{R}^3$. *Let* (ρ, u, d) *be a local strong solution to the problem* (1.1)–(1.6). *If* (1.8) *holds true with* $0 < T < \infty$, *then the solution* (ρ, u, d) *can be extended beyond* $T > 0$.

Remark 1.1. *For the case that* $\inf \rho_0 > 0$, *Theorem* 1.1 *is proved recently in* [8]. *Concerning related studies, we refer to* [4, 9, 10, 20] *and references therein.*

Remark 1.2. *We can prove a similar regularity criterion*

$$u, \nabla d \in L^2(0, T; \dot{B}^0_{\infty,\infty}).$$

Remark 1.3. *We can prove some similar regularity criteria in a bounded and smooth domain.*

In the proofs, we will use the following Gagliardo-Nirenberg inequality

$$\|\nabla d\|^2_{L^{4p/(p-2)}} \le C \|d\|_{L^\infty} \|\Delta d\|_{L^{2p/(p-2)}}, \tag{1.9}$$

and the following interpolation inequalities [11, 17, 19]:

$$\|\nabla d\|_{L^p} \le C \|\nabla d\|^{1-\theta}_{\dot{B}^{-r}_{\infty,\infty}} \|d\|^{\theta}_{\dot{H}^{r+2}}, \tag{1.10}$$

$$\|\nabla^2 d\|_{L^{2p/(p-2)}} \le C \|\nabla d\|^{\theta}_{\dot{B}^{-r}_{\infty,\infty}} \|d\|^{1-\theta}_{\dot{H}^{r+2}} \tag{1.11}$$

with $0 < r < 1$ and $\theta := \frac{2}{p}$, and $p = \frac{2(1+2r)}{r}$.

We will also use the following bilinear estimates [13]:

$$\|u \cdot \nabla u\|_{L^2} \le C\|u\|_{\dot{B}^{-r}_{\infty,\infty}} \|u\|_{\dot{H}^{1+r}}, \tag{1.12}$$

$$\|\nabla(u \cdot \nabla d)\|_{L^2} \le C\|u\|_{\dot{B}^{-r}_{\infty,\infty}} \|\nabla d\|_{\dot{H}^{1+r}} + C\|\nabla d\|_{\dot{B}^{-r}_{\infty,\infty}} \|u\|_{\dot{H}^{1+r}} \tag{1.13}$$

with $0 < r < 1$.

2. Proof of Theorem 1.1

Due to the aforementioned existence and uniqueness theorem for local strong solutions, by using a standard continuation argument, we only need to establish a priori estimates on smooth enough local solutions. Hence, by using a standard argument we assume the existence of strong solutions in the maximal interval $(0, T)$, and all the calculations we are going to perform are completely justified. By proving uniform bounds for all $t \in [0, T)$, for appropriate norms of the unknowns, we will show that the solution can be continued beyond T, which is in contradiction to its maximality.

First, observe that, thanks to the maximum principle, it follows from (1.1) and (1.4) that

$$0 \le \rho \le C. \tag{2.1}$$

Testing (1.2) by u and using (1.1) and (1.4), we see that

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2 dx + \int |\nabla u|^2 dx = -\int (u \cdot \nabla) d \cdot \Delta d \, dx. \tag{2.2}$$

Testing (1.3) by $-\Delta d$ and using $d \cdot \Delta d = -|\nabla d|^2$ and $|d| = 1$, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx + \int |\Delta d|^2 dx \\ &\quad = \int (u \cdot \nabla) d \cdot \Delta d \, dx + \int (d \cdot \Delta d)^2 dx \\ &\quad \le \int (u \cdot \nabla) d \cdot \Delta d \, dx + \int |\Delta d|^2 dx. \end{aligned} \tag{2.3}$$

Summing up (2.2) and (2.3), we get

$$\frac{1}{2}\int (\rho|u|^2 + |\nabla d|^2)(t) dx + \int_0^T \int |\nabla u|^2 dx dt \le \frac{1}{2}\int (\rho_0|u_0|^2 + |\nabla d_0|^2) dx. \tag{2.4}$$

Testing (1.2) by $\partial_t u$ and using (1.1) and (1.4), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2dx+\int\rho|\partial_t u|^2dx\\ &=-\int\rho u\cdot\nabla u\cdot\partial_t u\,dx+\int\nabla\partial_t u:\nabla d\odot\nabla d\,dx\\ &=:I_1+I_2.\end{aligned}\tag{2.5}$$

Using (1.12) and (2.1), we bound I_1 as follows.

