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On a generalized Stokes system with slip boundary conditions in the half-space

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Abstract. We are interested in a system of Stokes type, where the divergence-free constraint is modified by adding a term proportional to the pressure. The domain is the half-space with nonhomogeneous Navier's boundary conditions. The weighted Sobolev spaces yield a natural functional framework to envisage a wide class of behavior at infinity for data and solutions. So, we can give a range of solutions from strong to very weak depending on the regularity of the data. All along this study, we take the bridge between this system and the linear elasticity system.

1. Introduction and preliminaries

In this paper we investigate the system of Stokes type

$$\begin{cases} -\nu\Delta \boldsymbol{u} - \mu\nabla \operatorname{div} \boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \mathbf{R}^n_+, \\ \lambda\pi + \operatorname{div} \boldsymbol{u} = h & \text{in } \mathbf{R}^n_+, \\ u_n = g_n & \text{on } \Gamma, \\ \partial_n \boldsymbol{u}' = \boldsymbol{g}' & \text{on } \Gamma, \end{cases} \tag{1}$$

where the constants ν, μ and λ satisfy the assumptions $\nu > 0$, $\lambda \geq 0$ and $\mu + \nu > 0$. This system was recently studied by Beirão da Veiga in bounded domains and also in $\mathbf{R}^n_+$; see references [10, 11, 12]. First, we notice that the elasticity term $-\mu\nabla \operatorname{div} \boldsymbol{u}$, added in the first equation, may be eliminated by using the second equation. However, the calculations made under the assumption $\mu \neq 0$ seem to be useful in studying some problems related to compressible fluids.

The additional term $\lambda\pi$, which relax the divergence-free constraint in the second equation relatively to the classic Stokes system, is the central point. The introduction of this term, for sufficiently small values of λ, appears for instance in numerical approximation with the penalization method.

The aim of this work is to establish some existence and uniqueness results for different type of data in weighted Sobolev spaces. Indeed, these spaces

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provide a natural framework in unbounded domains in order to express the regularity and the behavior at infinity of data and solutions.

For positive values of λ, the generalized Stokes system is equivalent to the linear elasticity system; see e.g., Kobel'kov [21]. We will make precise this point in Subsection 2.5, and we will exploit it all along this paper with the assumption $\lambda > 0$. The case $\lambda = 0$, that is the classic Stokes system, was treated in [9] with similar tools, without appealing to the linear elasticity system.

So, at this stage two remarks are essential. On the one hand the results obtained in [9] and here are complementary and then the case $\lambda = 0$ can be included in all the statements on the Stokes system in the present paper; this is why we call it "generalized" Stokes system. On the other hand we may regard the classic system as the limit case $\lambda = 0$—in a sense that will be made clear in Remark 3.12—of this generalized Stokes system.

About the choice of the slip boundary conditions, let us recall that it takes its source in the recent developments of micro- and nanofluidic techniques. So, the slip assumption is validated by numerous experiments and simulations as well as theoretical studies; see e.g., Einzel, Panzer and Liu [13], Jäger and Mikelić [20], Lauga, Brenner and Stone [22], Priezjev, Darhuber and Troian [23], Priezjev and Troian [24], Qian, Wang and Sheng [25], and Zhu and Granick [27].

Another contribution of this work concerns the existence of a class of very weak solutions corresponding to singular boundary conditions. Several authors have considered this type of solutions for the Stokes and Navier-Stokes systems in bounded domains; see e.g., Giga [18], Amann [2], Galdi, Simader and Sohr [17], Farwig, Galdi and Sohr [15, 16], and Schumacher [26].

This paper is organized as follows. The remaining part of this section is devoted to the notations and functional setting.

In Section 2, we investigate the problem in the whole space, in order to use the reflection principle for the half-space. After the characterization of the kernel in Lemma 2.1, we establish the existence of generalized solutions in Theorem 2.3. Next, we give a regularity result in Theorem 2.4. Last, we translate these results for the linear elasticity system in Theorems 2.5 and 2.6.

In Section 3, we explore various aspects of the problem in the half-space. Lemma 3.1, Proposition 3.2 and Proposition 3.3 characterize the kernel in this geometry; then Theorem 3.4, Corollary 3.6 and Theorem 3.7 show the existence of strong solutions for the two linked problems. Last, Propositions 3.8 and 3.9 show the existence of generalized solutions in the homogeneous case and allow to establish Theorems 3.10 and 3.11 for the nonhomogeneous case.

To finish this study, in Section 4, we are interested in the case where very low regularity of the boundary data yields very weak solutions. By means of two technical lemmas—Lemmas 4.1 and 4.2—, we obtain Theorem 4.3.

For any real number $p > 1$, p' always stands for the Hölder conjugate of p, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

For any integer $n \geq 2$, writing a typical point $x \in \mathbf{R}^n$ as $x = (x', x_n)$, we denote by $\mathbf{R}_+^n$ the upper half-space of $\mathbf{R}^n$ and $\Gamma \equiv \mathbf{R}^{n-1}$ its boundary. We will use the two basic weights $\varrho = (1 + |x|^2)^{1/2}$ and $\lg \varrho = \ln(2 + |x|^2)$, where $|x|$ is the Euclidean norm of x.

For any integer q, $\mathscr{P}_q$ stands for the space of polynomials of degree smaller than or equal to q; $\mathscr{P}_q^{\Delta}$ (resp. $\mathscr{P}_q^{\Delta^2}$) is the subspace of harmonic (resp. biharmonic) polynomials in $\mathscr{P}_q$; $\mathscr{A}_q^{\Delta}$ (resp. $\mathscr{N}_q^{\Delta}$) is the subspace of polynomials in $\mathscr{P}_q^{\Delta}$, odd (resp. even) with respect to x_n, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$ (resp. $\partial_n \varphi(x', 0) = 0$); with the convention that these spaces are reduced to $\{0\}$ if $q < 0$. For any real number s, we denote by $[s]$ the largest integer less than or equal to s.

Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X. For any $k \in \mathbf{Z}$, we will denote by $\{1, \dots, k\}$ the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

Throughout this paper, bold characters are used for the vector fields; depending on the context, $\boldsymbol{f} \in \boldsymbol{X}$ stands for $\boldsymbol{f} = (f_1, \dots, f_n) \in \boldsymbol{X} = X^n$ and $\boldsymbol{g}' \in \boldsymbol{X}$ stands for $\boldsymbol{g}' = (g_1, \dots, g_{n-1}) \in \boldsymbol{X} = X^{n-1}$. We denote by C a generic positive real constant, and the symbol $\stackrel{\simeq}{\to}$ is reserved for isomorphisms between two spaces.

For weighted Sobolev spaces, we refer the reader to Hanouzet's classic article [19] and more especially to [3] for logarithmic weights. Let Ω be an open set of $\mathbf{R}^n$. For any $m \in \mathbf{N}$, $p \in \,]1, \infty[$, $(\alpha, \beta) \in \mathbf{R}^2$, we define the following space:

$$
\begin{aligned}
W_{\alpha,\beta}^{m,p}(\Omega) = \{u \in \mathscr{D}'(\Omega); 0 \leq |\lambda| \leq k, \varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta - 1} \partial^{\lambda} u \in L^p(\Omega); \\
k + 1 \leq |\lambda| \leq m, \varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta} \partial^{\lambda} u \in L^p(\Omega)\}, \qquad (2)
\end{aligned}
$$

where $k = m - n/p - \alpha$ if $n/p + \alpha \in \{1, \dots, m\}$, and $k = -1$ otherwise. In the case $\beta = 0$, we simply denote the space by $W_{\alpha}^{m,p}(\Omega)$. Note that $W_{\alpha,\beta}^{m,p}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$
\|u\|_{W_{\alpha,\beta}^{m,p}(\Omega)} = \left(\sum_{0 \leq |\lambda| \leq k} \|\varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta - 1} \partial^{\lambda} u\|_{L^p(\Omega)}^p \right.
$$

$$
\left. + \sum_{k+1 \leq |\lambda| \leq m} \|\varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta} \partial^{\lambda} u\|_{L^p(\Omega)}^p \right)^{1/p}.
$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \left(\sum_{|\lambda|=m} \|\varrho^{\alpha}(\lg \varrho)^{\beta}\partial^{\lambda}u\|^{p}_{L^{p}(\Omega)} \right)^{1/p}.$$

The weights in definition (2) are chosen so that $\mathscr{D}(\overline{\mathbf{R}^n_+})$ is dense in $W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)$ and so that the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)$ (see [5]): Let $q^* = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)$. If $n/p + \alpha \notin \{1, \ldots, m\}$ or $(\beta - 1)p \neq -1$, then

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+), \qquad \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)/\mathscr{P}_{q^*}} \leq C|u|_{W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)},$$

and

$$\forall u \in \mathring{W}^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+) = \overline{\mathscr{D}(\mathbf{R}^n_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)}}, \qquad \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)} \leq C|u|_{W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)}.$$

We denote by $W^{-m,p'}_{-\alpha,-\beta}(\mathbf{R}^n_+)$ the dual space of $\mathring{W}^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+)$ and we notice that it is a space of distributions. If $n/p + \alpha \notin \{1, \ldots, m\}$, we have the imbeddings

$$W^{m,p}_{\alpha,\beta}(\mathbf{R}^n_+) \hookrightarrow W^{m-1,p}_{\alpha-1,\beta}(\mathbf{R}^n_+) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta}(\mathbf{R}^n_+).$$

If $n/p + \alpha = j \in \{1, \ldots, m\}$, then we have

$$W^{m,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{m-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{m-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta-1}.$$

In order to define the traces of functions of $W^{m,p}_{\alpha}(\mathbf{R}^n_+)$ (here we do not consider the case $\beta \neq 0$), for any $\sigma \in]0,1[$, we introduce the space

$$W^{\sigma,p}_{\alpha}(\mathbf{R}^n) = \Big\{ u \in \mathscr{D}'(\mathbf{R}^n); w^{\alpha-\sigma}u \in L^p(\mathbf{R}^n),$$
$$\int_{\mathbf{R}^n \times \mathbf{R}^n} \frac{|\varrho^{\alpha}(x)u(x) - \varrho^{\alpha}(y)u(y)|^p}{|x-y|^{n+\sigma p}} \, \mathrm{d}x\mathrm{d}y < \infty \Big\},$$

where $w = \varrho$ if $n/p + \alpha \neq \sigma$ and $w = \varrho(\lg \varrho)^{1/(\sigma-\alpha)}$ if $n/p + \alpha = \sigma$. For any $s \in \mathbf{R}^+$, we set

$$W^{s,p}_{\alpha}(\mathbf{R}^n) = \{ u \in \mathscr{D}'(\mathbf{R}^n); 0 \leq |\lambda| \leq k, \varrho^{\alpha-s+|\lambda|}(\lg \varrho)^{-1}\partial^{\lambda}u \in L^p(\mathbf{R}^n);$$
$$k+1 \leq |\lambda| \leq [s]-1, \varrho^{\alpha-s+|\lambda|}\partial^{\lambda}u \in L^p(\mathbf{R}^n); \partial^{[s]}u \in W^{\sigma,p}_{\alpha}(\mathbf{R}^n) \},$$

where $k = s - n/p - \alpha$ if $n/p + \alpha \in \{\sigma, \ldots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. In the same way, we also define, for any real number β, the space $W^{s,p}_{\alpha,\beta}(\mathbf{R}^n) = \{ v \in \mathscr{D}'(\mathbf{R}^n); (\lg \varrho)^{\beta} v \in W^{s,p}_{\alpha}(\mathbf{R}^n) \}$. These spaces are reflexive Banach spaces equipped with their natural norms.

