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Terminating q-Kampé de Fériet Series $\Phi^{1:3;\lambda}_{1:2;\mu}$ and $\Phi^{2:2;\lambda}_{2:1;\mu}$

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Abstract. By means of the transformations of Sears and Watson for the terminating balanced ${}_4\phi_3$-series, we investigate the two terminating q-Kampé de Fériet series $\Phi^{1:3;\lambda}_{1:2;\mu}$ and $\Phi^{2:2;\lambda}_{2:1;\mu}$. Several reduction and summation formulae are established. They extend the corresponding known results about the double q-Clausen series.

1. Introduction and motivation

For the two indeterminates x and q, the shifted factorial of x with base q reads as

$$(x;q)_0 = 1 \qquad \text{and} \qquad (x;q)_n = (1-x)(1-qx)\dots(1-q^{n-1}x) \qquad \text{for } n \in \mathbf{N}.$$

When $|q| < 1$, we have two well–defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \qquad \text{and} \qquad (x;q)_n = (x;q)_\infty/(q^n x;q)_\infty.$$

The product and fraction of shifted factorials are abbreviated respectively to

$$[\alpha,\beta,\dots,\gamma;q]_n = (\alpha;q)_n(\beta;q)_n\dots(\gamma;q)_n,$$
$$\left[\begin{matrix}\alpha,\beta,\dots,\gamma\\ A,B,\dots,C\end{matrix}\middle| q\right]_n = \frac{(\alpha;q)_n(\beta;q)_n\dots(\gamma;q)_n}{(A;q)_n(B;q)_n\dots(C;q)_n}.$$

Following Gasper–Rahman [8], the basic hypergeometric series is defined by

$${}_{1+r}\phi_s\left[\begin{matrix}a_0,a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix}\middle| q;z\right] = \sum_{n=0}^{\infty}\{(-1)^n q^{\binom{n}{2}}\}^{s-r}\left[\begin{matrix}a_0,a_1,\dots,a_r\\ q,b_1,\dots,b_s\end{matrix}\middle| q\right]_n z^n$$

where the base q will be restricted to $|q| < 1$ for nonterminating q-series. Most of the q-series results concern the case $r = s$. When the parameters satisfy the

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condition $qa_0a_1a_2\dots a_r = b_1b_2\dots b_r$, the series is said to be balanced. On the other hand, when $qa_0 = a_1b_1 = \cdots = a_rb_r$, the corresponding series is well–poised.

There are different extensions of q-series in two variables. The easier case is about the q-extensions of Appell functions (cf. Bailey [2, Chapter 9]), which is considered by Jain [10]. The more general bivariate basic hypergeometric function is defined by Srivastava and Karlsson [24, P 349]

$$\Phi^{\lambda:r;s}_{\mu:u;v}\left[\begin{matrix}\alpha_1,\dots,\alpha_\lambda : a_1,\dots,a_r; c_1,\dots,c_s; q : x, y\\ \beta_1,\dots,\beta_\mu : b_1,\dots,b_u; d_1,\dots,d_v; i, j, k\end{matrix}\right]$$
$$= \sum_{m,n=0}^{\infty} \frac{[\alpha_1,\dots,\alpha_\lambda;q]_{m+n}}{[\beta_1,\dots,\beta_\mu;q]_{m+n}} \frac{[a_1,\dots,a_r;q]_m[c_1,\dots,c_s;q]_n}{[b_1,\dots,b_u;q]_m[d_1,\dots,d_v;q]_n} \frac{x^my^nq^{i\binom{m}{2}+j\binom{n}{2}+kmn}}{(q;q)_m(q;q)_n}$$

which is generally called the q-analogue of Kampé de Fériet function. When $i, j, k \in \mathbf{N}_0$, this double series $\Phi^{\lambda:r;s}_{\mu:u;v}$ is convergent for $|x| < 1$, $|y| < 1$ and $|q| < 1$. The literatures on basic double series have been less extensive. Apart from the aforementioned works by Jain [10] and Srivastava–Karlsson [24], most of the published papers on this topic deal with only the case corresponding to $\lambda + r = 3$ and $\mu + u = 2$, i.e., the q-Clausen series. Early works can be found in Al-Salam [1], Carlitz [3, 4], Sinhal [21] and Srivastava–Jain [23]. Chu–Srivastava [7], Gasper [9], Karlsson [14–17], Van der Jeugt *et al.* [18, 19, 25, 26], Singh [20], Sinhal [22] made further research and obtained several interesting results. Recently, by applying exclusively the Sears transformations for ${}_3\phi_2$-series, Chu–Jia [6] and Jia *et al.* [11–13] examined systematically transformation and reduction formulae for the q-Clausen series.

After the q-Clausen series, the next natural target should be the q-Kampé de Fériet series with $\lambda + r = 4$ and $\mu + u = 3$. In the terminating case, there are two non–equivalent types of such series $\Phi^{1:3;\lambda}_{1:2;\mu}$ and $\Phi^{2:2;\lambda}_{2:1;\mu}$. Having escaped from the aforementioned works, these series seem to be the only remaining instances of q-Kampé de Fériet function $\Phi^{\lambda:r;s}_{\mu:u;v}$, that can be attacked by available univariate q-series transformations. The purpose of this paper is to investigate thoroughly these series which will result in several transformation, reduction and summation formulae with some of them involving double q-series even beyond the class of q-Kampé de Fériet functions. To our knowledge, there does not seem to exist much mathematical literature dedicated specifically to these series. Therefore, most of the results presented here seem unlikely to have previously appeared, except for some particular cases contained in [6, 11, 13, 25].

The basic tools for carrying out our investigation will consist of three transformations on the terminating balanced ${}_4\phi_3$-series plus the q-Pfaff–

Saalschütz theorem. For the sake of subsequent references, we reproduce them below. The first one is the Watson transformation (cf. Bailey [2, §8.5] and Gasper–Rahman [8, III-18]) which is fundamental in the q-series theory:

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, & b, & c, & d, & e, & q^{-n} \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{1+n}a \end{matrix}\middle| q; \frac{q^{2+n}a^2}{bcde}\right] \\
&\quad = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix}\middle| q\right]_n \times {}_4\phi_3\left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix}\middle| q; q\right].
\end{aligned}
$$

In order to shorten the lengthy expressions, we shall employ the following abbreviated notation for the very–well–poised series

$$
W(a:b,c,d,e,f;z) := {}_8\phi_7\left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, & b, & c, & d, & e, & f \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e, & qa/f \end{matrix}\middle| q; z\right].
$$

Then the last transformation can equivalently be reformulated as

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, e, q^{-n} \\ b, d, q^{1-n}ace/bd \end{matrix}\middle| q; q\right] \\
&\quad = \left[\begin{matrix} bd/ae, bd/ce \\ bd/e, bd/ace \end{matrix}\middle| q\right]_n W\left(\frac{bd}{qe} : a, c, \frac{b}{e}, \frac{d}{e}, q^{-n}; q^n\frac{bd}{ac}\right). \qquad (1)
\end{aligned}
$$

Two other transformations on the terminating balanced ${}_4\phi_3$-series due to Sears [8, III-15 and III-16] will frequently be utilized

$$
{}_4\phi_3\left[\begin{matrix} a, c, e, q^{-n} \\ b, d, q^{1-n}ace/bd \end{matrix}\middle| q; q\right] = \left[\begin{matrix} d/a, bd/ce \\ d, bd/ace \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} a, b/c, b/e, q^{-n} \\ b, bd/ce, q^{1-n}a/d \end{matrix}\middle| q; q\right], \qquad (2)
$$