$$\begin{aligned}I_1&\le\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\|u\cdot\nabla u\|_{L^2}\\ &\le C\|\sqrt{\rho}\partial_t u\|_{L^2}\|u\|_{\dot B^{-r}_{\infty,\infty}}\|u\|_{\dot H^{1+r}}\\ &\le C\|\sqrt{\rho}\partial_t u\|_{L^2}\|u\|_{\dot B^{-r}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-r}\|\nabla^2 u\|_{L^2}^{r}\\ &\le\varepsilon\|\sqrt{\rho}\partial_t u\|_{L^2}^2+\varepsilon\|\nabla^2 u\|_{L^2}^2+C\|u\|_{\dot B^{-r}_{\infty,\infty}}^{2/(1-r)}\|\nabla u\|_{L^2}^2\end{aligned}\tag{2.6}$$

for any $0<\varepsilon<1$.

On the other hand, because (u,π) is a solution of the Stokes system,

$$-\Delta u+\nabla\pi=f:=-\nabla\cdot(\nabla d\odot\nabla d)-\rho\partial_t u-\rho u\cdot\nabla u.\tag{2.7}$$

It follows from the $\dot H^2$-theory of the Stokes system and (1.12) that

$$\begin{aligned}\|\nabla^2 u\|_{L^2}&\le C\|f\|_{L^2}\\ &\le C\|\nabla\cdot(\nabla d\odot\nabla d)\|_{L^2}+C\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|u\cdot\nabla u\|_{L^2}\\ &\le C\|\nabla d\|_{\dot B^{-r}_{\infty,\infty}}\|d\|_{\dot H^{2+r}}+C\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|u\|_{\dot B^{-r}_{\infty,\infty}}\|u\|_{\dot H^{1+r}}\\ &\le C\|\nabla d\|_{\dot B^{-r}_{\infty,\infty}}\|\nabla^2 d\|_{L^2}^{1-r}\|\nabla\Delta d\|_{L^2}^{r}+C\|\sqrt{\rho}\partial_t u\|_{L^2}\\ &\quad+C\|u\|_{\dot B^{-r}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-r}\|\nabla^2 u\|_{L^2}^{r}\\ &\le\frac{1}{2}\|\nabla^2 u\|_{L^2}+C\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|\nabla\Delta d\|_{L^2}+C\|\nabla d\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\nabla^2 d\|_{L^2}\\ &\quad+C\|u\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\nabla u\|_{L^2}\end{aligned}$$

which gives

$$\begin{aligned}\|\nabla^2 u\|_{L^2}&\le C\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|\nabla\Delta d\|_{L^2}\\ &\quad+C\|\nabla d\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\nabla^2 d\|_{L^2}+C\|u\|_{\dot B^{-r}_{\infty,\infty}}^{1/(1-r)}\|\nabla u\|_{L^2}.\end{aligned}\tag{2.8}$$

Here we used the following bilinear estimate [13]:

$$\|\nabla\cdot(\nabla d\odot\nabla d)\|_{L^2}\le C\|\nabla d\|_{\dot B^{-r}_{\infty,\infty}}\|d\|_{\dot H^{2+r}}\qquad\text{with }0<r<1.\tag{2.9}$$

By integration by parts, the term I_2 can be rewritten as

$$\begin{aligned}I_2&=\frac{d}{dt}\int\nabla u:\nabla d\odot\nabla d\,dx-2\int\nabla u:\nabla d\odot\partial_t\nabla d\,dx\\&=:\frac{d}{dt}\int\nabla u:\nabla d\odot\nabla d\,dx+I_3.\end{aligned}\tag{2.10}$$

Using (1.3), (1.9), (1.10), (1.11) and (1.13), we bound I_3 as follows.