If $n/p + \alpha \notin \{\sigma, \dots, \sigma + [s] - 1\}$, we have the imbeddings

$$W^{s,p}_{\alpha,\beta}(\mathbf{R}^n) \hookrightarrow W^{s-1,p}_{\alpha-1,\beta}(\mathbf{R}^n) \hookrightarrow \cdots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta}(\mathbf{R}^n),$$

$$W^{s,p}_{\alpha,\beta}(\mathbf{R}^n) \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta}(\mathbf{R}^n) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-s,\beta}(\mathbf{R}^n).$$

If $n/p + \alpha = j \in \{\sigma, \dots, \sigma + [s] - 1\}$, then we have

$$W^{s,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{s-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{s-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta-1},$$

$$W^{s,p}_{\alpha,\beta} \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta} \hookrightarrow \cdots \hookrightarrow W^{[s]-j+1,p}_{\alpha-\sigma-j+1,\beta} \hookrightarrow W^{[s]-j,p}_{\alpha-\sigma-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-s,\beta-1}.$$

If u is a function on $\mathbf{R}^n_+$, we denote its trace of order j on the hyperplane Γ by

$$\forall j \in \mathbf{N}, \qquad \gamma_j u : x' \in \mathbf{R}^{n-1} \mapsto \partial^j_n u(x', 0).$$

Let us recall the following trace lemma due to Hanouzet (see [19]) and extended by Amrouche and Nečasová (see [5]) to the critical values with logarithmic weights.

Lemma 1.1 (The Trace Lemma). *For any integer $m \geq 1$ and real number α, we have the linear continuous mapping*

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{m-1}) : W^{m,p}_{\alpha}(\mathbf{R}^n_+) \to \prod_{j=0}^{m-1} W^{m-j-1/p,p}_{\alpha}(\mathbf{R}^{n-1}).$$

Moreover γ is surjective and $\operatorname{Ker} \gamma = \mathring{W}^{m,p}_{\alpha}(\mathbf{R}^n_+)$.

Remark 1.2. As we saw in [8] and [9], it is possible to give a sense to traces in such spaces $W^{s,p}_{\alpha}(\mathbf{R}^n)$ with $s < 0$ for particular classes of functions or distributions. For instance, if $u \in W^{0,p}_{\alpha}(\mathbf{R}^n_+)$—that is a weighted L^p space—with $\Delta u = 0$, then we have $\gamma_0 u \in W^{-1/p,p}_{\alpha}(\mathbf{R}^{n-1})$.

2. The generalized Stokes system in $\mathbf{R}^n$

As usual, our method for the half-space requires the extension of problems to the whole space. Then a necessary step is to consider the corresponding Stokes system in $\mathbf{R}^n$:

$$\begin{cases} -\nu \Delta \boldsymbol{u} - \mu \nabla \operatorname{div} \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \mathbf{R}^n, \\ \lambda \pi + \operatorname{div} \boldsymbol{u} = h & \text{in } \mathbf{R}^n. \end{cases} \tag{3}$$

In this section, we essentially adapt to this problem, with minor modifications, the arguments used by Alliot and Amrouche for the classic Stokes system in the whole space (see [1]). However, in the last subsection, we shed new light on this generalized system when $\lambda \neq 0$, regarding it now as the system of linear elasticity, with the pressure being a function of the velocity field. Let us denote by T_λ the corresponding operator:

$$T_\lambda : (\boldsymbol{u}, \pi) \mapsto (-\nu \Delta \boldsymbol{u} - \mu \nabla \operatorname{div} \boldsymbol{u} + \nabla \pi, -\lambda \pi - \operatorname{div} \boldsymbol{u}).$$

2.1. Existence and uniqueness results. We consider (3) with $(\boldsymbol{f}, h) = (\boldsymbol{0}, 0)$ through the operator T_λ defined on the space of tempered distributions $\mathscr{S}'(\mathbf{R}^n) \times \mathscr{S}'(\mathbf{R}^n)$. Using the second equation in order to substitute $-\lambda\pi$ for $\operatorname{div} \boldsymbol{u}$ in the first equation, we get

$$-\nu \Delta \boldsymbol{u} + (1 + \lambda\mu) \nabla \pi = \boldsymbol{0} \qquad \text{in } \mathbf{R}^n. \tag{4}$$

Applying the divergence operator to this equation, we obtain $\Delta \pi = 0$ in $\mathbf{R}^n$. Finally, applying the Laplacian to the same equation, we find $\Delta^2 \boldsymbol{u} = \boldsymbol{0}$ in $\mathbf{R}^n$. So, π and $\boldsymbol{u}$ are respectively tempered harmonic and biharmonic distributions, thus polynomials. Consequently, the kernel of T_λ is quite similar to the kernel of the classic Stokes operator. For any $k \in \mathbf{Z}$, let us introduce the space

$$\mathscr{S}_k^\lambda = \{(\boldsymbol{\chi}, q) \in \mathscr{P}_k^{\Delta^2} \times \mathscr{P}_{k-1}^{\Delta}; T_\lambda(\boldsymbol{\chi}, q) = \boldsymbol{0}\}.$$

According to the maximum degree of polynomials in weighted Sobolev spaces (see [3]), we have the following uniqueness result:

Lemma 2.1. *Let $\ell \in \mathbf{Z}$, $m \in \mathbf{N}$ and assume that $n/p \notin \{1, \dots, -\ell - m\}$, then the kernel of T_λ defined on $\boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbf{R}^n) \times W_{m+\ell}^{m,p}(\mathbf{R}^n)$ is the space $\mathscr{S}_{[1-\ell-n/p]}^\lambda$.*

Now, we are interested in the question of existence of solutions. Let $(\boldsymbol{u}, \pi) \in \mathscr{S}'(\mathbf{R}^n) \times \mathscr{S}'(\mathbf{R}^n)$ be a solution pair to problem (3). The second equation of (3) allows us to substitute $h - \lambda\pi$ for $\operatorname{div} \boldsymbol{u}$ in the first equation. Then, we obtain

$$\nu \Delta \boldsymbol{u} = (1 + \lambda\mu) \nabla \pi - \boldsymbol{f} - \mu \nabla h. \tag{5}$$

Next, taking the divergence of (5) and replacing angain $\operatorname{div} \boldsymbol{u}$ by $h - \lambda\pi$, we obtain

$$(1 + \lambda(\nu + \mu)) \Delta \pi = \operatorname{div} \boldsymbol{f} + (\nu + \mu) \Delta h. \tag{6}$$

Thus, similarly to the classic Stokes system—i.e., the case $\lambda = 0$—in $\mathbf{R}^n$, it suffices to solve these two Poisson's equations. Indeed, if $(\boldsymbol{v}, \tau)$ verifies (5) and (6), then we get

$$-\nu \Delta \boldsymbol{v} - \mu \nabla (h - \lambda\tau) + \nabla \tau = \boldsymbol{f} \quad \text{in } \mathscr{S}'(\mathbf{R}^n),$$

$$\Delta \operatorname{div} \boldsymbol{v} = \Delta (h - \lambda\tau) \quad \text{in } \mathscr{S}'(\mathbf{R}^n),$$

and thus, $\operatorname{div} \boldsymbol{v} - h + \lambda\tau = \varphi$, where φ is a harmonic polynomial. So we can use the following lemma proved in [1]:

Lemma 2.2. *For any $k \in \mathbf{N}$, $\mathscr{P}_k^{\Delta} = \operatorname{div}(\boldsymbol{\mathscr{P}}_{k+1}^{\Delta})$.*

Therefore, if $\varphi \in \mathscr{P}_{k-1}^{\Delta}$ with $k \geq 1$, there exists $\boldsymbol{\chi} \in \boldsymbol{\mathscr{P}}_k^{\Delta}$ such that $\varphi = \operatorname{div} \boldsymbol{\chi}$ and the pair $(\boldsymbol{v} - \boldsymbol{\chi}, \tau)$ satisfies the initial problem (3).

2.2. Generalized solutions. We now give the main result for (3).

Theorem 2.3. *Let $\ell \in \mathbf{Z}$ and assume that*

$$n/p' \notin \{1, \dots, \ell\} \qquad \text{and} \qquad n/p \notin \{1, \dots, -\ell\}. \tag{7}$$

For any $(\boldsymbol{f}, h) \in (\boldsymbol{W}_\ell^{-1,p}(\mathbf{R}^n) \times W_\ell^{0,p}(\mathbf{R}^n)) \perp \mathscr{S}_{[1+\ell-n/p']}^{\lambda}$, problem (3) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n) \times W_\ell^{0,p}(\mathbf{R}^n)$, unique up to an element of $\mathscr{S}_{[1-\ell-n/p]}^{\lambda}$, with the estimate

$$\inf_{(\boldsymbol{\chi}, q) \in \mathscr{S}_{[1-\ell-n/p]}^{\lambda}} (\|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n)} + \|\pi + q\|_{W_\ell^{0,p}(\mathbf{R}^n)}) \leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}_\ell^{-1,p}(\mathbf{R}^n)} + \|h\|_{W_\ell^{0,p}(\mathbf{R}^n)}).$$

Proof. We proceed in three steps. First we solve the case $\ell = 0$, then we consider the negative weights to avoid troubles with the compatibility conditions and last, we obtain the solutions for positive weights by a duality argument.

(i) The Stokes operator T_λ satisfies

$$T_\lambda : (\boldsymbol{W}_0^{1,p}(\mathbf{R}^n) \times L^p(\mathbf{R}^n))/\mathscr{S}_{[1-n/p]}^{\lambda} \overset{\simeq}{\to} (\boldsymbol{W}_0^{-1,p}(\mathbf{R}^n) \times L^p(\mathbf{R}^n)) \perp \mathscr{S}_{[1-n/p']}^{\lambda}.$$

The operator T_λ is clearly continuous, moreover it is injective by Lemma 2.1, then by the Banach Theorem, it remains to show that it is surjective. For that we naturally use the splitting of this problem in (5) and (6). Let us consider a pair $(\boldsymbol{f}, h) \in (\boldsymbol{W}_0^{-1,p}(\mathbf{R}^n) \perp \boldsymbol{\mathscr{P}}_{[1-n/p']}) \times L^p(\mathbf{R}^n)$, then $\operatorname{div} \boldsymbol{f} \in W_0^{-2,p}(\mathbf{R}^n)$. Moreover, for any $\varphi \in \mathscr{P}_{[2-n/p']}$, we have

$$\langle \operatorname{div} \boldsymbol{f}, \varphi \rangle_{W_0^{-2,p}(\mathbf{R}^n) \times W_0^{2,p'}(\mathbf{R}^n)} = \langle \boldsymbol{f}, \nabla\varphi \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbf{R}^n) \times \boldsymbol{W}_0^{1,p'}(\mathbf{R}^n)} = 0,$$

i.e., $\operatorname{div} \boldsymbol{f} \in W_0^{-2,p}(\mathbf{R}^n) \perp \mathscr{P}_{[2-n/p']}$. The same arguments hold for h and yield $\Delta h \in W_0^{-2,p}(\mathbf{R}^n) \perp \mathscr{P}_{[2-n/p']}$. According to the results on the Laplacian in $\mathbf{R}^n$ (see [3]), we know that

$$\Delta : L^p(\mathbf{R}^n) \overset{\simeq}{\to} W_0^{-2,p}(\mathbf{R}^n) \perp \mathscr{P}_{[2-n/p']}$$

and thus, there exists $\pi \in L^p(\mathbf{R}^n)$ solution to (6). On the other hand, for any $\psi \in W_0^{1,p'}(\mathbf{R}^n)$ and $1 \leq i \leq n$, we have

$$\langle \partial_i \pi, \psi \rangle_{W_0^{-1,p}(\mathbf{R}^n) \times W_0^{1,p'}(\mathbf{R}^n)} = -\langle \pi, \partial_i \psi \rangle_{L^p(\mathbf{R}^n) \times L^{p'}(\mathbf{R}^n)}.$$

That implies $\partial_i \pi \perp \mathbf{R}$ if $n/p' \leq 1$—indeed, $\mathbf{R} = \mathscr{P}_{[1-n/p']} \subset W_0^{1,p'}(\mathbf{R}^n)$ if $n/p' \leq 1$—, and the same argument holds for $\partial_i h$. Thanks to the fact (see [3]) that

$$\Delta : W_0^{1,p}(\mathbf{R}^n)/\mathscr{P}_{[1-n/p]} \xrightarrow{\simeq} W_0^{-1,p}(\mathbf{R}^n) \perp \mathscr{P}_{[1-n/p']},$$

there exists a solution $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\mathbf{R}^n)$ of (5). In addition, as we have seen above, $\operatorname{div} \boldsymbol{u} - h + \lambda\pi$ is a harmonic polynomial. Since it belongs to $L^p(\mathbf{R}^n)$, it is actually zero. So $(\boldsymbol{u}, \pi)$ verifies $T_\lambda(\boldsymbol{u}, \pi) = (\boldsymbol{f}, -h)$, which proves the surjectivity of T_λ.