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, e, q^{-n} \\ b, d, q^{1-n}ace/bd \end{matrix}\middle| q; q\right] \\
&\quad = \left[\begin{matrix} a, bd/ac, bd/ae \\ b, d, bd/ace \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} b/a, d/a, bd/ace, q^{-n} \\ bd/ac, bd/ae, q^{1-n}/a \end{matrix}\middle| q; q\right] \qquad (3)
\end{aligned}
$$

When one pair of numerator and denominator parameters cancels, the ${}_4\phi_3$-series becomes a balanced ${}_3\phi_2$-series, which can be evaluated by the well–known q-Pfaff–Saalschütz summation theorem (cf. [2, §8.4] and [8, II-12])

$$
{}_3\phi_2\left[\begin{matrix} q^{-n}, a, & b \\ & c, q^{1-n}ab/c \end{matrix}\middle| q; q\right] = \left[\begin{matrix} c/a, c/b \\ c, c/ab \end{matrix}\middle| q\right]_n. \qquad (4)
$$

The rest of the paper will be organized as follows. In the next section, we shall investigate the series $\Phi^{1:3;\lambda}_{1:2;\mu}$ by considering all the applicable possibilities of three transformations (1), (2) and (3). The results are presented as six transformation theorems and twelve specific reduction formulae. Then the analogous examination for $\Phi^{2:2;\lambda}_{2:1;\mu}$ will be worked out in the third section. Finally, the paper will be concluded with comments on the series $\Phi^{3:1;\lambda}_{3:0;\mu}$.

2. Bivariate terminating series $\Phi^{1:3;\lambda}_{1:2;\mu}$

In this section, we shall investigate the bivariate terminating series

$$\sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a,\ c, & e \\ q,\ d,\ q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j) \tag{5}$$

$$= \sum_{j\geq 0} \Omega(j) \frac{(q^{-n};q)_j}{(b;q)_j} {}_4\phi_3\left[\begin{matrix} q^{j-n},\ a,\ c,\ e \\ q^jb, d, q^{1-n}ace/bd \end{matrix}\middle| q;q\right]. \tag{6}$$

By reformulating the ${}_4\phi_3$-series just displayed with (1), (2) and (3), six transformation theorems will be established. Then they will be employed to derive several reduction and summation formulae, that are mainly done by carrying out the following replacements

$$\text{Eq}(5) \Rightarrow \left[\begin{matrix} b/\beta, b/\gamma \\ b, b/\beta\gamma \end{matrix}\middle| q\right]_n {}_5\phi_4\left[\begin{matrix} q^{-n},\ a,\ \ c,\ \ e, & b/\beta\gamma \\ & d,\ b/\beta,\ b/\gamma,\ q^{1-n}ace/bd \end{matrix}\middle| q;q\right] \tag{7}$$

$$\text{when} \qquad \Omega(j) = \left[\begin{matrix} \beta, & \gamma \\ q,\ q^{1-n}\beta\gamma/b \end{matrix}\middle| q\right]_j q^j. \tag{8}$$

The reduced expression (7) is obtained by specifying the $\Omega(j)$-sequence in (5) with (8) and then evaluating the corresponding sum with respect to j by means of the q-Saalschütz theorem (4).

§ 2.1. Transformation into $\Phi^{1:3;1+\lambda}_{1:2;1+\mu}$. According to (2), the ${}_4\phi_3$-series in (6) can be expressed as

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{j-n},\ a,\ c,\ e \\ q^jb, d, q^{1-n}ace/bd \end{matrix}\middle| q;q\right] \\ &\quad = \left[\begin{matrix} q^jb/a, q^jbd/ce \\ q^jb, q^jbd/ace \end{matrix}\middle| q\right]_{n-j} {}_4\phi_3\left[\begin{matrix} q^{j-n},\ a,\ d/c,\ d/e \\ d, q^jbd/ce, q^{1-n}a/b \end{matrix}\middle| q;q\right]. \end{aligned}$$

Therefore we have the equality

$$\text{Eq}(5) = \sum_{j\geq 0} \Omega(j) \frac{(q^{-n};q)_j}{(b;q)_j} \left[\begin{matrix} q^jb/a, q^jbd/ce \\ q^jb, q^jbd/ace \end{matrix}\middle| q\right]_{n-j} {}_4\phi_3\left[\begin{matrix} q^{j-n},\ a,\ d/c,\ d/e \\ d, q^jbd/ce, q^{1-n}a/b \end{matrix}\middle| q;q\right]$$

which leads to the following transformation theorem.

Theorem 1 (Bivariate terminating series transformation).

$$\left[\begin{matrix} bd/ace, b \\ bd/ce, b/a \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a, \ c, & e \\ q, \ d, & q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)$$

$$= \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(bd/ce;q)_{i+j}} \left[\begin{matrix} a, d/c, d/e \\ q, d, q^{1-n}a/b \end{matrix}\middle| q\right]_i \frac{(bd/ace;q)_j}{(b/a;q)_j} \Omega(j).$$

This theorem contains the two transformations due to Chu–Jia [6, Theorems 4.1 and 4.2] as special cases, which can be confirmed by making the simultaneous replacements $e \to be$ and $a \to ab$ and then letting $b \to \infty$.

In order to reduce the double sum on the right hand side, rewrite it as

$$\sum_{i=0}^{n} q^i \left[\begin{matrix} q^{-n}, a, d/c, d/e \\ q, d, q^{1-n}a/b, bd/ce \end{matrix}\middle| q\right]_i \sum_{j=0}^{n-i} \left[\begin{matrix} q^{i-n}, bd/ace \\ b/a, q^i bd/ce \end{matrix}\middle| q\right]_j \Omega(j).$$

Then specifying the $\Omega(j)$-sequence in Theorem 1 by

$$\Omega(j) = \left[\begin{matrix} \beta, & \gamma, & b/a \\ q, & bd/ace, & q^{1-n}ce\beta\gamma/bd \end{matrix}\middle| q\right]_j q^j$$

and evaluating the corresponding sum just displayed by means of (4), we find after some trivial simplification the following reduction formula to balanced ${}_5\phi_4$-series.

Proposition 2 (Balanced reduction formula).

$$\sum_{i,j\geq 0} q^{i+j} \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a, c, e \\ q, d, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \left[\begin{matrix} \beta, \gamma, b/a \\ q, bd/ace, q^{1-n}ce\beta\gamma/bd \end{matrix}\middle| q\right]_j$$

$$= \left[\begin{matrix} b/a, bd/ce\beta, bd/ce\gamma \\ b, bd/ace, bd/ce\beta\gamma \end{matrix}\middle| q\right]_n {}_5\phi_4\left[\begin{matrix} q^{-n}, a, d/c, d/e, bd/ce\beta\gamma \\ d, bd/ce\beta, bd/ce\gamma, q^{1-n}a/b \end{matrix}\middle| q; q\right].$$

Replacing e by be and then letting $b \to \infty$, we recover from this proposition the two transformations due to Chu–Jia [6, Propositions 4.6 and 4.11], where the ${}_5\phi_4$-series needs to be inverted for the latter.

When $\beta = b/e$ and $\gamma = 1/c$, the last ${}_5\phi_4$-series reduces to a ${}_3\phi_2$-series, which has a closed form in view of (4). This leads to the following summation formula, which is equivalent to Gasper [9, Equations 5.5 and 5.6].