$$\begin{aligned}I_3&\le C\|\nabla u\|_{L^{2p/(p-2)}}\|\nabla d\|_{L^p}\|\nabla\partial_t d\|_{L^2}\\&\le C\|\nabla d\|_{L^p}\|\nabla u\|_{L^{2p/(p-2)}}(\|\nabla\Delta d\|_{L^2}+\|\nabla d\|_{L^p}\|\nabla^2 d\|_{L^{2p/(p-2)}}\\&\quad+\|\nabla d\|_{L^p}\|\nabla d\|^2_{L^{4p/(p-2)}}+\|\nabla(u\cdot\nabla d)\|_{L^2})\\&\le C\|\nabla d\|_{L^p}\|\nabla u\|_{L^{2p/(p-2)}}(\|\nabla\Delta d\|_{L^2}+\|\nabla d\|_{L^p}\|\nabla^2 d\|_{L^{2p/(p-2)}}+\|\nabla(u\cdot\nabla d)\|_{L^2})\\&\le\varepsilon\|\nabla\Delta d\|^2_{L^2}+C\|\nabla d\|^2_{L^p}\|\nabla^2 d\|^2_{L^{2p/(p-2)}}+C\|\nabla(u\cdot\nabla d)\|^2_{L^2}\\&\quad+C\|\nabla d\|^2_{L^p}\|\nabla u\|^2_{L^{2p/(p-2)}}\\&\le\varepsilon\|\nabla\Delta d\|^2_{L^2}+C\|\nabla d\|^2_{\dot B^{-r}_{\infty,\infty}}\|d\|^2_{\dot H^{2+r}}+C\|u\|^2_{\dot B^{-r}_{\infty,\infty}}\|d\|^2_{\dot H^{2+r}}\\&\quad+C\|\nabla d\|^2_{\dot B^{-r}_{\infty,\infty}}\|u\|^2_{\dot H^{1+r}}+C\|\nabla d\|^{2(1-\theta)}_{\dot B^{-r}_{\infty,\infty}}\|d\|^{2\theta}_{\dot H^{2+r}}\|u\|^{2\theta}_{\dot B^{-r}_{\infty,\infty}}\|u\|^{2(1-\theta)}_{\dot H^{1+r}}\\&\le\varepsilon\|\nabla\Delta d\|^2_{L^2}+C(\|u\|^2_{\dot B^{-r}_{\infty,\infty}}+\|\nabla d\|^2_{\dot B^{-r}_{\infty,\infty}})(\|u\|^2_{\dot H^{1+r}}+\|d\|^2_{\dot H^{2+r}})\\&\le 2\varepsilon\|\nabla\Delta d\|^2_{L^2}+\varepsilon\|\Delta u\|^2_{L^2}\\&\quad+C(\|u\|^{2/(1-r)}_{\dot B^{-r}_{\infty,\infty}}+\|\nabla d\|^{2/(1-r)}_{\dot B^{-r}_{\infty,\infty}})(\|\nabla u\|^2_{L^2}+\|\Delta d\|^2_{L^2})\end{aligned}\tag{2.11}$$

for any $0<\varepsilon<1$. Here we used the interpolation inequality [11, 17, 19]:

$$\|\nabla u\|_{L^{2p/(p-2)}}\le C\|u\|^{\theta}_{\dot B^{-r}_{\infty,\infty}}\|u\|^{1-\theta}_{\dot H^{1+r}}\qquad\text{with }0<r<1.\tag{2.12}$$

Applying Δ to (1.3), testing by Δd and using (1.9), (1.10), (1.11) and (1.13), we reach

$$\begin{aligned}&\frac12\frac{d}{dt}\int|\Delta d|^2dx+\int|\nabla\Delta d|^2dx\\&\quad=-\int\nabla(d|\nabla d|^2)\nabla\Delta d\,dx+\int\nabla(u\cdot\nabla d)\nabla\Delta d\,dx\end{aligned}$$

$$\begin{aligned}
&\leq C\|\nabla d\|_{L^p}(\|\nabla d\|^2_{L^{4p/(p-2)}} + \|\nabla^2 d\|_{L^{2p/(p-2)}})\|\nabla \Delta d\|_{L^2} \\
&\quad + C\|\nabla(u\cdot\nabla d)\|_{L^2}\|\nabla\Delta d\|_{L^2} \\
&\leq C\|\nabla d\|_{L^p}\|\nabla^2 d\|_{L^{2p/(p-2)}}\|\nabla\Delta d\|_{L^2} + C\|\nabla(u\cdot\nabla d)\|_{L^2}\|\nabla\Delta d\|_{L^2} \\
&\leq \varepsilon\|\nabla\Delta d\|^2_{L^2} + C\|\nabla d\|^2_{\dot B^{-r}_{\infty,\infty}}\|d\|^2_{\dot H^{2+r}} + C\|u\|^2_{\dot B^{-r}_{\infty,\infty}}\|d\|^2_{\dot H^{2+r}} + C\|\nabla d\|^2_{\dot B^{-r}_{\infty,\infty}}\|u\|^2_{\dot H^{1+r}} \\
&\leq 2\varepsilon\|\nabla\Delta d\|^2_{L^2} + \varepsilon\|\Delta u\|^2_{L^2} \\
&\quad + C(\|u\|^{2/(1-r)}_{\dot B^{-r}_{\infty,\infty}} + \|\nabla d\|^{2/(1-r)}_{\dot B^{-r}_{\infty,\infty}})(\|\nabla u\|^2_{L^2} + \|\nabla^2 d\|^2_{L^2}) \qquad (2.13)
\end{aligned}$$

for any $0 < \varepsilon < 1$.