(ii) For any $\ell < 0$, assuming that $n/p \notin \{1, \dots, -\ell\}$, we have

$$T_\lambda : (\boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n) \times W_\ell^{0,p}(\mathbf{R}^n))/\mathscr{S}^\lambda_{[1-\ell-n/p]} \xrightarrow{\simeq} \boldsymbol{W}_\ell^{-1,p}(\mathbf{R}^n) \times W_\ell^{0,p}(\mathbf{R}^n). \tag{8}$$

It is the same reasoning to solve the two Poisson's equations (6) and (5), but using this time successively the isomorphisms

$$\Delta : W_\ell^{0,p}(\mathbf{R}^n)/\mathscr{P}^\Delta_{[-\ell-n/p]} \xrightarrow{\simeq} W_\ell^{-2,p}(\mathbf{R}^n) \perp \mathscr{P}_{[2+\ell-n/p']},$$

and

$$\Delta : W_\ell^{1,p}(\mathbf{R}^n)/\mathscr{P}^\Delta_{[1-\ell-n/p]} \xrightarrow{\simeq} W_\ell^{-1,p}(\mathbf{R}^n),$$

valid under these assumptions (see [3, 4]). Finally, modifying this solution with a polynomial constructed by means of Lemma 2.2, we get a solution to (3).

(iii) For any $\ell > 0$, the adjoint operator T_λ^* of T_λ satisfies,

$$T_\lambda^* : \boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n) \times W_\ell^{0,p}(\mathbf{R}^n) \xrightarrow{\simeq} (\boldsymbol{W}_\ell^{-1,p}(\mathbf{R}^n) \times W_\ell^{0,p}(\mathbf{R}^n)) \perp \mathscr{S}^\lambda_{[1+\ell-n/p']}. \tag{9}$$

We get it by duality, replacing $-\ell$ by ℓ and p' by p. In addition, by a density argument, we show that

$$T_\lambda^*(\boldsymbol{v}, \vartheta) = (-\nu\Delta\boldsymbol{v} - \mu\nabla \operatorname{div} \boldsymbol{v} + \nabla\vartheta, -\lambda\vartheta - \operatorname{div} \boldsymbol{v}),$$

i.e., T_λ is selfadjoint and the proof is complete. □

2.3. The classic problem as limit case. In order to answer to the question of the limit as λ goes to zero, it is interesting to yield an estimate in Theorem 2.3 where the constant C does not depend on the parameter λ. We start with the central case of weight zero. On the one hand, problem (6) yields the estimate

$$\|\pi\|_{L^p(\mathbf{R}^n)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbf{R}^n)} + (\nu + \mu)\|h\|_{L^p(\mathbf{R}^n)}),$$

where $C = C(n, p)$; and on the other hand, (5) yields

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbf{R}^n)/\mathscr{P}_{[1-n/p]}} \leq C[|1 + \lambda\mu|\, \|\pi\|_{L^p(\mathbf{R}^n)} + \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbf{R}^n)} + |\mu|\, \|h\|_{L^p(\mathbf{R}^n)}],$$

where $C = C(\nu, n, p)$. Hence, we obtain the global estimate for problem (3)

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbf{R}^n)/\mathscr{P}_{[1-n/p]}} + \|\pi\|_{L^p(\mathbf{R}^n)} \\ \le C[(2 + |1+\lambda\mu|)\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbf{R}^n)} + ((1+|1+\lambda\mu|)(\nu+\mu) + |\mu|)\|h\|_{L^p(\mathbf{R}^n)}],$$

where $C = C(\nu, n, p)$ is a constant depending only on ν, n and p. Next, it is clear that this estimate also holds for a weight $\ell > 0$, because the kernel is then reduced to zero; that is

$$\|(\boldsymbol{u}, \pi)\|_{\boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n)\times W_\ell^{0,p}(\mathbf{R}^n)} \\ \le C[(2 + |1+\lambda\mu|)\|\boldsymbol{f}\|_{\boldsymbol{W}_\ell^{-1,p}(\mathbf{R}^n)} + ((1+|1+\lambda\mu|)(\nu+\mu) + |\mu|)\|h\|_{W_\ell^{0,p}(\mathbf{R}^n)}],$$

where $C = C(\nu, n, p)$. Since our goal here is to observe the behavior of this estimate as λ tends to 0, we can assume that $0 \le \lambda \le 1$. Then, we can write

$$\|(\boldsymbol{u}, \pi)\|_{\boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n)\times W_\ell^{0,p}(\mathbf{R}^n)} \le C(\|\boldsymbol{f}\|_{\boldsymbol{W}_\ell^{-1,p}(\mathbf{R}^n)} + \|h\|_{W_\ell^{0,p}(\mathbf{R}^n)}),$$

where $C = C(\nu, \mu, n, p)$. Concerning the isomorphisme T_λ^*, we know that the best constant for this estimate is in fact $\|(T_\lambda^*)^{-1}\|$. In addition, since T_λ is selfadjoint, we have $\|(T_\lambda^*)^{-1}\| = \|T_\lambda^{-1}\|$.

Thus, in the case $\ell < 0$, we can deduce by duality that we also have

$$\|(\boldsymbol{u}, \pi)\|_{(\boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n)\times W_\ell^{0,p}(\mathbf{R}^n))/\mathscr{S}_{[1-\ell-n/p]}^{\lambda}} \le C\|(\boldsymbol{f}, h)\|_{\boldsymbol{W}_\ell^{-1,p}(\mathbf{R}^n)\times W_\ell^{0,p}(\mathbf{R}^n)}, \tag{10}$$

where $C = C(\nu, \mu, n, p)$ for any $0 \le \lambda \le 1$. So, this estimate, where the constant C does not depend on $\lambda \in [0, 1]$, holds for any $\ell \in \mathbf{Z}$.

Now, let us consider fixed data $\boldsymbol{f}$ and h and a sequence of parameters $\lambda_i \in [0, 1]$ which tends to zero (ν and μ are also prescribed), such that the family of generalized problems admits solutions $(\boldsymbol{u}_i, \pi_i)$ and the classic system—that is for $\lambda = 0$—a solution $(\boldsymbol{u}, \pi)$. We naturally expect that $(\boldsymbol{u}_i, \pi_i)$ converges to $(\boldsymbol{u}, \pi)$ as λ_i goes to zero.

So, for any $i \in \mathbf{N}$, we have

$$\begin{aligned} -\nu\Delta\boldsymbol{u}_i - \mu\nabla \operatorname{div} \boldsymbol{u}_i + \nabla\pi_i &= \boldsymbol{f} \quad \text{in } \mathbf{R}^n, \\ \lambda_i\pi_i + \operatorname{div} \boldsymbol{u}_i &= h \quad \text{in } \mathbf{R}^n, \end{aligned}$$

and

$$\begin{aligned} -\nu\Delta\boldsymbol{u} - \mu\nabla \operatorname{div} \boldsymbol{u} + \nabla\pi &= \boldsymbol{f} \quad \text{in } \mathbf{R}^n, \\ \operatorname{div} \boldsymbol{u} &= h \quad \text{in } \mathbf{R}^n. \end{aligned}$$

By difference, we see that the pair $(\boldsymbol{u} - \boldsymbol{u}_i, \pi - \pi_i)$ verifies the problem

$$-\nu\Delta \boldsymbol{v} - \mu\nabla \operatorname{div} \boldsymbol{v} + \nabla\tau = \mathbf{0} \quad \text{in } \mathbf{R}^n,$$

$$\lambda_i\tau + \operatorname{div} \boldsymbol{v} = \lambda_i\pi \quad \text{in } \mathbf{R}^n;$$

and the estimate (10) is written here as

$$\|(\boldsymbol{v}, \tau)\|_{(\boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n)\times W_\ell^{0,p}(\mathbf{R}^n))/\mathscr{S}^\lambda_{[1-\ell-n/p]}} \le C\lambda_i\|\pi\|_{W_\ell^{0,p}(\mathbf{R}^n)},$$

where $C = C(\nu, \mu, n, p)$ is a constant which does not depend on λ_i.

So, by passing to the limit as $\lambda_i \to 0$, we deduce that $(\boldsymbol{u}_i, \pi_i) \to (\boldsymbol{u}, \pi)$ in $(\boldsymbol{W}_\ell^{1,p}(\mathbf{R}^n) \times W_\ell^{0,p}(\mathbf{R}^n))/\mathscr{S}^\lambda_{[1-\ell-n/p]}$.

2.4. Regularity of solutions. In this subsection, we establish a global regularity result.

THEOREM 2.4. *Let $\ell \in \mathbf{Z}$ and $m \ge 1$ be two integers and assume that*

$$n/p' \notin \{1, \dots, \ell + 1\} \qquad \text{and} \qquad n/p \notin \{1, \dots, -\ell - m\}. \tag{11}$$

For any $(\boldsymbol{f}, h) \in (\boldsymbol{W}_{m+\ell}^{m-1,p}(\mathbf{R}^n) \times W_{m+\ell}^{m,p}(\mathbf{R}^n)) \perp \mathscr{S}^\lambda_{[1+\ell-n/p']}$, problem (3) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbf{R}^n) \times W_{m+\ell}^{m,p}(\mathbf{R}^n)$, unique up to an element of $\mathscr{S}^\lambda_{[1-\ell-n/p]}$, with the estimate

$$\inf_{(\boldsymbol{\chi}, q) \in \mathscr{S}^\lambda_{[1-\ell-n/p]}} (\|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbf{R}^n)} + \|\pi + q\|_{W_{m+\ell}^{m,p}(\mathbf{R}^n)})$$
$$\le C(\|\boldsymbol{f}\|_{\boldsymbol{W}_{m+\ell}^{m-1,p}(\mathbf{R}^n)} + \|g\|_{W_{m+\ell}^{m,p}(\mathbf{R}^n)}).$$

PROOF. For the negative weights, the proof follows the same reasoning as for the generalized solutions, except that the regularity results for the Laplacian (see [3, 4]) are employed. Namely, we use

$$\Delta : W_{m+\ell}^{m,p}(\mathbf{R}^n)/\mathscr{P}^\Delta_{[-\ell-n/p]} \xrightarrow{\simeq} W_{m+\ell}^{m-2,p}(\mathbf{R}^n) \qquad \text{if } \ell \le -2 \text{ and (11)},$$

or in the case $\ell = -1$,

$$\Delta : W_{m-1}^{m,p}(\mathbf{R}^n)/\mathscr{P}_{[1-n/p]} \xrightarrow{\simeq} W_{m-1}^{m-2,p}(\mathbf{R}^n) \perp \mathscr{P}_{[1-n/p']} \quad \text{if } n/p' \ne 1 \text{ or } m = 1, \tag{12}$$

to solve (6); and use

$$\Delta : W_{m+\ell}^{m+1,p}(\mathbf{R}^n)/\mathscr{P}^\Delta_{[1-\ell-n/p]} \xrightarrow{\simeq} W_{m+\ell}^{m-1,p}(\mathbf{R}^n),$$

to solve (5). However, the case $n = p'$ for $\ell = -1$ and $m \ge 2$ is critical for the isomorphism (12), and the use of a critical result on the Laplace operator is required to solve (6). According to [4], we have

$$\Delta : W_m^{1+m,p}(\mathbf{R}^n)/\mathscr{P}_{[1-n/p]} \xrightarrow{\simeq} X_m^{m-1,p}(\mathbf{R}^n) \perp \mathbf{R} \qquad \text{if } n = p' \text{ and } m \ge 1, \tag{13}$$

where the family of spaces X is defined as follows. For any $m \in \mathbf{Z}$, $\ell \in \mathbf{N}$,

$$X_{\ell}^{m+\ell,p}(\mathbf{R}^n) = \{u \in W_0^{m,p}(\mathbf{R}^n); \forall \lambda \in \mathbf{N}^n, 0 \le |\lambda| \le \ell,$$

$$x^{\lambda} u \in W_0^{m+|\lambda|,p}(\mathbf{R}^n); u \in W_{\mathrm{loc}}^{m+\ell,p}(\mathbf{R}^n)\},$$

and its dual space is denoted by $X_{-\ell}^{-m-\ell,p'}(\mathbf{R}^n)$.