Corollary 3 (Summation formula of double series).

$$\left[\begin{matrix} d/c, bd/ae \\ d, bd/ace \end{matrix}\middle| q\right]_n$$

$$= \sum_{i,j\geq 0} \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a, c, e \\ q, d, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \left[\begin{matrix} b/a, b/e, 1/c \\ q, bd/ace, q^{1-n}/d \end{matrix}\middle| q\right]_j q^{i+j}.$$

The informed reader will notice that this corollary contains the formula due to Van der Jeugt [25, Equation 11] (cf. Chu–Jia [6, Corollary 4.12]) as the limiting case $b \to \infty$ after having replaced e by be. Another limiting case $n \to \infty$ has been conjectured by Van der Jeugt and confirmed by Gasper [9, Equation 5.3].

§ 2.2. Transformation into $\Phi^{3:1;\,2+\lambda}_{3:0;\,2+\mu}$. Instead, the ${}_4\phi_3$-series displayed in (6) can also be reformulated by (2) into

$$
{}_4\phi_3\left[\begin{matrix} q^{j-n}, & a, & c, & e \\ & q^jb, & d, & q^{1-n}ace/bd \end{matrix}\middle| q;q\right]
$$
$$
= \left[\begin{matrix} q^jbd/ce, d/a \\ q^jbd/ace, d \end{matrix}\middle| q\right]_{n-j} \times {}_4\phi_3\left[\begin{matrix} q^{j-n}, & a, & q^jb/c, & q^jb/e \\ & q^jb, & q^jbd/ce, & q^{1-n+j}a/d \end{matrix}\middle| q;q\right]
$$

which yields the following transformation formula, whose limiting case $e \to 0$ has been obtained by Jia–Wang [11, Theorem 4.1].

Theorem 4 (Bivariate terminating series transformation).

$$
\left[\begin{matrix} bd/ace, d \\ bd/ce, d/a \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a, & c, & e \\ q, & d, & q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)
$$
$$
= \sum_{i,j\geq 0} q^ia^j \left[\begin{matrix} q^{-n}, b/c, b/e \\ b, q^{1-n}a/d, bd/ce \end{matrix}\middle| q\right]_{i+j} \frac{(a;q)_i}{(q;q)_i} \left[\begin{matrix} q^{1-n}/d, bd/ace \\ b/c, b/e \end{matrix}\middle| q\right]_j \Omega(j).
$$

In this theorem, specifying the $\Omega(j)$-sequence by (8) and then applying (7), we find after some trivial simplification the following reduction formula.

Proposition 5 (Balanced reduction formula).

$$
\sum_{i,j\geq 0} q^{i+j} \left[\begin{matrix} q^{-n}, b/c, b/e \\ b, q^{1-n}a/d, bd/ce \end{matrix}\middle| q\right]_{i+j} \frac{(a;q)_i}{(q;q)_i} \left[\begin{matrix} \beta, \gamma, q^{1-n}/d, bd/ace \\ q, b/c, b/e, q^{1-n}\beta\gamma/b \end{matrix}\middle| q\right]_j a^j
$$
$$
= \left[\begin{matrix} d, b/\beta, b/\gamma, bd/ace \\ b, b/\beta\gamma, d/a, bd/ce \end{matrix}\middle| q\right]_n {}_5\phi_4\left[\begin{matrix} q^{-n}, & a, & c, & e, & b/\beta\gamma \\ & d, & b/\beta, & b/\gamma, & q^{1-n}ace/bd \end{matrix}\middle| q;q\right].
$$

When $\beta = a/d$ and $\gamma = b/a$, the last ${}_5\phi_4$-series reduces to a ${}_3\phi_2$-series, which has a closed form in view of (4). This leads to the following summation formula.

Corollary 6 (Summation formula of double series).

$$
\left[\begin{matrix} a, bd/ac, bd/ae \\ b, d/a, bd/ce \end{matrix}\middle| q\right]_n = \sum_{i,j\geq 0} \left[\begin{matrix} q^{-n}, b/c, b/e \\ b, q^{1-n}a/d, bd/ce \end{matrix}\middle| q\right]_{i+j} q^{i+j}
$$
$$
\times \frac{(a;q)_i}{(q;q)_i} \left[\begin{matrix} a/d, b/a, bd/ace \\ q, b/c, b/e \end{matrix}\middle| q\right]_j a^j.
$$

This identity extends, with an extra free parameter, the summation formula due to Jia–Zhang [13, Corollary 2.9], which corresponds to the case $e \to 0$ of Corollary 6.

§ 2.3. Transformation into $\Phi^{3:1;1+\lambda}_{3:0;1+\mu}$. By means of (3), we can reformulate the ${}_4\phi_3$-series in (6) as

$$
{}_4\phi_3\left[\begin{matrix} q^{j-n}, & a, & c, & e \\ & q^j b, & d, & q^{1-n}ace/bd \end{matrix}\middle| q;q\right]
= \left[\begin{matrix} a, q^j bd/ac, q^j bd/ae \\ d, q^j b, q^j bd/ace \end{matrix}\middle| q\right]_{n-j} \times {}_4\phi_3\left[\begin{matrix} q^{j-n}, d/a, q^j b/a, q^j bd/ace \\ q^j bd/ac, q^j bd/ae, q^{1-n+j}/a \end{matrix}\middle| q;q\right].
$$

This leads to the equality

$$
\mathrm{Eq}(5) = \sum_{j\ge 0} \frac{(q^{-n};q)_j}{(b;q)_j}\Omega(j)\left[\begin{matrix} a, q^j bd/ac, q^j bd/ae \\ d, q^j b, q^j bd/ace \end{matrix}\middle| q\right]_{n-j}
\times {}_4\phi_3\left[\begin{matrix} q^{j-n}, d/a, q^j b/a, q^j bd/ace \\ q^j bd/ac, q^j bd/ae, q^{1-n+j}/a \end{matrix}\middle| q;q\right].
$$

Writing the last double sum explicitly, we find the following transformation theorem, whose limiting case $e \to 0$ recovers that due to Jia–Wang [11, Theorem 4.2].

Theorem 7 (Bivariate terminating series transformation).

$$
\left[\begin{matrix} b, & d, & bd/ace \\ a, & bd/ac, & bd/ae \end{matrix}\middle| q\right]_n \sum_{i,j\ge 0} q^i \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a, & c, & e \\ q, & d, & q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)
$$
$$
= \sum_{i,j\ge 0} q^i \left[\begin{matrix} q^{-n}, b/a, bd/ace \\ q^{1-n}/a, bd/ac, bd/ae \end{matrix}\middle| q\right]_{i+j} \frac{(d/a;q)_i}{(q;q)_i} \frac{(q^{1-n}/d;q)_j}{(b/a;q)_j} \left(\frac{d}{a}\right)^j \Omega(j).
$$

Recalling (7–8), we can derive from this theorem the reduction formula below.

Proposition 8 (Balanced reduction formula).

$$
\sum_{i,j\ge 0} q^{i+j} \left[\begin{matrix} q^{-n}, b/a, bd/ace \\ q^{1-n}/a, bd/ac, bd/ae \end{matrix}\middle| q\right]_{i+j} \frac{(d/a;q)_i}{(q;q)_i} \left[\begin{matrix} q^{1-n}/d, \beta, \gamma \\ q, b/a, q^{1-n}\beta\gamma/b \end{matrix}\middle| q\right]_j \left(\frac{d}{a}\right)^j
$$
$$
= \left[\begin{matrix} b/\beta, b/\gamma, d, bd/ace \\ a, b/\beta\gamma, bd/ac, bd/ae \end{matrix}\middle| q\right]_n {}_5\phi_4\left[\begin{matrix} q^{-n}, a, c, e, b/\beta\gamma \\ b/\beta, b/\gamma, d, q^{1-n}ace/bd \end{matrix}\middle| q;q\right].
$$

When $\beta = b/c$ and $\gamma = b/e$, the last ${}_5\phi_4$-series reduces to a ${}_3\phi_2$-series, which has a closed form in view of (4). This leads to the following summation formula.