Combining (2.5), (2.6), (2.8), (2.10), (2.11) and (2.13), taking ε suitably small, noting that

$$\left|\int \nabla u : \nabla d \odot \nabla d\, dx\right| \leq \|\nabla u\|_{L^2}\|\nabla d\|^2_{L^4} \leq C\|\nabla u\|_{L^2}\|\Delta d\|_{L^2},$$

and using the Gronwall inequality, we conclude that

$$\begin{aligned}
&\|\nabla u\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^2(0,T;H^1)} + \|\sqrt{\rho}u_t\|_{L^2(0,T;L^2)} \\
&\quad + \|\nabla d\|_{L^\infty(0,T;H^1)} + \|\nabla d\|_{L^2(0,T;H^2)} \leq C. \qquad (2.14)
\end{aligned}$$

Testing (1.3) by d_t and using $d\cdot d_t = 0$ and (2.14), we deduce that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int|\nabla d|^2dx + \int|d_t|^2dx &= -\int u\cdot\nabla d\cdot d_t\,dx \\
&\leq \|u\|_{L^6}\|\nabla d\|_{L^3}\|d_t\|_{L^2} \leq C\|d_t\|_{L^2} \leq \frac{1}{2}\|d_t\|^2_{L^2} + C,
\end{aligned}$$

which gives

$$\|d_t\|_{L^2(0,T;L^2)} \leq C. \qquad (2.15)$$

Testing (1.3) by $-\Delta d_t$ and using (2.14), we derive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int|\Delta d|^2dx + \int|\nabla d_t|^2dx &= \int\nabla(d|\nabla d|^2 - u\cdot\nabla d)\cdot\nabla d_t\,dx \\
&\leq C(\|\nabla d\|^3_{L^6} + \|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^2} + \|u\|_{L^6}\|\nabla^2 d\|_{L^3} \\
&\quad + \|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty})\|\nabla d_t\|_{L^2} \\
&\leq C(1 + \|\nabla d\|_{L^\infty} + \|\nabla^2 d\|_{L^3})\|\nabla d_t\|_{L^2} \\
&\leq \frac{1}{2}\|\nabla d_t\|^2_{L^2} + C(1 + \|\nabla d\|^2_{L^\infty} + \|\nabla^2 d\|^2_{L^3}),
\end{aligned}$$

which leads to

$$\|d_t\|_{L^2(0,T;H^1)} \le C. \tag{2.16}$$

Taking ∂_t to (1.2), testing by u_t and using (1.1) and (1.4), we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 dx + \int |\nabla u_t|^2 dx &= \int \operatorname{div}(\rho u)u_t^2\, dx + \int \operatorname{div}(\rho u)(u\cdot\nabla)u\cdot u_t\, dx \\
&\quad - \int \rho(u_t\cdot\nabla u)\cdot u_t\, dx + \int \partial_t(\nabla d \odot \nabla d) : \nabla u_t\, dx \\
&=: I_4 + I_5 + I_6 + I_7.
\end{aligned} \tag{2.17}$$

Using (2.1) and (2.14), we bound I_i $(i = 4,5,6,7)$ as follows.

$$\begin{aligned}
I_4 &= -\int \rho u \nabla u_t^2\, dx \le 2\|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\|\sqrt{\rho}u_t\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2} \le C\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{3/2} \\
&\le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2. \\
I_5 &= -\int \rho u \nabla[(u\cdot\nabla)u\cdot u_t]dx \\
&\le \|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\|u_t\|_{L^6} + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6} \\
&\quad + \|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\le C\|\nabla^2 u\|_{L^2}\|\nabla u_t\|_{L^2} \le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2. \\
I_6 &\le \|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6} \\
&\le C\|\sqrt{\rho}u_t\|_{L^3}\|u_t\|_{L^6} \le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2. \\
I_7 &\le C\|\nabla d\|_{L^\infty}\|\nabla d_t\|_{L^2}\|\nabla u_t\|_{L^2} \le \frac{1}{16}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (2.17), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2 dx + \int |\nabla u_t|^2 dx \\
&\quad \le \frac{1}{4}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2.
\end{aligned} \tag{2.18}$$