So, replacing m by $m-1$ in (13), we get

$$\Delta : W_{m-1}^{m,p}(\mathbf{R}^n)/\mathscr{P}_{[1-n/p]} \xrightarrow{\simeq} X_{m-1}^{m-2,p}(\mathbf{R}^n) \perp \mathbf{R} \qquad \text{if } n = p' \text{ and } m \ge 2,$$

which precisely fills the gap in the isomorphism (12) for this critical case.

In addition, we can show that $X_{m-1}^{m-2,p}(\mathbf{R}^n) = W_{m-1}^{m-2,p}(\mathbf{R}^n) \cap W_0^{-1,p}(\mathbf{R}^n)$. Since $\boldsymbol{f} \in \boldsymbol{W}_{m-1}^{m-1,p}(\mathbf{R}^n)$, we have $\operatorname{div} \boldsymbol{f} \in W_{m-1}^{m-2,p}(\mathbf{R}^n)$, and thanks to the imbedding $W_{m-1}^{m-1,p}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$, we also have $\operatorname{div} \boldsymbol{f} \in W_0^{-1,p}(\mathbf{R}^n)$, hence $\operatorname{div} \boldsymbol{f} \in X_{m-1}^{m-2,p}(\mathbf{R}^n)$. In the same way, we have $\Delta h \in X_{m-1}^{m-2,p}(\mathbf{R}^n)$, and thus we are able to solve (6). The rest of the proof is quite similar.

For $\ell \ge 0$, contrary to the generalized solutions, the duality reasoning fails, however we can use a regularity argument similar to the one employed for the classic Stokes system (see [8, Corollary 5.5]). □

2.5. A change of point of view. As we remarked in the introduction, if $\lambda = 0$, we find the classic Stokes system. Now, if $\lambda \ne 0$, we can completely decouple the velocity field from the pressure in the main equation. Indeed, the system (3) in the whole space is clearly equivalent to

$$\begin{cases} -\nu \Delta \boldsymbol{u} - \left(\mu + \frac{1}{\lambda}\right) \nabla \operatorname{div} \boldsymbol{u} = \boldsymbol{f} - \frac{1}{\lambda} \nabla h & \text{in } \mathbf{R}^n, \\ \pi = \frac{1}{\lambda}(h - \operatorname{div} \boldsymbol{u}) & \text{in } \mathbf{R}^n. \end{cases} \tag{14}$$

So we recognize the equation of linear elasticity as main equation, which can be rewritten by means of Lamé operator $\mathrm{L} = -\nu \Delta - \left(\mu + \frac{1}{\lambda}\right) \nabla \operatorname{div}$, as

$$\mathrm{L} \boldsymbol{u} = \boldsymbol{F} \qquad \text{in } \mathbf{R}^n, \tag{15}$$

where $\boldsymbol{F} = \boldsymbol{f} - \frac{1}{\lambda} \nabla h$. Let us still notice that if $\lambda \mu = -1$, the operator L is nothing else but the Laplacian. Then solving (3) is equivalent to solving (15)—indeed, knowing the velocity field, we immediately get the pressure π; moreover, the kernel of L is the velocity field's part, decoupled from the pressure, in the kernel of T_{λ}, that is the polynomial space

$$\mathscr{L}_{[1-\ell-n/p]} = \{\boldsymbol{\chi} \in \boldsymbol{\mathscr{P}}_{[1-\ell-n/p]}^{\Delta^2}; \mathrm{L} \boldsymbol{\chi} = \mathbf{0}\},$$

if L is defined on $\boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbf{R}^n)$, for any $\ell \in \mathbf{Z}$ and $m \in \mathbf{N}$. So, we can express the results on system (3) in terms adapted to equation (15). For instance, Theorem 2.3 becomes:

Theorem 2.5. *Let $\ell \in \mathbf{Z}$ and assume* (7). *For any $\boldsymbol{F} \in \boldsymbol{W}_{\ell}^{-1,p}(\mathbf{R}^n) \perp \mathscr{L}_{[1+\ell-n/p']}$, problem* (15) *admits a solution $\boldsymbol{u} \in \boldsymbol{W}_{\ell}^{1,p}(\mathbf{R}^n)$, unique up to an element of $\mathscr{L}_{[1-\ell-n/p]}$, with the estimate*

$$\inf_{\chi \in \mathscr{L}_{[1-\ell-n/p]}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}_{\ell}^{1,p}(\mathbf{R}^n)} \le C\|\boldsymbol{F}\|_{\boldsymbol{W}_{\ell}^{-1,p}(\mathbf{R}^n)}.$$

Of course, we also have an equivalent to Theorem 2.4 for the regularity:

Theorem 2.6. *Let $\ell \in \mathbf{Z}$ and $m \ge 1$ be two integers and assume* (11). *For any $\boldsymbol{F} \in \boldsymbol{W}_{m+\ell}^{m-1,p}(\mathbf{R}^n) \perp \mathscr{L}_{[1+\ell-n/p']}$, problem* (15) *admits a solution $\boldsymbol{u} \in \boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbf{R}^n)$, unique up to an element of $\mathscr{L}_{[1-\ell-n/p]}$, with the corresponding estimate.*

3. The generalized Stokes system in $\mathbf{R}_+^n$

After the question of the kernel and the one of the compatibility condition for the data, we deal with strong and then generalized solutions to (1).

3.1. The kernel. Examining the reflection principle (see R. Farwig [14]) for the classic Stokes system with slip condition, we immediately see that it is unchanged for the generalized system. Namely, if $\ell \in \mathbf{Z}$ and $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{\ell}^{1,p}(\mathbf{R}_+^n) \times W_{\ell}^{0,p}(\mathbf{R}_+^n)$ is an element of the kernel of the generalized Stokes operator T_λ with slip boundary conditions, then the unique extension $(\tilde{\boldsymbol{u}}, \tilde{\pi})$ of $(\boldsymbol{u}, \pi)$ to the whole space, satisfying

$$-\nu\Delta\tilde{\boldsymbol{u}} - \mu\nabla \operatorname{div} \tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \mathbf{0} \quad \text{and} \quad \lambda\tilde{\pi} + \operatorname{div} \tilde{\boldsymbol{u}} = 0 \qquad \text{in } \mathbf{R}^n,$$

is given by the following continuation formulae: for all $x = (x', x_n) \in \mathbf{R}_-^n$,

$$\tilde{\boldsymbol{u}}'(x) = \boldsymbol{u}'(x^*), \quad \tilde{u}_n(x) = -u_n(x^*), \quad \tilde{\pi}(x) = \pi(x^*), \qquad \text{where } x^* = (x', -x_n).$$

Moreover, such $\tilde{\pi}$ and $\tilde{\boldsymbol{u}}$ are respectively harmonic and biharmonic tempered distributions in $\mathbf{R}^n$, thus polynomials. For all $k \in \mathbf{Z}$, let us denote

$$\begin{aligned} {}^{+}\mathscr{S}_k^{\lambda} = \{(\boldsymbol{\chi}, q) \in \mathscr{P}_k^{\Delta^2} \times \mathscr{P}_{k-1}^{\Delta}; -\nu\Delta\boldsymbol{\chi} - \mu\nabla \operatorname{div} \boldsymbol{\chi} + \nabla q = \mathbf{0} \text{ and} \\ \lambda q + \operatorname{div} \boldsymbol{\chi} = 0 \text{ in } \mathbf{R}_+^n, \partial_n\boldsymbol{\chi}' = \mathbf{0} \text{ and } \chi_n = 0 \text{ on } \Gamma\}. \end{aligned}$$

As in the whole space, we can show that this kernel does not depend on the regularity according to the Sobolev imbeddings and that it is characterized as follows.

Lemma 3.1. *Let $\ell \in \mathbf{Z}$, $m \in \mathbf{N}$ and assume that $n/p \notin \{1, \dots, -\ell - m\}$, then the kernel of the Stokes operator T_λ defined on $\boldsymbol{W}^{m+1,p}_{m+\ell}(\mathbf{R}^n_+) \times W^{m,p}_{m+\ell}(\mathbf{R}^n_+)$ with the homogeneous Navier boundary conditions is the space ${}^+\mathscr{S}^\lambda_{[1-\ell-n/p]}$.*

We also can express it in terms of the polynomial spaces $\mathscr{A}^\Delta_k$ and $\mathscr{N}^\Delta_k$ which respectively define the kernels of the Laplacian with Dirichlet and Neumann boundary conditions in the half-space. With this aim, we will use the operator Π_N—introduced in [7] for the biharmonic problem—defined as follows.

$$\forall s \in \mathscr{N}^\Delta_k, \qquad \Pi_N s(x', x_n) = \frac{1}{2} x_n \int_0^{x_n} s(x', t) \mathrm{d}t,$$

satisfies $\Delta \Pi_N s = s$ in $\mathbf{R}^n_+$ and $\Pi_N s = \partial_n \Pi_N s = 0$ on Γ.

Proposition 3.2. *Let $\ell \in \mathbf{Z}$. The pair $(\boldsymbol{\chi}, q) \in {}^+\mathscr{S}^\lambda_{[1-\ell-n/p]}$ if and only if there exists $\boldsymbol{\varphi} \in \mathscr{N}^\Delta_{[1-\ell-n/p]} \times \mathscr{A}^\Delta_{[1-\ell-n/p]}$ such that*

$$\boldsymbol{\chi} = \kappa_2 \boldsymbol{\varphi} - \kappa_1 \nabla \Pi_N \operatorname{div} \boldsymbol{\varphi} \qquad \textit{and} \qquad q = -\operatorname{div} \boldsymbol{\varphi}, \tag{16}$$

where $\kappa_1 = \frac{1+\lambda\mu}{\nu}$ and $\kappa_2 = \frac{1+\lambda(\mu+\nu)}{\nu}$.

Proof. Given $(\boldsymbol{\chi}, q) \in {}^+\mathscr{S}^\lambda_{[1-\ell-n/p]}$, we have $\Delta q = 0$ in $\mathbf{R}^n_+$. Now, taking the restriction to Γ of the n^{th} component of the main equation in ${}^+\mathscr{S}^\lambda_k$ with the boundary conditions and applying the operator ∂_n to the second equation in ${}^+\mathscr{S}^\lambda_k$, we obtain $\partial_n q = 0$ on Γ. Hence we deduce that $q \in \mathscr{N}^\Delta_{[-\ell-n/p]}$. So, by use of (4), we can write

$$\Delta(\nu \boldsymbol{\chi} - (1 + \lambda\mu) \nabla \Pi_N q) = \nu \Delta \boldsymbol{\chi} - (1 + \lambda\mu) \nabla q = \mathbf{0},$$

which implies the existence of $\boldsymbol{\varphi} \in \mathscr{P}^\Delta_{[1-\ell-n/p]}$ such that

$$\boldsymbol{\varphi} = \nu \boldsymbol{\chi} - (1 + \lambda\mu) \nabla \Pi_N q. \tag{17}$$

In fact, we can see that $\boldsymbol{\varphi} \in \mathscr{N}^\Delta_{[1-\ell-n/p]} \times \mathscr{A}^\Delta_{[1-\ell-n/p]}$ by considerations on the parity of $\boldsymbol{\chi}'$, χ_n and $\nabla \Pi_N q$. In addition, applying the operator div to (17), we get

$$\operatorname{div} \boldsymbol{\varphi} = \nu \operatorname{div} \boldsymbol{\chi} - (1 + \lambda\mu) q = -(1 + \lambda(\mu + \nu)) q,$$

which yieds $q = -\frac{1}{1+\lambda(\mu+\nu)} \operatorname{div} \boldsymbol{\varphi}$ and, by substitution in (17),

$$\boldsymbol{\chi} = \frac{1}{\nu} \left(\boldsymbol{\varphi} - \frac{1 + \lambda\mu}{1 + \lambda(\mu + \nu)} \nabla \Pi_N \operatorname{div} \boldsymbol{\varphi} \right).$$

Hence, integrating the constant $\frac{1}{1+\lambda(\mu+\nu)}$ in $\boldsymbol{\varphi}$, we get the equations (16).