Corollary 9 (Summation formula of double series).

$$\left[\begin{matrix} c, e, d/a, bd/ce \\ a, bd/ac, bd/ae, ce/b \end{matrix}\middle| q\right]_n = \sum_{i,j\geq 0}\left[\begin{matrix} q^{-n}, b/a, bd/ace \\ q^{1-n}/a, bd/ac, bd/ae \end{matrix}\middle| q\right]_{i+j} q^{i+j} \times \frac{(d/a;q)_i}{(q;q)_i}\left[\begin{matrix} b/c, b/e, q^{1-n}/d \\ q, b/a, q^{1-n}b/ce \end{matrix}\middle| q\right]_j \left(\frac{d}{a}\right)^j.$$

§ 2.4. Transformation into $\Phi^{1:3;2+\lambda}_{1:2;2+\mu}$. Alternatively, the ${}_4\phi_3$-series in (6) can be transformed by (3) into

$$ {}_4\phi_3\left[\begin{matrix} q^{j-n}, & a, & c, & e \\ & q^j b, & d, & q^{1-n}ace/bd \end{matrix}\middle| q;q\right] = \left[\begin{matrix} a, q^{1-n}c/b, q^{1-n}e/b \\ d, q^{1-n}/b, q^{1-n}ace/bd \end{matrix}\middle| q\right]_{n-j} \times {}_4\phi_3\left[\begin{matrix} q^{j-n}, d/a, q^{1-n}ce/bd, q^{1-n}/b \\ q^{1-n}c/b, q^{1-n}e/b, q^{1-n+j}/a \end{matrix}\middle| q;q\right]$$

which gives rise to the following transformation formula.

Theorem 10 (Bivariate terminating series transformation).

$$\left[\begin{matrix} b, d, bd/ace \\ a, b/c, b/e \end{matrix}\middle| q\right]_n \left(\frac{a}{d}\right)^n \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}}\left[\begin{matrix} a, & c, & e \\ q, & d, & q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)$$
$$= \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(q^{1-n}/a;q)_{i+j}}\left[\begin{matrix} d/a, q^{1-n}ce/bd, q^{1-n}/b \\ q, q^{1-n}c/b, q^{1-n}e/b \end{matrix}\middle| q\right]_i \left[\begin{matrix} q^{1-n}/d, bd/ace \\ b/c, b/e \end{matrix}\middle| q\right]_j \Omega(j).$$

This theorem contains a known transformation due to Chu–Jia [6, Theorem 4.3] as a special case, which can be seen by first replacing e by be and then letting $b \to \infty$.

Rewriting the second double sum in Theorem 10 as

$$\sum_{i=0}^{n} q^i \left[\begin{matrix} q^{-n}, d/a, q^{1-n}ce/bd, q^{1-n}/b \\ q, q^{1-n}/a, q^{1-n}c/b, q^{1-n}e/b \end{matrix}\middle| q\right]_i \sum_{j=0}^{n-i}\left[\begin{matrix} q^{i-n}, q^{1-n}/d, bd/ace \\ q^{1+i-n}/a, b/c, b/e \end{matrix}\middle| q\right]_j \Omega(j)$$

and then evaluating the inner sum with respect to j via (4) for the specific $\Omega(j)$-sequence

$$\Omega(j) = \left[\begin{matrix} \beta, & \gamma, & b/c, & b/e \\ q, & a\beta\gamma, & q^{1-n}/d, & bd/ace \end{matrix}\middle| q\right]_j q^j$$

we find after some trivial simplification the following reduction formula.

Proposition 11 (Balanced reduction formula).

$$\sum_{i,j\geq 0} q^{i+j}\frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}}\left[\begin{matrix} a,\ c, & e \\ q,\ d,\ q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \left[\begin{matrix} \beta,\ \ \gamma, & b/c, & b/e \\ q,\ a\beta\gamma,\ q^{1-n}/d,\ bd/ace \end{matrix}\middle| q\right]_j$$

$$= \left[\begin{matrix} d/a, bd/ce \\ d, bd/ace \end{matrix}\middle| q\right]_n {}_5\phi_4\left[\begin{matrix} q^{-n}, b/c, & b/e, & a\beta, & a\gamma \\ & b, & bd/ce, & a\beta\gamma, & q^{1-n}a/d \end{matrix}\middle| q; q\right].$$

This proposition contains the two transformations due to Chu–Jia [6, Propositions 4.9 and 4.13] as the limiting case $b \to \infty$ after having made the replacement e by be, where the ${}_5\phi_4$-series should be inverted for the former.

When $\beta = b/a$ and $\gamma = q^{1-n}/d$, the last ${}_5\phi_4$-series reduces to a ${}_3\phi_2$-series, which has a closed form in view of (4). This leads to the following summation formula.

Corollary 12 (Summation formula of double series).

$$\left[\begin{matrix} d/a, d/c, d/e \\ d, d/b, bd/ace \end{matrix}\middle| q\right]_n$$

$$= \sum_{i,j\geq 0}\frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}}\left[\begin{matrix} a, c, e \\ q, d, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \left[\begin{matrix} b/a, b/c, b/e \\ q, bd/ace, q^{1-n}b/d \end{matrix}\middle| q\right]_j q^{i+j}.$$

§ 2.5. Transformation into $\Phi_{3:2;1+\mu}^{3:3;1+\lambda}$ weighted with linear factor. In view of the Watson transformation (1), we can reformulate the ${}_4\phi_3$-series in (6) as the well–poised series

$${}_4\phi_3\left[\begin{matrix} q^{j-n}, & a, & c, & e \\ & q^jb, & d, & q^{1-n}ace/bd \end{matrix}\middle| q; q\right]$$

$$= \left[\begin{matrix} q^jbd/ae, q^jbd/ce \\ q^jbd/e, q^jbd/ace \end{matrix}\middle| q\right]_{n-j} \times W(q^{j-1}bd/e : a, c, d/e, q^jb/e, q^{j-n}; q^nbd/ac).$$

This leads to the expression

$$\mathrm{Eq}(5) = \sum_{j\geq 0}\Omega(j)\frac{(q^{-n};q)_j}{(b;q)_j}\left[\begin{matrix} q^jbd/ce, d/a \\ q^jbd/ace, d \end{matrix}\middle| q\right]_{n-j}$$

$$\times W(q^{j-1}bd/e : a, c, d/e, q^jb/e, q^{j-n}; q^nbd/ac).$$

Writing the last double sum explicitly, we get the following transformation.

Theorem 13 (Bivariate terminating series transformation).

$$\begin{bmatrix} bd/e, bd/ace \\ bd/ae, bd/ce \end{bmatrix} q \Bigg]_n \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a, & c, & e \\ q, & d, & q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)$$

$$= \sum_{i,j\geq 0} \frac{1-q^{2i+j-1}bd/e}{1-bd/qe} \left[\begin{matrix} a, c, d/e \\ q, d, q^n bd/e \end{matrix}\middle| q\right]_i \left(\frac{q^n bd}{ac}\right)^i$$

$$\times \left[\begin{matrix} q^{-n}, b/e, bd/qe \\ b, bd/ae, bd/ce \end{matrix}\middle| q\right]_{i+j} \frac{(bd/ace;q)_j}{(b/e;q)_j} \Omega(j).$$

Specifying the $\Omega(j)$-sequence in this theorem by (8) and then appealing to (7), we get the following formula.