Applying ∂_t to (1.3), testing by $-\Delta d_t$ and using $|d| = 1$ and (2.14), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla d_t|^2 dx + \int |\Delta d_t|^2 dx \\
&\quad = -\int (d_t|\nabla d|^2 + d\partial_t|\nabla d|^2 - u\cdot\nabla d_t)\Delta d_t\, dx + \int \nabla(u_t\cdot\nabla d) : \nabla d_t\, dx \\
&\quad \le C(\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|\nabla d\|_{L^6}\|\nabla d_t\|_{L^3} + \|u\|_{L^6}\|\nabla d_t\|_{L^3})\|\Delta d_t\|_{L^2} \\
&\qquad + C(\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^6} + \|u_t\|_{L^6}\|\nabla^2 d\|_{L^2})\|\nabla d_t\|_{L^3} \\
&\quad \le C(\|d_t\|_{L^6} + \|\nabla d_t\|_{L^3})\|\Delta d_t\|_{L^2} + C\|\nabla u_t\|_{L^2}\|\nabla d_t\|_{L^3} \\
&\quad \le \frac{1}{8}\|\Delta d_t\|_{L^2}^2 + \frac{1}{8}\|\nabla u_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2. \qquad (2.19)
\end{aligned}
$$

Summing up (2.18) and (2.19), we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int (\rho|u_t|^2 + |\nabla d_t|^2)dx + \int (|\nabla u_t|^2 + |\Delta d_t|^2)dx \\
&\quad \le C\|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2. \qquad (2.20)
\end{aligned}
$$

Multiplying (2.20) by t,

$$
\begin{aligned}
&\frac{d}{dt}\left(t\int (\rho|u_t|^2 + |\nabla d_t|^2)dx\right) + t\int (|\nabla u_t|^2 + |\Delta d_t|^2)dx \\
&\quad \le C\int (\rho|u_t|^2 + |\nabla d_t|^2)dx + C(1 + \|\nabla d\|_{L^\infty}^2)t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \\
&\qquad + Ct\|\nabla^2 u\|_{L^2}^2, \qquad (2.21)
\end{aligned}
$$

which implies

$$
\sup_{0\le t\le T} t\int (\rho|u_t|^2 + |\nabla d_t|^2)dx + \int_0^T t(\|\nabla u_t\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2)dt \le C. \qquad (2.22)
$$

Now it is easy to verify that

$$
\sup_{0\le t\le T} t(\|\nabla^2 u\|_{L^2}^2 + \|\nabla d\|_{H^2}^2) + \int_0^T t\|\nabla^2 u\|_{L^6}^2 dt \le C. \qquad (2.23)
$$

It is standard to find that

$$
\begin{aligned}
\int_0^T \|\nabla u\|_{L^\infty} dt &\le C\int_0^T \|\nabla u\|_{L^6}^{1/2}\|\nabla^2 u\|_{L^6}^{1/2} dt \\
&\le C\int_0^T \|\nabla^2 u\|_{L^2}^{1/2}(t\|\nabla^2 u\|_{L^6}^2)^{1/4} t^{-1/4}\, dt
\end{aligned}
$$

$$\leq C\left(\int_0^T \|\nabla^2 u\|_{L^2}^2 dt\right)^{1/4}\left(\int_0^T t\|\nabla^2 u\|_{L^6}^2 dt\right)^{1/4}\left(\int_0^T t^{-1/2}\, dt\right)^{1/2}$$

$$\leq C. \tag{2.24}$$

Taking ∇ to (1.1), testing by $|\nabla\rho|^{q-2}\nabla\rho$ $(2 < q < \infty)$ and using (1.4) and (2.24), we have

$$\frac{d}{dt}\|\nabla\rho\|_{L^q}^q \leq C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}^q,$$

which implies

$$\|\nabla\rho\|_{L^\infty(0,T;L^q)} \leq C \qquad (2 \leq q < \infty). \tag{2.25}$$

Similarly to (2.24), we have

$$\int_0^T \|u\|_{L^\infty}^4 dt \leq C\int_0^T \|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 dt \leq C,$$

thus

$$\int_0^T \|\rho_t\|_{L^q}^4 dt = \int_0^T \|u\nabla\rho\|_{L^q}^4 dt \leq \int_0^T \|u\|_{L^\infty}^4\|\nabla\rho\|_{L^q}^4 dt$$

$$\leq C\int_0^T \|u\|_{L^\infty}^4 dt \leq C. \tag{2.26}$$

This completes the proof. □

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