Conversely, we can verify that such a pair $(\boldsymbol{\chi}, q)$ belongs to ${}^+\mathscr{S}^\lambda_{[1-\ell-n/p]}$. □

Another look at the kernel. As in the whole space, we can readily see that if $\lambda \neq 0$, (1) in the half-space is equivalent to the equation of linear elasticity (15) combined with the Navier boundary conditions, i.e.,

$$\begin{cases} \mathrm{L}\boldsymbol{u} = \boldsymbol{F} & \text{in } \mathbf{R}_+^n, \\ u_n = g_n & \text{on } \Gamma, \\ \partial_n \boldsymbol{u}' = \boldsymbol{g}' & \text{on } \Gamma. \end{cases} \tag{18}$$

So, the kernel of the operator associated to this problem is the polynomial space

$$^+\mathscr{L}_{[1-\ell-n/p]} = \{\boldsymbol{\chi} \in \mathscr{P}_{[1-\ell-n/p]}^{\Delta^2}; \mathrm{L}\boldsymbol{\chi} = \mathbf{0} \ \text{ in } \ \mathbf{R}_+^n, \partial_n \boldsymbol{\chi}' = \mathbf{0} \ \text{ and } \ \chi_n = 0 \ \text{ on } \ \Gamma\}.$$

Likewise, we have the following characterization:

PROPOSITION 3.3. *Let $\ell \in \mathbf{Z}$. The polynomial $\boldsymbol{\chi} \in {}^+\mathscr{L}_{[1-\ell-n/p]}$ if and only if there exists $\boldsymbol{\varphi} \in \mathscr{N}_{[1-\ell-n/p]}^{\Delta} \times \mathscr{A}_{[1-\ell-n/p]}^{\Delta}$ such that $\boldsymbol{\chi} = \kappa_2 \boldsymbol{\varphi} - \kappa_1 \nabla \Pi_N \operatorname{div} \boldsymbol{\varphi}$.*

3.2. The compatibility condition. We notice that for $\ell > 0$ the kernel reduces to $\{0\}$ and in turn there appears a compatibility condition among the data $\boldsymbol{f}$, h, g_n, $\boldsymbol{g}'$. Let $(\boldsymbol{u}, \pi)$ be a solution to (1), then by means of Green's formula, we get after simplification:

$$\forall (\boldsymbol{\chi}, q) \in {}^+\mathscr{S}_{[1+\ell-n/p']}^{\lambda},$$

$$\int_{\mathbf{R}_+^n} (-\nu \Delta \boldsymbol{u} - \mu \nabla \operatorname{div} \boldsymbol{u} + \nabla \pi) \cdot \boldsymbol{\chi} \, \mathrm{d}x - \int_{\mathbf{R}_+^n} (\lambda \pi + \operatorname{div} \boldsymbol{u}) q \, \mathrm{d}x$$

$$= -\nu \int_\Gamma u_n \partial_n \chi_n \, \mathrm{d}x' + \nu \langle \partial_n \boldsymbol{u}', \boldsymbol{\chi}' \rangle_{\boldsymbol{W}_\ell^{-1/p,p}(\Gamma) \times \boldsymbol{W}_{-\ell}^{1-1/p',p'}(\Gamma)}$$

$$- \mu \int_\Gamma u_n \operatorname{div} \boldsymbol{\chi} \, \mathrm{d}x' + \int_\Gamma u_n q \, \mathrm{d}x'.$$

Hence we obtain a first formulation of the compatibility condition:

$$\forall (\boldsymbol{\chi}, q) \in {}^+\mathscr{S}_{[1+\ell-n/p']}^{\lambda},$$

$$\int_{\mathbf{R}_+^n} \boldsymbol{f} \cdot \boldsymbol{\chi} \, \mathrm{d}x - \int_{\mathbf{R}_+^n} hq \, \mathrm{d}x$$

$$= \int_\Gamma g_n(-\nu \partial_n \chi_n - \mu \operatorname{div} \boldsymbol{\chi} + q) \mathrm{d}x' + \langle \boldsymbol{g}', \nu \boldsymbol{\chi}' \rangle_{\boldsymbol{W}_\ell^{-1/p,p}(\Gamma) \times \boldsymbol{W}_{-\ell}^{1-1/p',p'}(\Gamma)}. \tag{19}$$

Now, in order to use Proposition 3.2, we can observe that

$$\int_{\mathbf{R}_+^n} \boldsymbol{f} \cdot (\nabla \Pi_N \operatorname{div} \boldsymbol{\varphi}) \mathrm{d}x = \langle -\operatorname{div} \boldsymbol{f}, \Pi_N \operatorname{div} \boldsymbol{\varphi} \rangle_{W_{\ell+1}^{-1,p}(\mathbf{R}_+^n) \times \mathring{W}_{-\ell-1}^{1,p'}(\mathbf{R}_+^n)}$$

and

$$\int_{\mathbf{R}^n_+} h \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x = -\int_{\mathbf{R}^n_+} \nabla h \cdot \boldsymbol{\varphi} \, \mathrm{d}x.$$

On the other hand, for the trace terms we have

$$\boldsymbol{\chi}' = \kappa_2 \boldsymbol{\varphi}' \qquad \text{and} \qquad \nu \partial_n \chi_n + \mu \operatorname{div} \boldsymbol{\chi} - q = \nu \kappa_2 \partial_n \varphi_n \qquad \text{on } \Gamma.$$

This yields the second formulation for the compatibility condition:

$$\forall \boldsymbol{\varphi} \in \mathscr{N}^{\Delta}_{[1+\ell-n/p']} \times \mathscr{A}^{\Delta}_{[1+\ell-n/p']},$$

$$\int_{\mathbf{R}^n_+} \left(\boldsymbol{f} - \frac{1}{\kappa_2} \nabla h \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \frac{\kappa_1}{\kappa_2} \langle \operatorname{div} \boldsymbol{f}, \Pi_N \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}_{\ell+1}(\mathbf{R}^n_+) \times \mathring{W}^{1,p'}_{-\ell-1}(\mathbf{R}^n_+)}$$
$$+ \nu \int_{\Gamma} g_n \partial_n \varphi_n \, \mathrm{d}x' - \nu \langle \boldsymbol{g}', \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma) \times \boldsymbol{W}^{1-1/p',p'}_{-\ell}(\Gamma)} = 0. \tag{20}$$

The linear elasticity system. For problem (18), the conditions (19) and (20) are respectively equivalent to

$$\forall \boldsymbol{\chi} \in {}^{+}\mathscr{L}_{[1+\ell-n/p']}, \qquad \int_{\mathbf{R}^n_+} \boldsymbol{F} \cdot \boldsymbol{\chi} \, \mathrm{d}x = \int_{\Gamma} g_n \left(-\nu \partial_n \chi_n - \left(\mu + \frac{1}{\lambda} \right) \operatorname{div} \boldsymbol{\chi} \right) \mathrm{d}x'$$
$$+ \langle \boldsymbol{g}', \nu \boldsymbol{\chi}' \rangle_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma) \times \boldsymbol{W}^{1-1/p',p'}_{-\ell}(\Gamma)} \tag{21}$$

and

$$\forall \boldsymbol{\varphi} \in \mathscr{N}^{\Delta}_{[1+\ell-n/p']} \times \mathscr{A}^{\Delta}_{[1+\ell-n/p']},$$

$$\int_{\mathbf{R}^n_+} \boldsymbol{F} \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \frac{\kappa_1}{\kappa_2} \langle \operatorname{div} \boldsymbol{F}, \Pi_N \operatorname{div} \boldsymbol{\varphi} \rangle_{W^{-1,p}_{\ell+1}(\mathbf{R}^n_+) \times \mathring{W}^{1,p'}_{-\ell-1}(\mathbf{R}^n_+)}$$
$$+ \nu \int_{\Gamma} g_n \partial_n \varphi_n \, \mathrm{d}x' - \nu \langle \boldsymbol{g}', \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma) \times \boldsymbol{W}^{1-1/p',p'}_{-\ell}(\Gamma)} = 0. \tag{22}$$

3.3. Strong solutions. In this case it is convenient to solve the linear elasticity system in order to get the solutions of the generalized Stokes system. We start with homogeneous boundary conditions.

THEOREM 3.4. *Let $\ell \in \mathbf{Z}$ and assume that*

$$n/p' \notin \{1, \ldots, \ell + 1\} \qquad \text{and} \qquad n/p \notin \{1, \ldots, -\ell - 1\}. \tag{23}$$

For any $\boldsymbol{F} \in \boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+) \perp {}^{+}\mathscr{L}_{[1+\ell-n/p']}$, the linear elasticity system (18), *with $(\boldsymbol{g}', g_n) = \mathbf{0}$, has a solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)$, unique up to an element of ${}^{+}\mathscr{L}_{[1-\ell-n/p]}$, with the corresponding estimate.*

Proof. First, we extend $\boldsymbol{F}$ to the whole space by $\tilde{\boldsymbol{F}} \in \boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n)$ as follows.

$$\forall \boldsymbol{\varphi} \in \mathscr{D}(\mathbf{R}^n), \qquad \int_{\mathbf{R}^n} \tilde{\boldsymbol{F}} \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_{\mathbf{R}^n_+} \boldsymbol{F} \cdot (\boldsymbol{\varphi}' + \boldsymbol{\varphi}'^*, \varphi_n - \varphi_n^*) \mathrm{d}x, \tag{24}$$

where $\varphi_i^*(x) = \varphi_i(x^*)$ for any $x = (x', x_n) \in \mathbf{R}^n$ with $x^* = (x', -x_n)$. That is to say, the functional expression of this extension is given by

$$\tilde{\boldsymbol{F}}(x', x_n) = \begin{cases} \boldsymbol{F}(x', x_n) & \text{if } x_n > 0, \\ (\boldsymbol{F}', -F_n)(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Next, by Theorem 2.6, with $m = 1$ and hypothesis (23), we know that there exists a solution $\boldsymbol{w} \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n)$ to the problem $\mathrm{L}\boldsymbol{w} = \tilde{\boldsymbol{F}}$ in $\mathbf{R}^n$, provided the condition $\tilde{\boldsymbol{F}} \perp \mathscr{L}_{[1+\ell-n/p']}$ is fulfilled. According to (24), we can write this last condition as

$$\forall \boldsymbol{\chi} \in \mathscr{L}_{[1+\ell-n/p']}, \qquad \int_{\mathbf{R}^n_+} \boldsymbol{F} \cdot (\boldsymbol{\chi}' + \boldsymbol{\chi}'^*, \chi_n - \chi_n^*) \mathrm{d}x = 0.$$

Now, thanks to $(\boldsymbol{\chi}' + \boldsymbol{\chi}'^*, \chi_n - \chi_n^*) \in {}^+\mathscr{L}_{[1+\ell-n/p']}$, we see that the last condition is a simple consequence of the condition on $\boldsymbol{F}$ in our statement. Then, the function $\boldsymbol{u}$ defined in $\mathbf{R}^n_+$ by

$$\boldsymbol{u} = \frac{1}{2}(\boldsymbol{w}' + \boldsymbol{w}'^*, w_n - w_n^*)$$

belongs to $\boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)$ and we can see by a straightforward calculation that it is solution to our problem. □

Remark 3.5. In the same way, we can directly demonstrate the counterpart of Theorem 3.4 for Problem (1) with $(\boldsymbol{g}', g_n) = \mathbf{0}$. In this case, the extension of $\boldsymbol{f}$ is identical to the one for $\boldsymbol{F}$ in the linear elasticity system; and it is even with respect to x_n for h. That is,

$$\forall \psi \in \mathscr{D}(\mathbf{R}^n), \qquad \int_{\mathbf{R}^n} \tilde{h} \psi \, \mathrm{d}x = \int_{\mathbf{R}^n_+} h(\psi + \psi^*) \mathrm{d}x.$$

Then a solution to (1) is given by the pair of functions $(\boldsymbol{u}, \pi)$ defined in $\mathbf{R}^n_+$ by

$$(\boldsymbol{u}, \pi) = \frac{1}{2}(\boldsymbol{w}' + \boldsymbol{w}'^*, w_n - w_n^*, \vartheta + \vartheta^*),$$

where $(\boldsymbol{w}, \vartheta)$ is the solution to the extended problem in $\mathbf{R}^n$.