Proposition 14 (Balanced reduction formula).

$$\sum_{i,j\geq 0} \frac{1-q^{2i+j-1}bd/e}{1-bd/qe} \left[\begin{matrix} q^{-n}, b/e, bd/qe \\ b, bd/ae, bd/ce \end{matrix}\middle| q\right]_{i+j} \left(\frac{q^n bd}{ac}\right)^i$$

$$\times \left[\begin{matrix} a, c, d/e \\ q, d, q^n bd/e \end{matrix}\middle| q\right]_i \left[\begin{matrix} bd/ace, \beta, \gamma \\ q, b/e, q^{1-n}\beta\gamma/b \end{matrix}\middle| q\right]_j q^j$$

$$= \left[\begin{matrix} b/\beta, b/\gamma, bd/e, bd/ace \\ b, b/\beta\gamma, bd/ae, bd/ce \end{matrix}\middle| q\right]_n {}_5\phi_4\left[\begin{matrix} q^{-n}, a, c, e, b/\beta\gamma \\ d, b/\beta, b/\gamma, q^{1-n}ace/bd \end{matrix}\middle| q; q\right].$$

When $\beta = a/d$ and $\gamma = b/a$, the last ${}_5\phi_4$-series reduces to a ${}_3\phi_2$-series. Evaluating this last ${}_3\phi_2$-series by (4) yields the following summation formula.

Corollary 15 (Summation formula of double series).

$$\left[\begin{matrix} a, bd/ac, bd/e \\ b, d, bd/ce \end{matrix}\middle| q\right]_n = \sum_{i,j\geq 0} \frac{1-q^{2i+j-1}bd/e}{1-bd/qe} \left[\begin{matrix} q^{-n}, b/e, bd/qe \\ b, bd/ae, bd/ce \end{matrix}\middle| q\right]_{i+j} \left(\frac{q^n bd}{ac}\right)^i$$

$$\times \left[\begin{matrix} a, c, d/e \\ q, d, q^n bd/e \end{matrix}\middle| q\right]_i \left[\begin{matrix} a/d, b/a, bd/ace \\ q, b/e, q^{1-n}/d \end{matrix}\middle| q\right]_j q^j.$$

§ 2.6. Transformation into double series with strange factorial quotient. Alternatively, expressing the same ${}_4\phi_3$-series via (1) as

$${}_4\phi_3\left[\begin{matrix} q^{j-n}, a, c, e \\ q^j b, d, q^{1-n}ace/bd \end{matrix}\middle| q; q\right]$$

$$= \left[\begin{matrix} q^j b/a, q^j b/c \\ q^j b/ac, q^j b \end{matrix}\middle| q\right]_{n-j} \times W(q^{-n}ac/b : a, c, d/e, q^{1-n}ac/bd, q^{j-n}; q^{1-j}e/b)$$

would lead to another transformation formula.

Theorem 16 (Bivariate terminating series transformation).

$$\left[\begin{matrix} b, b/ac \\ b/a, b/c \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}} \left[\begin{matrix} a,\ c, & e \\ q,\ d,\ q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)$$

$$= \sum_{i,j\geq 0} \frac{1-q^{2i-n}ac/b}{1-q^{-n}ac/b} \frac{(q^{-n};q)_{i+j}}{(qac/b;q)_{i-j}} \frac{(-b/ac)^j}{[b/a,b/c;q]_j} q^{\binom{j}{2}}$$

$$\times \left(\frac{q^{1-j}e}{b}\right)^i \left[\begin{matrix} q^{-n}ac/b, a, c, d/e, q^{1-n}ac/bd \\ q, d, q^{1-n}ace/bd, q^{1-n}a/b, q^{1-n}c/b \end{matrix}\middle| q\right]_i \Omega(j).$$

By invoking (7–8), we find from this theorem the following reduction formula.

Proposition 17 (Balanced reduction formula).

$$\sum_{i,j\geq 0} \frac{1-q^{2i-n}ac/b}{1-q^{-n}ac/b} \left[\begin{matrix} q^{-n}ac/b, a, c, d/e, q^{1-n}ac/bd \\ q, d, q^{1-n}ace/bd, q^{1-n}a/b, q^{1-n}c/b \end{matrix}\middle| q\right]_i$$

$$\times \left(\frac{q^{1-j}e}{b}\right)^i \frac{(q^{-n};q)_{i+j}}{(qac/b;q)_{i-j}} \frac{[\beta,\gamma;q]_j(-qb/ac)^j}{[q,b/a,b/c,q^{1-n}\beta\gamma/b;q]_j} q^{\binom{j}{2}}$$

$$= \left[\begin{matrix} b/\beta, b/\gamma, b/ac \\ b/a, b/c, b/\beta\gamma \end{matrix}\middle| q\right]_n {}_5\phi_4 \left[\begin{matrix} q^{-n}, a, & c, & e, & b/\beta\gamma \\ & d, b/\beta, b/\gamma, q^{1-n}ace/bd \end{matrix}\middle| q;q\right].$$

Similarly for $\beta = b/e$ and $\gamma = e/d$, the ${}_5\phi_4$-series in Proposition 17 reduces again to a ${}_3\phi_2$-series, which has a closed form in view of (4). This leads to another summation formula.

Corollary 18 (Summation formula of double series).

$$\left[\begin{matrix} e, b/ac, bd/ae, bd/ce \\ b/a, b/c, d, bd/ace \end{matrix}\middle| q\right]_n$$

$$= \sum_{i,j\geq 0} \frac{1-q^{2i-n}ac/b}{1-q^{-n}ac/b} \frac{(q^{-n};q)_{i+j}}{(qac/b;q)_{i-j}} \left(\frac{q^{1-j}e}{b}\right)^i$$

$$\times \left[\begin{matrix} q^{-n}ac/b, a, c, d/e, q^{1-n}ac/bd \\ q, d, q^{1-n}ace/bd, q^{1-n}a/b, q^{1-n}c/b \end{matrix}\middle| q\right]_i \frac{[b/e, e/d;q]_j(-qb/ac)^j}{[q,b/a,b/c,q^{1-n}/d;q]_j} q^{\binom{j}{2}}.$$

3. Bivariate terminating series $\Phi_{2:1;\mu}^{2:2;\lambda}$

In this section, we shall investigate another bivariate terminating series

$$\sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j) \tag{9}$$

$$= \sum_{j\geq 0} \Omega(j) \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_j {}_4\phi_3\left[\begin{matrix} q^{j-n}, q^j a, c, e \\ q^j b, q^j d, q^{1-n}ace/bd \end{matrix}\middle| q; q\right]. \tag{10}$$

By reformulating the ${}_4\phi_3$-series just displayed with (1), (2) and (3), six transformation theorems will be established. Then as consequences, we shall derive several reduction formulae by simplifying the double sum displayed in (9) to a single one in the following manner:

$$\mathrm{Eq}(9) \Rightarrow a^n \left[\begin{matrix} b/a, d/a \\ b, d \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ q^{1-n}a/b, & q^{1-n}a/d, & q^{1-n}ace/bd \end{matrix}\middle| q; \frac{q^{2-n}a}{bd}\right] \tag{11}$$

$$\text{when } \Omega(j) = \frac{(q^{n-1}bd/a;q)_j}{(q;q)_j} q^j. \tag{12}$$

The expression in (11) is obtained by applying the q-Saalschütz theorem to the sum with respect to j displayed in (9) under the specification (12).