Corollary 3.6. *Let $\ell \in \mathbf{Z}$ and assume* (23). *For any $\boldsymbol{F} \in \boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)$, $g_n \in W^{2-1/p,p}_{\ell+1}(\Gamma)$ and $\boldsymbol{g}' \in \boldsymbol{W}^{1-1/p,p}_{\ell+1}(\Gamma)$, satisfying the compatibility condition* (22)*, the linear elasticity system* (18) *has a solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)$, unique up to an element of ${}^{+}\mathscr{L}_{[1-\ell-n/p]}$, with the corresponding estimate.*

Proof. It is a consequence of Theorem 3.4 and Lemma 1.1. Indeed, according to Lemma 1.1, there exists a lifting function $\boldsymbol{u_g} = (\boldsymbol{u}_{\boldsymbol{g}'}, u_{g_n}) \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)$ of $\boldsymbol{g} = (\boldsymbol{g}', g_n)$ such that $\partial_n \boldsymbol{u}_{\boldsymbol{g}'} = \boldsymbol{g}'$ and $u_{g_n} = g_n$ on Γ. Then, if we put $\mathfrak{F} = \boldsymbol{F} - \mathrm{L}\boldsymbol{u_g}$ and $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u_g}$, system (18) is equivalent to

$$\mathrm{L}\boldsymbol{v} = \mathfrak{F} \text{ in } \mathbf{R}^n_+, \qquad (\partial_n \boldsymbol{v}', v_n) = \mathbf{0} \text{ in } \Gamma$$

and we can easily verify that condition (22)—more precisely its alternative form (21)—is equivalent to $\mathfrak{F} \perp {}^{+}\mathscr{L}_{[1+\ell-n/p']}$. Finally, this problem is solved by Theorem 3.4. □

Hence the corresponding result for the generalized Stokes system is given as follows.

Theorem 3.7. *Let $\ell \in \mathbf{Z}$ and assume* (23). *For any $\boldsymbol{f} \in \boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)$, $h \in W^{1,p}_{\ell+1}(\mathbf{R}^n_+)$, $g_n \in W^{2-1/p,p}_{\ell+1}(\Gamma)$ and $\boldsymbol{g}' \in \boldsymbol{W}^{1-1/p,p}_{\ell+1}(\Gamma)$, satisfying the compatibility condition* (20)*, problem* (1) *has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+) \times W^{1,p}_{\ell+1}(\mathbf{R}^n_+)$, unique up to an element of ${}^{+}\mathscr{S}^{\lambda}_{[1-\ell-n/p]}$, with the estimate*

$$\inf_{(\boldsymbol{\chi}, q) \in {}^{+}\mathscr{S}^{\lambda}_{[1-\ell-n/p]}} (\|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)} + \|\pi + q\|_{W^{1,p}_{\ell+1}(\mathbf{R}^n_+)})$$
$$\leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)} + \|h\|_{W^{1,p}_{\ell+1}(\mathbf{R}^n_+)} + \|g_n\|_{W^{2-1/p,p}_{\ell+1}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{1-1/p,p}_{\ell+1}(\Gamma)}).$$

3.4. Generalized solutions. We start with the homogeneous problem. Unfortunately, Lemma 1.1 does not yield any lifting function for such boundary conditions—we will see how to bypass this difficulty for very weak solutions in the last section. Here, we intend to directly show the existence of generalized solutions to the homogeneous Stokes system in the half-space. The method is directly adapted from the one introduced for the classic Stokes system in [9].

Proposition 3.8. *Let $\ell \in \mathbf{Z}$ and assume* (7). *For any $g_n \in W^{1-1/p,p}_{\ell}(\Gamma)$ and $\boldsymbol{g}' \in \boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma)$, satisfying the compatibility condition*

$$\forall \boldsymbol{\varphi} \in \mathscr{N}^{\Delta}_{[1+\ell-n/p']} \times \mathscr{A}^{\Delta}_{[1+\ell-n/p']},$$
$$\int_{\Gamma} g_n \partial_n \varphi_n \, \mathrm{d}x' - \langle \boldsymbol{g}', \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma) \times \boldsymbol{W}^{1-1/p',p'}_{-\ell}(\Gamma)} = 0, \tag{25}$$

the Stokes problem

$$-\nu\Delta \boldsymbol{u} - \mu\nabla \operatorname{div} \boldsymbol{u} + \nabla\pi = \boldsymbol{0} \qquad \text{in } \mathbf{R}^n_+, \tag{26a}$$

$$\lambda\pi + \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbf{R}^n_+, \tag{26b}$$

$$u_n = g_n \qquad \text{on } \Gamma, \tag{26c}$$

$$\partial_n \boldsymbol{u}' = \boldsymbol{g}' \qquad \text{on } \Gamma, \tag{26d}$$

has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_\ell(\mathbf{R}^n_+) \times W^{0,p}_\ell(\mathbf{R}^n_+)$, *unique up to an element of* ${}^+\mathscr{S}^\lambda_{[1-\ell-n/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\chi}, q) \in {}^+\mathscr{S}^\lambda_{[1-\ell-n/p]}} (\|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}^{1,p}_\ell(\mathbf{R}^n_+)} + \|\pi + q\|_{W^{0,p}_\ell(\mathbf{R}^n_+)})$$
$$\leq C(\|g_n\|_{W^{1-1/p,p}_\ell(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1/p,p}_\ell(\Gamma)}).$$

Proof. First, let us notice a particular case, which is naturally included in this result, but which requires a particular treatment. Indeed, if $\lambda\mu = -1$, we simply get a Dirichlet problem for the Laplacian on the normal component of the velocity field u_n and a Neumann problem on its tangential components $\boldsymbol{u}'$. Then, applying the results of [5] and [6]—which are recalled in [7]—, respectively for u_n and $\boldsymbol{u}'$, we find the orthogonality condition and the kernel of our statement. Moreover, we directly find the pressure from the velocity field thanks to the second equation. In the sequel of the proof, we will assume that $\lambda\mu \neq -1$.

(i) Reduction of system (26).

As for the question of the uniqueness in the whole space—see the start of Subsection 2.1—, we deduce from (26a) and (26b) that we have both $\Delta\pi = 0$ and $\Delta^2\boldsymbol{u} = \boldsymbol{0}$ in $\mathbf{R}^n_+$.

Then, we have $\Delta^2 u_n = 0$ in $\mathbf{R}^n_+$ and $u_n = g_n$ on Γ.

Now, let us extract another boundary condition on Δu_n from this system. From (26b), we get

$$\lambda\partial_n\pi + \partial_n \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbf{R}^n_+, \tag{27}$$

that we substitute in the n^{th} component of (26a), to obtain

$$\lambda\nu\Delta u_n + (1 + \lambda\mu)\partial_n \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbf{R}^n_+.$$

We can deduce that

$$\lambda\nu\Delta u_n + (1 + \lambda\mu)(\operatorname{div}' \boldsymbol{g}' + \partial_n^2 u_n) = 0 \qquad \text{on } \Gamma,$$

$$\lambda\nu\Delta u_n + (1 + \lambda\mu)(\operatorname{div}' \boldsymbol{g}' + \Delta u_n - \Delta' u_n) = 0 \qquad \text{on } \Gamma,$$

$$(1 + \lambda(\mu + \nu))\Delta u_n + (1 + \lambda\mu)(\operatorname{div}' \boldsymbol{g}' - \Delta' g_n) = 0 \qquad \text{on } \Gamma,$$

hence,

$$\Delta u_n = \frac{1+\lambda\mu}{1+\lambda(\mu+\nu)}(\Delta' g_n - \operatorname{div}' \boldsymbol{g}') \qquad \text{on } \Gamma.$$

About the pressure, looking again at the n^{th} component of (26a), with (27), we have

$$\partial_n \pi = \nu \Delta u_n - \lambda\mu\partial_n\pi \qquad \text{in } \mathbf{R}^n_+,$$

hence (since $\lambda\mu \neq -1$),

$$\partial_n \pi = \frac{\nu}{1+\lambda\mu}\Delta u_n \qquad \text{on } \Gamma.$$

Finally, from (26b), we also get

$$\lambda\nabla'\pi + \nabla' \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in } \mathbf{R}^n_+,$$

that we substitute in the tangential components of (26a), to obtain

$$\Delta \boldsymbol{u}' = \frac{1+\lambda\mu}{\nu}\nabla'\pi \qquad \text{in } \mathbf{R}^n_+.$$

So, thanks to the two constants $\kappa_1 = \frac{1+\lambda\mu}{\nu}$ and $\kappa_2 = \frac{1+\lambda(\mu+\nu)}{\nu}$ introduced above, this yields the following three problems

$$\Delta^2 u_n = 0 \text{ in } \mathbf{R}^n_+, \qquad u_n = g_n \qquad \text{and} \qquad \Delta u_n = \frac{\kappa_1}{\kappa_2}(\Delta' g_n - \operatorname{div}' \boldsymbol{g}') \text{ on } \Gamma, \tag{28}$$

$$\Delta \pi = 0 \text{ in } \mathbf{R}^n_+, \qquad \partial_n \pi = \frac{1}{\kappa_1}\Delta u_n \text{ on } \Gamma, \tag{29}$$

$$\Delta \boldsymbol{u}' = \kappa_1 \nabla'\pi \text{ in } \mathbf{R}^n_+, \qquad \partial_n \boldsymbol{u}' = \boldsymbol{g}' \text{ on } \Gamma. \tag{30}$$

(ii) Solution of these three problems.

Step 1: We deal with problem (28). Denoting $z_n = \Delta u_n$, we can split it in the following two Dirichlet problems:

$$\Delta z_n = 0 \text{ in } \mathbf{R}^n_+, \qquad z_n = \frac{\kappa_1}{\kappa_2}(\Delta' g_n - \operatorname{div}' \boldsymbol{g}') \text{ on } \Gamma, \tag{31}$$

$$\Delta u_n = z_n \text{ in } \mathbf{R}^n_+, \qquad u_n = g_n \text{ on } \Gamma. \tag{32}$$

Concerning (31), we notice that $\Delta' g_n - \operatorname{div}' \boldsymbol{g}' \in W^{-1-1/p,p}_{\ell}(\Gamma)$, then we can apply the result on the singular boundary conditions for the homogeneous Dirichlet problem (see [8, Theorem 3.5]), provided the following orthogonality condition is satisfied:

$$\forall \psi \in \mathscr{A}^{\Delta}_{[3+\ell-n/p']}, \qquad \langle \Delta' g_n - \operatorname{div}' \boldsymbol{g}', \partial_n \psi \rangle_{\boldsymbol{W}^{-1-1/p,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{2-1/p',p'}_{-\ell}(\Gamma)} = 0. \tag{33}$$

By means of Green's formulae, we can rewrite it

$$\forall \psi \in \mathscr{A}^{\Delta}_{[3+\ell-n/p']}, \qquad \int_{\Gamma} g_n \partial_n \Delta' \psi \, \mathrm{d}x' + \langle \boldsymbol{g}', \partial_n \nabla' \psi \rangle_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma) \times \boldsymbol{W}^{1-1/p',p'}_{-\ell}(\Gamma)} = 0.$$

Now, to see that this follows from (25), it suffices to remark that

$$\forall \psi \in \mathscr{A}^{\Delta}_{[3+\ell-n/p']}, \qquad \Delta' \psi \in \mathscr{A}^{\Delta}_{[1+\ell-n/p']} \qquad \text{and} \qquad \partial_n \nabla' \psi \in \mathscr{N}^{\Delta}_{[1+\ell-n/p']}.$$

So, we get a solution $z_n \in W^{-1,p}_{\ell}(\mathbf{R}^n_+)/\mathscr{A}^{\Delta}_{[-1-\ell-n/p]}$ to (31).

For (32), the compatibility condition is

$$\forall \psi \in \mathscr{A}^{\Delta}_{[1+\ell-n/p']}, \qquad \langle z_n, \psi \rangle_{W^{-1,p}_{\ell}(\mathbf{R}^n_+) \times \mathring{W}^{1,p'}_{-\ell}(\mathbf{R}^n_+)} = \int_{\Gamma} g_n \partial_n \psi \, \mathrm{d}x'. \tag{34}$$

(see [7, Theorem 2.5]). First, (25) implies $\int_{\Gamma} g_n \partial_n \psi \, \mathrm{d}x' = 0$ for any $\psi \in \mathscr{A}^{\Delta}_{[1+\ell-n/p']}$. It remains to show that the left-hand term of (34) is also zero. For this, we need to express $\mathscr{A}^{\Delta}_{[1+\ell-n/p']}$ by means of the kernel, denoted by $\mathscr{B}_k$, of the biharmonic operator—that is the space of polynomials ζ of degree less than or equal to k, such that $\Delta^2 \zeta = 0$ in $\mathbf{R}^n_+$ and $\zeta = \partial_n \zeta = 0$ on Γ. We showed in [7, Lemma 4.4] that

$$\forall k \in \mathbf{Z}, \qquad \mathscr{B}_{k+2} = \Pi_D \mathscr{A}^{\Delta}_k \oplus \Pi_N \mathscr{N}^{\Delta}_k, \tag{35}$$

where Π_D is defined as follows.

$$\forall r \in \mathscr{A}^{\Delta}_k, \qquad \Pi_D r(x', x_n) = \frac{1}{2} \int_0^{x_n} t r(x', t) \mathrm{d}t,$$

satisfies $\Delta \Pi_D r = r$ in $\mathbf{R}^n_+$ and $\Pi_D r = \partial_n \Pi_D r = 0$ on Γ.

From (35), we get for any $\psi \in \mathscr{A}^{\Delta}_{[1+\ell-n/p']}$, $\Pi_D \psi = \zeta \in \mathscr{B}_{[3+\ell-n/p']}$ and thus we have $\psi = \Delta \zeta$. So, by means of a Green's formula (see [8, Lemma 3.7] for the justification), we can write

$$\forall \psi \in \mathscr{A}^{\Delta}_{[1+\ell-n/p']}, \qquad \exists \zeta \in \mathscr{B}_{[3+\ell-n/p']} \text{ such that}$$

$$\langle z_n, \psi \rangle_{W^{-1,p}_{\ell}(\mathbf{R}^n_+) \times \mathring{W}^{1,p'}_{-\ell}(\mathbf{R}^n_+)} = \langle z_n, \Delta \zeta \rangle = \langle \Delta z_n, \zeta \rangle = 0.$$

So (34) is proved and we get a solution $u_n \in W^{1,p}_{\ell}(\mathbf{R}^n_+)/\mathscr{A}^{\Delta}_{[1-\ell-n/p]}$ to (32).

Step 2: Next we study problem (29). According to [8, Lemma 3.7], we can check that $\Delta u_n \in W^{-1-1/p,p}_{\ell}(\Gamma)$ and the compatibility condition is

$$\forall \psi \in \mathscr{N}^{\Delta}_{[2+\ell-n/p']}, \qquad \langle \Delta u_n, \psi \rangle_{W^{-1-1/p,p}_{\ell}(\Gamma) \times W^{2-1/p',p'}_{-\ell}(\Gamma)} = 0 \tag{36}$$

(see [8, Theorem 3.3]). For any $\psi \in \mathscr{N}^{\Delta}_{[2+\ell-n/p']}$, if we put $\zeta = \int_0^{x_n} \psi(x', t) \mathrm{d}t$, this yields $\psi = \partial_n \zeta$ with $\zeta \in \mathscr{A}^{\Delta}_{[3+\ell-n/p']}$. Since $\Delta u_n = \Delta' g_n - \operatorname{div}' \boldsymbol{g}'$ on Γ, we see that (36) is exactly written as the condition (33), which is satisfied.

So, we get a solution $\pi \in W_\ell^{0,p}(\mathbf{R}_+^n)/\mathscr{N}_{[-\ell-n/p]}^{\Delta}$ to (29).

Step 3: Finally, to treat problem (30), we can split it in

$$\Delta \boldsymbol{v}' = \kappa_1 \nabla' \pi \ \text{ in } \mathbf{R}_+^n, \qquad \partial_n \boldsymbol{v}' = \mathbf{0} \ \text{ on } \Gamma, \tag{37}$$

$$\Delta \boldsymbol{z}' = \mathbf{0} \ \text{ in } \mathbf{R}_+^n, \qquad \partial_n \boldsymbol{z}' = \boldsymbol{g}' \ \text{ on } \Gamma; \tag{38}$$

and in order to solve (37), we introduce the auxiliary problem

$$\Delta w = \kappa_1 \pi \ \text{ in } \mathbf{R}_+^n, \qquad \partial_n w = 0 \ \text{ on } \Gamma. \tag{39}$$

For (39), the compatibility condition is

$$\forall \psi \in \mathscr{N}_{[\ell-n/p']}^{\Delta}, \qquad \int_{\mathbf{R}_+^n} \pi \psi \, \mathrm{d}x = 0 \tag{40}$$

(see [6, Theorem 3.1]). According to (35), we also have for any $\psi \in \mathscr{N}_{[\ell-n/p']}^{\Delta}$, $\Pi_N \psi = \zeta \in \mathscr{B}_{[2+\ell-n/p']}$ and thus $\psi = \Delta\zeta$. So, we have

$$\forall \psi \in \mathscr{N}_{[\ell-n/p']}^{\Delta}, \qquad \exists \zeta \in \mathscr{B}_{[2+\ell-n/p']} \ \text{such that}$$

$$\int_{\mathbf{R}_+^n} \pi \psi \, \mathrm{d}x = \int_{\mathbf{R}_+^n} \pi \Delta \zeta \, \mathrm{d}x = \langle \Delta \pi, \zeta \rangle = 0.$$

Thus (40) is proved and we get a solution $w \in W_\ell^{2,p}(\mathbf{R}_+^n)/\mathscr{N}_{[2-\ell-n/p]}^{\Delta}$ to (39). Consequently, $\boldsymbol{v}' = \nabla' w \in \boldsymbol{W}_\ell^{1,p}(\mathbf{R}_+^n)/\mathscr{N}_{[1-\ell-n/p]}^{\Delta}$ is a solution to problem (37).

Finally, for problem (38), the compatibility condition is

$$\forall \boldsymbol{\varphi}' \in \mathscr{N}_{[1+\ell-n/p']}^{\Delta}, \qquad \langle \boldsymbol{g}', \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}_\ell^{-1/p,p}(\Gamma) \times \boldsymbol{W}_{-\ell}^{1-1/p',p'}(\Gamma)} = 0 \tag{41}$$

(see [8, Theorem 3.4]). It is clear that (41) is included in (25) and then we get a solution $\boldsymbol{z}' \in \boldsymbol{W}_\ell^{1,p}(\mathbf{R}_+^n)/\mathscr{N}_{[1-\ell-n/p]}^{\Delta}$ to (38).

So $\boldsymbol{u}' = \boldsymbol{v}' + \boldsymbol{z}' \in \boldsymbol{W}_\ell^{1,p}(\mathbf{R}_+^n)/\mathscr{N}_{[1-\ell-n/p]}^{\Delta}$ is a solution to (30).

(iii) Conversely, solving (28), (29) and (30) yields a solution $(\boldsymbol{u}, \pi)$ to the original problem (26). So, we must show that $(\boldsymbol{u}, \pi)$ satisfies the n^{th} component of (26a) and (26b). The idea is based on the nonuniqueness of the solutions u_n, π and $\boldsymbol{u}'$ constructed in (ii), to select a "good one".

The first equation of (30) is written

$$-\nu \Delta \boldsymbol{u}' + (1 + \lambda\mu) \nabla' \pi = 0 \qquad \text{in } \mathbf{R}_+^n. \tag{42}$$

Thanks to the first equations of (28) and (29), we get

$$\Delta(-\nu \Delta u_n + (1 + \lambda\mu) \partial_n \pi) = 0 \qquad \text{in } \mathbf{R}_+^n.$$

In addition, the boundary condition of (29) can be written

$$-\nu\Delta u_n + (1+\lambda\mu)\partial_n\pi = 0 \qquad \text{on } \Gamma.$$

Since $-\nu\Delta u_n + (1+\lambda\mu)\partial_n\pi \in W_\ell^{-1,p}(\mathbf{R}_+^n)$, according to [8, Theorem 3.5], we get $-\nu\Delta u_n + (1+\lambda\mu)\partial_n\pi \in \mathscr{A}_{[-1-\ell-n/p]}^{\Delta}$. As π is defined up to an element of $\mathscr{N}_{[-\ell-n/p]}^{\Delta}$, $\partial_n\pi$ is defined up to an element of $\mathscr{A}_{[-1-\ell-n/p]}^{\Delta}$ and thus we can choose π such that

$$-\nu\Delta u_n + (1+\lambda\mu)\partial_n\pi = 0 \qquad \text{in } \mathbf{R}_+^n. \tag{43}$$

The boundary condition of (30) implies $\partial_n \operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ. Besides, the boundary conditions of (28) yield

$$\frac{\kappa_2}{\kappa_1}\Delta u_n - \Delta' g_n = -\operatorname{div}' \boldsymbol{g}' \qquad \text{on } \Gamma,$$

$$\partial_n^2 u_n + \frac{\lambda}{\kappa_1}\Delta u_n = -\operatorname{div}' \boldsymbol{g}' \qquad \text{on } \Gamma;$$

hence, with the boundary condition of (29),

$$\partial_n^2 u_n + \lambda\partial_n\pi = -\operatorname{div}' \boldsymbol{g}' \qquad \text{on } \Gamma.$$

We can deduce

$$\begin{aligned}\partial_n \operatorname{div} \boldsymbol{u} &= \partial_n \operatorname{div}' \boldsymbol{u}' + \partial_n^2 u_n \qquad \text{on } \Gamma,\\ &= \operatorname{div}' \boldsymbol{g}' - \operatorname{div}' \boldsymbol{g}' - \lambda\partial_n\pi \qquad \text{on } \Gamma,\end{aligned}$$

that is

$$\partial_n(\lambda\pi + \operatorname{div} \boldsymbol{u}) = 0 \qquad \text{on } \Gamma.$$

Moreover, from (42) and (43), we obtain $\operatorname{div}(-\nu\Delta\boldsymbol{u}) = 0$ in $\mathbf{R}_+^n$, hence

$$\Delta(\lambda\pi + \operatorname{div} \boldsymbol{u}) = 0 \qquad \text{in } \mathbf{R}_+^n.$$

Since $\lambda\pi + \operatorname{div} \boldsymbol{u} \in W_\ell^{0,p}(\mathbf{R}_+^n)$, by [8, Theorem 3.3], we get $\lambda\pi + \operatorname{div} \boldsymbol{u} \in \mathscr{N}_{[-\ell-n/p]}^{\Delta}$. As $\boldsymbol{u}'$ is defined up to an element of $\mathscr{N}_{[1-\ell-n/p]}^{\Delta}$, $\operatorname{div}' \boldsymbol{u}'$ is defined up to an element of $\mathscr{N}_{[-\ell-n/p]}^{\Delta}$ and thus we can choose $\boldsymbol{u}'$ such that $\lambda\pi + \operatorname{div} \boldsymbol{u} = 0$ in $\mathbf{R}_+^n$, that is the equation (26b). Finally, substituting this last relation in (42) and (43), we find the first equation (26a) of this system. □

So, by equivalence, we get the version for the linear elasticity system:

Proposition 3.9. *Let $\ell \in \mathbf{Z}$ and assume* (7). *For any $g_n \in W_\ell^{1-1/p,p}(\Gamma)$ and $\boldsymbol{g}' \in \boldsymbol{W}_\ell^{-1/p,p}(\Gamma)$, satisfying the compatibility condition* (25), *the linear elasti-*

city system (18), *with* $\boldsymbol{F}=\mathbf{0}$, *has a solution* $\boldsymbol{u}\in \boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)$, *unique up to an element of* ${}^{+}\mathscr{L}_{[1-\ell-n/p]}$, *with the estimate*

$$\inf_{\chi\in {}^{+}\mathscr{L}_{[1-\ell-n/p]}} \|\boldsymbol{u}+\boldsymbol{\chi}\|_{\boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)} \le C(\|g_n\|_{W^{1-1/p,p}_{\ell}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma)}).$$

Now our task is to combine this result with Theorem 3.4 in order to get generalized solutions of the non-homogeneous linear elasticity system.