§3.1. Self–reciprocal transformation for $\Phi^{2:2;\lambda}_{2:1;\mu}$. According to (3), we can reformulate the ${}_4\phi_3$-series in (10) as

$${}_4\phi_3\left[\begin{matrix} q^{j-n}, q^j a, c, e \\ q^j b, q^j d, q^{1-n}ace/bd \end{matrix}\middle| q; q\right]$$
$$= \left[\begin{matrix} q^j a, q^j bd/ac, q^j bd/ae \\ q^j b, q^j d, q^j bd/ace, d \end{matrix}\middle| q\right]_{n-j} \times {}_4\phi_3\left[\begin{matrix} q^{j-n}, b/a, d/a, q^j bd/ace \\ q^j bd/ac, q^j bd/ae, q^{1-n}/a \end{matrix}\middle| q; q\right]$$

which leads to the equality

$$\mathrm{Eq}(9) = \sum_{j\geq 0} \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_j \Omega(j) \left[\begin{matrix} q^j a, q^j bd/ac, q^j bd/ae \\ q^j b, q^j d, q^j bd/ace, d \end{matrix}\middle| q\right]_{n-j}$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{j-n}, b/a, d/a, q^j bd/ace \\ q^j bd/ac, q^j bd/ae, q^{1-n}/a \end{matrix}\middle| q; q\right].$$

Writing the last double sum explicitly, we get the following self–reciprocal transformation in the sense that if it is applied to the double sum on the right hand side, the resulting double sum turns back to that on the left hand side.

THEOREM 19 (Bivariate terminating series transformation).

$$\left[\begin{matrix} b, d, bd/ace \\ a, bd/ac, bd/ae \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)$$

$$= \sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, bd/ace \\ bd/ae, bd/ac \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} b/a, d/a \\ q, q^{1-n}/a \end{matrix}\middle| q\right]_i \Omega(j).$$

In this theorem, specifying the $\Omega(j)$-sequence by (12) and then appealing to (11), we find the following reduction formula.

Proposition 20 (Reduction formula).

$$\sum_{i,j\geq 0} q^{i+j} \left[\begin{matrix} q^{-n}, bd/ace \\ bd/ac, bd/ae \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} b/a, d/a \\ q, q^{1-n}/a \end{matrix}\middle| q\right]_i \frac{(q^{n-1}bd;q)_j}{(q;q)_j}$$

$$= a^n \left[\begin{matrix} b/a, d/a, bd/ace \\ a, bd/ac, bd/ae \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ q^{1-n}a/b, & q^{1-n}a/d, & q^{1-n}ace/bd \end{matrix}\middle| q; \frac{q^{2-n}a}{bd}\right].$$

§3.2. Transformation into $\Phi^{2:2;2+\lambda}_{2:1;2+\mu}$. By means of the Sears transformation (2), we can reformulate the ${}_4\phi_3$-series in (10) as the well–poised series

$${}_4\phi_3\left[\begin{matrix} q^{j-n}, & c, & q^j a, & e \\ & q^j b, & q^{1-n}ace/bd, & q^j d \end{matrix}\middle| q; q\right]$$

$$= \left[\begin{matrix} q^{1-n}c/d, q^{1-n}ae/bd \\ q^{1-n}ace/bd, q^{1-n}/d \end{matrix}\middle| q\right]_{n-j} \times {}_4\phi_3\left[\begin{matrix} q^{j-n}, & b/a, & c, & q^j b/e \\ & q^j b, & q^{1-n}c/d, & q^j bd/ae \end{matrix}\middle| q; q\right].$$

Substituting this expression into (9–10) and then simplifying the resulting double sum, we get the following transformation formula.

Theorem 21 (Bivariate terminating series transformation).

$$\left[\begin{matrix} bd/ace, d \\ bd/ae, d/c \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)$$

$$= \sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, b/e \\ b, bd/ae \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} b/a, c \\ q, q^{1-n}c/d \end{matrix}\middle| q\right]_i \left[\begin{matrix} a, bd/ace \\ b/e, d/c \end{matrix}\middle| q\right]_j \Omega(j).$$

By applying (11–12) to this theorem, we find the reduction formula below.

Proposition 22 (Reduction formula).

$$\sum_{i,j\geq 0} \left[\begin{matrix} q^{-n}, b/e \\ b, bd/ae \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} b/a, c \\ q, q^{1-n}c/d \end{matrix}\middle| q\right]_i \left[\begin{matrix} a, bd/ace, q^{n-1}bd/a \\ q, b/e, d/c \end{matrix}\middle| q\right]_j q^{i+j}$$

$$= a^n \left[\begin{matrix} b/a, d/a, bd/ace \\ b, d/c, bd/ae \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ q^{1-n}a/b, & q^{1-n}a/d, & q^{1-n}ace/bd \end{matrix}\middle| q; \frac{q^{2-n}a}{bd}\right].$$

§ 3.3. Transformation into double series with strange factorial denominator. In view of (2), the ${}_4\phi_3$-series displayed in (10) can be reformulated as

$$
{}_4\phi_3\left[\begin{matrix} q^{j-n},\, q^j a,\quad c,\qquad e \\ q^j b,\, q^j d,\, q^{1-n}ace/bd \end{matrix}\middle| q; q\right]
$$

$$
= \left[\begin{matrix} q^{2j}bd/ce, d/a \\ q^j bd/ace, q^j d \end{matrix}\middle| q\right]_{n-j} \times {}_4\phi_3\left[\begin{matrix} q^{j-n},\, q^j a,\quad q^j b/c,\qquad q^j b/e \\ q^j b,\, q^{2j}bd/ce,\, q^{1-n+j}a/d \end{matrix}\middle| q; q\right].
$$

Therefore we have the equality

$$
\mathrm{Eq}(9) = \sum_{j\ge 0} \Omega(j) \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_j \left[\begin{matrix} q^{2j}bd/ce, d/a \\ q^j bd/ace, q^j d \end{matrix}\middle| q\right]_{n-j}
$$

$$
\times {}_4\phi_3\left[\begin{matrix} q^{j-n},\, q^j a,\quad q^j b/c,\qquad q^j b/e \\ q^j b,\, q^{2j}bd/ce,\, q^{1-n+j}a/d \end{matrix}\middle| q; q\right].
$$

Writing the last double sum explicitly and taking into account

$$
\frac{(d/a;q)_{n-j}}{(d;q)_{n-j}} = a^j \frac{(d/a;q)_n}{(d;q)_n} \frac{(q^{1-n}/d;q)_j}{(q^{1-n}a/d;q)_j},
$$

we get the following transformation theorem.

Theorem 23 (Bivariate terminating series transformation).

$$
\left[\begin{matrix} bd/ace, d \\ bd/ce, d/a \end{matrix}\middle| q\right]_n \sum_{i,j\ge 0} q^i \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j)
$$

$$
= \sum_{i,j\ge 0} \left[\begin{matrix} q^{-n}, a, b/c, b/e \\ b, q^{1-n}a/d \end{matrix}\middle| q\right]_{i+j} \frac{(-qa/d)^j q^{i-nj+\binom{j}{2}}}{(q;q)_i (bd/ce;q)_{i+2j}} \left[\begin{matrix} q^n bd/ce, bd/ace \\ b/c, b/e \end{matrix}\middle| q\right]_j \Omega(j).
$$

Recalling (11–12), we derive from this theorem the following reduction formula.

Proposition 24 (Reduction formula).

$$
\sum_{i,j\ge 0} \left[\begin{matrix} q^{-n}, a, b/c, b/e \\ b, q^{1-n}a/d \end{matrix}\middle| q\right]_{i+j} \frac{q^{i+\binom{j}{2}}(-q^{2-n}a/d)^j}{(q;q)_i (bd/ce;q)_{i+2j}} \left[\begin{matrix} q^n bd/ce, bd/ace, q^{n-1}bd/a \\ q,\quad b/c,\quad b/e \end{matrix}\middle| q\right]_j
$$

$$
= a^n \left[\begin{matrix} bd/ace, b/a \\ bd/ce, b \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n},\quad a,\quad c,\quad e \\ q^{1-n}a/b, q^{1-n}a/d, q^{1-n}ace/bd \end{matrix}\middle| q; \frac{q^{2-n}a}{bd}\right].
$$

§3.4. Transformation into double series with strange factorial quotient. Taking account of the Watson transformation (1), we can reformulate the ${}_4\phi_3$-series in (10) as the well–poised series

$$\begin{aligned}{}_4\phi_3&\left[\begin{matrix}q^{j-n}, q^j a, \quad c, \qquad e\\ q^j b,\, q^j d,\, q^{1-n}ace/bd\end{matrix}\middle| q; q\right]\\ &=\left[\begin{matrix}q^j bd/ae, q^{2j}bd/ce\\ q^{2j}bd/e, q^j bd/ace\end{matrix}\middle| q\right]_{n-j}\\ &\quad\times W(q^{2j-1}bd/e : q^j a, c, q^j d/e, q^j b/e, q^{j-n}; q^n bd/ac).\end{aligned}$$

This leads to the expression

$$\begin{aligned}\mathrm{Eq}(9) = \sum_{j\ge 0}\Omega(j)&\left[\begin{matrix}q^{-n}, a\\ b, d\end{matrix}\middle| q\right]_{i+j}\left[\begin{matrix}q^j bd/ae, q^{2j}bd/ce\\ q^{2j}bd/e, q^j bd/ace\end{matrix}\middle| q\right]_{n-j}\\ &\times W(q^{2j-1}bd/e : q^j a, c, q^j d/e, q^j b/e, q^{j-n}; q^n bd/ac).\end{aligned}$$

Writing the last double sum explicitly, we get the following transformation.

Theorem 25 (Bivariate terminating series transformation).

$$\begin{aligned}&\left[\begin{matrix}bd/e, bd/ace\\ bd/ae, bd/ce\end{matrix}\middle| q\right]_n \sum_{i,j\ge 0} q^i \left[\begin{matrix}q^{-n}, a\\ b, d\end{matrix}\middle| q\right]_{i+j}\left[\begin{matrix}c, e\\ q, q^{1-n}ace/bd\end{matrix}\middle| q\right]_i \Omega(j)\\ &= \sum_{i,j\ge 0}\frac{1-q^{2i+2j-1}bd/e}{1-bd/qe}\left[\begin{matrix}q^{-n}, a, d/e, b/e\\ b, d, bd/ae, q^n bd/e\end{matrix}\middle| q\right]_{i+j}\Omega(j)\\ &\quad\times\frac{(bd/qe;q)_{i+2j}}{(bd/ce;q)_{i+2j}}\frac{(c;q)_i}{(q;q)_i}\left(\frac{q^n bd}{ac}\right)^i\left[\begin{matrix}bd/ace, q^n bd/ce\\ b/e, d/e\end{matrix}\middle| q\right]_j.\end{aligned}$$

Letting $a \to ad$ and then $d \to 0$ in this theorem, we obtain a formula equivalent to Jia–Wang [11, Theorem 4.1]. Analogously, for $a \to ad$ and $d \to \infty$, we derive another slightly different transformation which is similar to Jia–Wang [11, Theorem 4.1].

Proposition 26 (Terminating double series transformation).

$$\begin{aligned}&c^n\frac{(b/ace;q)_n}{(b/ae;q)_n}\sum_{i,j\ge 0}(qa)^i\frac{(q^{-n};q)_{i+j}}{(b;q)_{i+j}}\left[\begin{matrix}c, e\\ q, q^{1-n}ace/b\end{matrix}\middle| q\right]_i\Omega(j)\\ &= \sum_{i,j\ge 0} q^i\left[\begin{matrix}q^{-n}, b/e\\ b, b/ae\end{matrix}\middle| q\right]_{i+j}\frac{(c;q)_i}{(q;q)_i}\frac{(b/ace;q)_j}{(b/e;q)_j}c^j\Omega(j).\end{aligned}$$

By utilizing (11–12), we get, from Theorem 25, the following reduction formula.

Proposition 27 (Reduction formula).

$$\begin{aligned}&\left[\begin{matrix}b,d\\ b/a,d/a\end{matrix}\middle|q\right]_n\sum_{i,j\geq 0}\frac{1-q^{2i+2j-1}bd/e}{1-bd/qe}\left[\begin{matrix}q^{-n},a,b/e,d/e\\ b,d,bd/ae,q^nbd/e\end{matrix}\middle|q\right]_{i+j}\\ &\qquad\times\frac{(c;q)_i}{(q;q)_i}\left(\frac{q^nbd}{ac}\right)^i\left[\begin{matrix}bd/ace,q^nbd/ce,q^{n-1}ba/a\\ q,\quad b/e,\quad d/e\end{matrix}\middle|q\right]_j\frac{(bd/qe;q)_{i+2j}}{(bd/ce;q)_{i+2j}}q^j\\ &\qquad=a^n\left[\begin{matrix}bd/e,bd/ace\\ bd/ae,bd/ce\end{matrix}\middle|q\right]_n{}_4\phi_3\left[\begin{matrix}q^{-n},\quad a,\quad c,\quad e\\ q^{1-n}a/b,q^{1-n}a/d,q^{1-n}ace/bd\end{matrix}\middle|q;\frac{q^{2-n}a}{bd}\right].\end{aligned}$$

In addition, specifying the $\Omega(j)$-sequence in Theorem 25 by

$$\Omega(j)=\frac{(q^{1-n}a/bd;q)_j}{(q;q)_j}(q^nbd/a)^j$$

and then simplifying, with the help of (4), the first double sum by letting $i+j=k$, we obtain another similar reduction formula.

Proposition 28 (Reduction formula).

$$\begin{aligned}&\sum_{i,j\geq 0}\left(\frac{q^nbd}{ac}\right)^i\frac{1-q^{2i+2j-1}bd/e}{1-bd/qe}\left[\begin{matrix}q^{-n},a,b/e,d/e\\ b,d,bd/ae,q^nbd/e\end{matrix}\middle|q\right]_{i+j}\\ &\qquad\times\frac{(c;q)_i}{(q;q)_i}\left[\begin{matrix}bd/ace,q^nbd/ce,q^{1-n}a/bd\\ q,\quad b/e,\quad d/e\end{matrix}\middle|q\right]_j\frac{(bd/qe;q)_{i+2j}}{(bd/ce;q)_{i+2j}}(q^nbd/a)^j\\ &\qquad=\left[\begin{matrix}bd/e,bd/ace\\ bd/ae,bd/ce\end{matrix}\middle|q\right]_n{}_4\phi_3\left[\begin{matrix}q^{-n},\ a,\ q^{1-n}ac/bd,\ q^{1-n}ae/bd\\ b,\quad d,\quad q^{1-n}ace/bd\end{matrix}\middle|q;q^n\frac{bd}{a}\right].\end{aligned}$$

§3.5. Transformation into $\Phi^{4:2;\lambda}_{2:3;2+\mu}$ weighted with linear factor. Expressing the same ${}_4\phi_3$-series in (10) further as

$$\begin{aligned}&{}_4\phi_3\left[\begin{matrix}q^{j-n},\ q^ja,\ c,\ e\\ q^jb,q^jd,q^{1-n}ace/bd\end{matrix}\middle|q;q\right]\\ &\qquad=\left[\begin{matrix}q^{1-n}c/d,q^{1-n+j}a/d\\ q^{1-n+j}ac/d,q^{1-n}/d\end{matrix}\middle|q\right]_{n-j}\\ &\qquad\qquad\times W(q^{j-n}ac/d:q^ja,c,q^jb/e,q^{1-n}ac/bd,q^{j-n};q^{1-j}e/b)\end{aligned}$$

would lead to another transformation formula.

Theorem 29 (Bivariate terminating series transformation).

$$\begin{aligned}
&\left[\begin{matrix} d, d/ac \\ d/a, d/c \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j) \\
&= \sum_{i,j\geq 0} \frac{1-q^{2i+j-n}ac/d}{1-q^{-n}ac/d} \left[\begin{matrix} q^{-n}, a, q^{-n}ac/d, b/e \\ b, q^{1-n}a/d \end{matrix}\middle| q\right]_{i+j} \\
&\times \left[\begin{matrix} c, \quad q^{1-n}ac/bd \\ q, qac/d, q^{1-n}ace/bd, q^{1-n}c/d \end{matrix}\middle| q\right]_i \frac{(qe/d)^i (q^i c)^{-j}}{(b/e, d/c; q)_j} \Omega(j).
\end{aligned}$$

Letting $a \to ab$ and then $b \to 0$ in this theorem, we get a formula equivalent to Chu–Jia [6, Theorem 4.1]. Analogously, for $a \to ab$ and $b \to \infty$, we derive another slightly different transformation which is similar to Chu–Jia [6, Theorem 4.1].

Proposition 30 (Terminating double series transformation).

$$\begin{aligned}
&\frac{(d/c;q)_n}{(d;q)_n} \sum_{i,j\geq 0} q^i \frac{(q^{-n};q)_{i+j}[c, q^{1-n}ac/d;q]_i}{[q, q^{1-n}ace/d, q^{1-n}c/d;q]_i} \frac{\Omega(j)}{(d/c;q)_j} \\
&= \sum_{i,j\geq 0} (qa)^i \frac{(q^{-n};q)_{i+j}}{(d;q)_{i+j}} \left[\begin{matrix} c, e \\ q, q^{1-n}ace/d \end{matrix}\middle| q\right]_i \Omega(j).
\end{aligned}$$

Taking the equality (11–12) into account, we may also derive from Theorem 29 the following reduction formula.

Proposition 31 (Reduction formula).

$$\begin{aligned}
&\sum_{i,j\geq 0} \left(\frac{q^{1-j}e}{d}\right)^i \frac{1-q^{2i+j-n}ac/d}{1-q^{-n}ac/d} \left[\begin{matrix} q^{-n}, a, q^{-n}ac/d, b/e \\ b, \quad q^{1-n}a/d \end{matrix}\middle| q\right]_{i+j} \\
&\times \left[\begin{matrix} c, \quad q^{1-n}ac/bd \\ q, qac/d, q^{1-n}ace/bd, q^{1-n}c/d \end{matrix}\middle| q\right]_i \left[\begin{matrix} q^{n-1}bd/a \\ q, b/e, d/c \end{matrix}\middle| q\right]_j (q/c)^j \\
&= a^n \left[\begin{matrix} b/a, d/ac \\ b, d/c \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, \quad a, \quad c, \quad e \\ q^{1-n}a/b, q^{1-n}a/d, q^{1-n}ace/bd \end{matrix}\middle| q; \frac{q^{2-n}a}{bd}\right].
\end{aligned}$$

§3.6. Transformation into $\Phi^{2:4;2+\lambda}_{4:1;\mu}$ weighted with linear factor. By appealing to the Watson transformation (1), we can reformulate the ${}_4\phi_3$-series in (10) as the well–poised series

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{j-n},\ c,\ e,\ q^j a \\ q^j b, q^j d, q^{1-n}ace/bd \end{matrix}\middle| q;q\right] \\
&\quad = \left[\begin{matrix} q^j bd/ac, q^j bd/ae \\ q^j bd/a, q^j bd/ace \end{matrix}\middle| q\right]_{n-j} \\
&\qquad \times W(q^{j-1}bd/a : b/a, d/a, c, e, q^{j-n}; q^{n+j}bd/ce).
\end{aligned}
$$

Substituting this expression into (9–10) and then simplifying the resulting double sum, we get the following transformation formula.

Theorem 32 (Bivariate terminating series transformation).

$$
\begin{aligned}
&\left[\begin{matrix} bd/a, bd/ace \\ bd/ac, bd/ae \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, a \\ b, d \end{matrix}\middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q, q^{1-n}ace/bd \end{matrix}\middle| q\right]_i \Omega(j) \\
&\quad = \sum_{i,j\geq 0} \frac{1-q^{2i+j-1}bd/a}{1-bd/qa} \left[\begin{matrix} q^{-n}, bd/qa \\ b, d, bd/ac, bd/ae \end{matrix}\middle| q\right]_{i+j} \\
&\qquad \times \left[\begin{matrix} b/a, d/a, c, e \\ q, q^n bd/a \end{matrix}\middle| q\right]_i \left(\frac{q^{n+j}bd}{ce}\right)^i [a, bd/ace; q]_j \Omega(j).
\end{aligned}
$$

In view of (11–12), we have analogously the reduction formula from Theorem 32.

Proposition 33 (Reduction formula).

$$
\begin{aligned}
&\left[\begin{matrix} bd/ac, bd/ae \\ bd/a, bd/ace \end{matrix}\middle| q\right]_n \sum_{i,j\geq 0} \frac{1-q^{2i+j-1}bd/a}{1-bd/qa} \left[\begin{matrix} q^{-n}, \quad bd/qa \\ b, d, bd/ac, bd/ae \end{matrix}\middle| q\right]_{i+j} \\
&\qquad \times \left[\begin{matrix} b/a, d/a, c, e \\ q, q^n bd/a \end{matrix}\middle| q\right]_i \left(\frac{q^{n+j}bd}{ce}\right)^i \left[\begin{matrix} a, bd/ace, q^{n-1}bd/a \\ q \end{matrix}\middle| q\right]_j q^j \\
&\quad = a^n \left[\begin{matrix} b/a, d/a \\ b, d \end{matrix}\middle| q\right]_n {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & e \\ q^{1-n}a/b, & q^{1-n}a/d, & q^{1-n}ace/bd \end{matrix}\middle| q; \frac{q^{2-n}a}{bd}\right].
\end{aligned}
$$

Concluding comments. There is the third class of bivariate terminating series $\Phi^{3:1;\lambda}_{3:0;\mu}$ given explicitly by

$$
\sum_{i,j\geq 0} q^i \left[\begin{matrix} q^{-n}, a, c \\ b, d, q^{1-n}ace/bd \end{matrix}\middle| q\right]_{i+j} \frac{(e;q)_i}{(q;q)_i} \Omega(j) \tag{13}
$$

$$
= \sum_{j\geq 0} \Omega(j) \left[\begin{matrix} q^{-n}, a, c \\ b, d, q^{1-n}ace/bd \end{matrix}\middle| q\right]_j {}_4\phi_3\left[\begin{matrix} q^{j-n}, q^j a, q^j c, e \\ q^j b, q^j d, q^{1-n+j}ace/bd \end{matrix}\middle| q; q\right] \tag{14}
$$

which has already appeared in the transformations displayed in Theorems 4 and 7. Following similar procedure to those carried out in the second and third sections, transformation theorems and reduction formulae can be derived by reformulating the ${}_4\phi_3$-series just displayed through the Sears and Watson transformations. However, this research will not be conducted because the double series in (13) is equivalent, under the involution $i \to n-i-j$, to the series $\Phi^{1:3;\lambda}_{1:2;\mu}$ displayed in (5), which has been exhaustively investigated.

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