Theorem 3.10. *Let* $\ell\in\mathbf{Z}$ *and assume that*

$$n/p'\notin\{1,\dots,\ell+1\} \qquad \textit{and} \qquad n/p\notin\{1,\dots,-\ell\}. \tag{44}$$

For any $\boldsymbol{F}\in \boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)$ *and* $\boldsymbol{g}=(\boldsymbol{g}',g_n)\in \boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma)\times W^{1-1/p,p}_{\ell}(\Gamma)$, *satisfying the compatibility condition* (22), *problem* (18) *admits a solution* $\boldsymbol{u}\in \boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)$, *unique up to an element of* ${}^{+}\mathscr{L}_{[1-\ell-n/p]}$, *with the estimate*

$$\inf_{\chi\in {}^{+}\mathscr{L}_{[1-\ell-n/p]}} \|\boldsymbol{u}+\boldsymbol{\chi}\|_{\boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)} \le C(\|\boldsymbol{F}\|_{\boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)} + \|g_n\|_{W^{1-1/p,p}_{\ell}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma)}).$$

Proof. If $\frac{n}{p'}>\ell+1$, the compatibility condition is trivial and then, according to Proposition 3.9, there exists $\boldsymbol{v}\in \boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)$ satisfying

$$\mathrm{L}\boldsymbol{v}=\mathbf{0} \text{ in } \mathbf{R}^n_+, \qquad (\partial_n\boldsymbol{v}', v_n)=\boldsymbol{g} \text{ on } \Gamma.$$

In addition, by Theorem 3.4, there exists $\boldsymbol{w}\in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)$ satisfying

$$\mathrm{L}\boldsymbol{v}=\boldsymbol{F} \text{ in } \mathbf{R}^n_+, \qquad (\partial_n\boldsymbol{v}', v_n)=\mathbf{0} \text{ on } \Gamma.$$

The function $\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{w}\in \boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)$ gives the desired solution.

If $\frac{n}{p'}<\ell+1$, we cannot directly construct a solution as above because the compatibility conditions are now non-trivial. Let N be the dimension of the subspace ${}^{+}\mathscr{L}_{[1+\ell-n/p']}$ of $\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$, which is imbedded in $\boldsymbol{W}^{0,p'}_{-\ell-1}(\mathbf{R}^n_+)$ and $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_N\}$ a basis of ${}^{+}\mathscr{L}_{[1+\ell-n/p']}$. According to Hahn-Banach Theorem, there exists a family $\{\boldsymbol{e}_1^*,\dots,\boldsymbol{e}_N^*\}$ of elements of $\boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)$, which extends the dual basis of the dual space $({}^{+}\mathscr{L}_{[1+\ell-n/p']})'$. First, let us rewrite more compactly the compatibility condition (21)—which is equivalent to (22)—as:

$$\forall\boldsymbol{\chi}\in {}^{+}\mathscr{L}_{[1+\ell-n/p']}, \qquad \int_{\mathbf{R}^n_+}\boldsymbol{F}\cdot\boldsymbol{\chi}\,\mathrm{d}x = \left\langle \boldsymbol{g}, \left(\nu\boldsymbol{\chi}', -\nu\partial_n\chi_n - \left(\mu+\frac{1}{\lambda}\right)\operatorname{div}\boldsymbol{\chi}\right)\right\rangle_{\Gamma}.$$

We denote the corresponding trace mapping by

$$\begin{aligned} \tau : \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+) &\to \boldsymbol{W}^{2-1/p',p'}_{-\ell+1}(\Gamma)\times W^{1-1/p',p'}_{-\ell+1}(\Gamma)\\ \boldsymbol{\chi} &\mapsto \left(\nu\boldsymbol{\chi}', -\nu\partial_n\chi_n - \left(\mu+\frac{1}{\lambda}\right)\operatorname{div}\boldsymbol{\chi}\right) \end{aligned}$$

and $\varepsilon_i = \tau(\boldsymbol{e}_i)$. With a suitable numbering, the subfamily $\{\varepsilon_1, \dots, \varepsilon_d\}$ form a basis of the subspace $\tau({}^+\mathscr{L}_{[1+\ell-n/p']})$ of $\boldsymbol{W}^{2-1/p',p'}_{-\ell+1}(\Gamma) \times W^{1-1/p',p'}_{-\ell+1}(\Gamma) \hookrightarrow \boldsymbol{W}^{-1/p',p'}_{-\ell-1}(\Gamma) \times W^{-1-1/p',p'}_{-\ell-1}(\Gamma)$, and $\varepsilon_i = \mathbf{0}$ for $i \in \{d+1, \dots, N\}$. Here again, according to Hahn-Banach Theorem, there exists a family $\{\varepsilon_1^*, \dots, \varepsilon_d^*\}$ of elements of $\boldsymbol{W}^{1-1/p,p}_{\ell+1}(\Gamma) \times W^{2-1/p,p}_{\ell+1}(\Gamma)$ which extends the dual basis of $\{\varepsilon_1, \dots, \varepsilon_d\}$. Now, let us consider the functions defined by

$$\mathfrak{F} = \sum_{i=1}^{N} \boldsymbol{e}_i^* \langle \boldsymbol{F}, \boldsymbol{e}_i \rangle \qquad \text{and} \qquad \boldsymbol{G} = \sum_{i=1}^{d} \varepsilon_i^* \langle \boldsymbol{g}, \varepsilon_i \rangle.$$

They satisfy

$$\begin{aligned} \langle \mathfrak{F}, e_k \rangle &= \langle \boldsymbol{F}, e_k \rangle = \langle \boldsymbol{g}, \varepsilon_k \rangle && \text{for } k \in \{1, \dots, N\}, \\ \langle \mathfrak{F}, e_k \rangle &= \langle \boldsymbol{G}, \varepsilon_k \rangle = \langle \boldsymbol{g}, \varepsilon_k \rangle && \text{for } k \in \{1, \dots, d\}, \\ \langle \mathfrak{F}, e_k \rangle &= \langle \boldsymbol{G}, \varepsilon_k \rangle = 0 && \text{for } k \in \{d+1, \dots, N\}. \end{aligned}$$

By Theorem 3.4, there exists $\boldsymbol{v} \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)$ satisfying

$$\mathrm{L}\boldsymbol{v} = \boldsymbol{F} - \mathfrak{F} \ \text{ in } \ \mathbf{R}^n_+, \qquad (\partial_n \boldsymbol{v}', v_n) = \mathbf{0} \ \text{ on } \ \Gamma.$$

By Proposition 3.9, there exists $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)$ satisfying

$$\mathrm{L}\boldsymbol{w} = \mathbf{0} \ \text{ in } \ \mathbf{R}^n_+, \qquad (\partial_n \boldsymbol{w}', w_n) = \boldsymbol{g} - \boldsymbol{G} \ \text{ on } \ \Gamma.$$

By Corollary 3.6, there exists $\boldsymbol{z} \in \boldsymbol{W}^{2,p}_{\ell+1}(\mathbf{R}^n_+)$ satisfying

$$\mathrm{L}\boldsymbol{z} = \mathfrak{F} \ \text{ in } \ \mathbf{R}^n_+, \qquad (\partial_n \boldsymbol{z}', z_n) = \boldsymbol{G} \ \text{ on } \ \Gamma.$$

Finally, the function $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w} + \boldsymbol{z} \in \boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)$ gives the desired solution. □

Hence we obtain the corresponding result for the generalized Stokes system:

THEOREM 3.11. *Let $\ell \in \mathbf{Z}$ and assume* (44). *For any $\boldsymbol{f} \in \boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)$, $h \in W^{1,p}_{\ell+1}(\mathbf{R}^n_+)$, and $\boldsymbol{g} = (\boldsymbol{g}', g_n) \in \boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma) \times W^{1-1/p,p}_{\ell}(\Gamma)$, satisfying the compatibility condition* (20), *problem* (1) *has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+) \times W^{0,p}_{\ell}(\mathbf{R}^n_+)$, unique up to an element of ${}^+\mathscr{S}^{\lambda}_{[1-\ell-n/p]}$, with the estimate*

$$\begin{aligned} &\inf_{(\boldsymbol{\chi}, q) \in {}^+\mathscr{S}^{\lambda}_{[1-\ell-n/p]}} (\|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}^{1,p}_{\ell}(\mathbf{R}^n_+)} + \|\pi + q\|_{W^{0,p}_{\ell}(\mathbf{R}^n_+)}) \\ &\quad \le C(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)} + \|h\|_{W^{1,p}_{\ell+1}(\mathbf{R}^n_+)} + \|g_n\|_{W^{1-1/p,p}_{\ell}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1/p,p}_{\ell}(\Gamma)}). \end{aligned}$$

REMARK 3.12. As we mentioned in the introduction, Theorem 3.11 was established in the case $\lambda = 0$ in [9]. Now, if we consider fixed data $\boldsymbol{f}$, h, $\boldsymbol{g}'$,

g_n and a sequence of parameters $\lambda_i \in [0,1]$ which tends to zero, such that the family of generalized problems admits solutions $(\boldsymbol{u}_i, \pi_i)$ and the classic one a solution $(\boldsymbol{u}, \pi)$; then $(\boldsymbol{u}_i, \pi_i) \to (\boldsymbol{u}, \pi)$ as $\lambda_i \to 0$.

In fact, the essential part of the proof was given in Subsection 2.3 for the whole space. For the half-space, we start to lift the boundary data $\boldsymbol{g}'$ and g_n in these problems. Next, we extend the lifted problems to the whole space, and to finish, we consider the solutions as defined in Remark 3.5.

4. Very weak solutions

We now intend to explore the case of very singular boundary conditions in the homogeneous problem. We again adopt the point of view of the linear elasticity system to retrieve as a consequence the generalized Stokes system. First, we need to give a sense to traces and next to establish a Green's formula.

For any $\ell \in \mathbf{Z}$, let us introduce the spaces

$$\boldsymbol{U}_\ell(\mathbf{R}^n_+) = \{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+); \mathrm{L}\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)\},$$

$$\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+) = \{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+); \mathrm{L}\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell+1,1}(\mathbf{R}^n_+)\}.$$

They are reflexive Banach spaces equipped with their natural norms:

$$\|\boldsymbol{v}\|_{\boldsymbol{U}_\ell(\mathbf{R}^n_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+)} + \|\mathrm{L}\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)},$$

$$\|\boldsymbol{v}\|_{\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+)} + \|\mathrm{L}\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell+1,1}(\mathbf{R}^n_+)}.$$

LEMMA 4.1. *Let $\ell \in \mathbf{Z}$ and assume that*

$$n/p' \notin \{1, \dots, \ell - 1\} \qquad \text{and} \qquad n/p \notin \{1, \dots, -\ell + 1\}. \tag{45}$$

The space $\mathscr{D}(\overline{\mathbf{R}^n_+})$ is dense in $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$ and in $\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)$.

PROOF. We give the proof for $\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)$, but it is similar for the space $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$. For every continuous linear form $\boldsymbol{z} \in (\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+))'$, there exists a unique pair $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbf{R}^n_+) \times \boldsymbol{W}^{0,p'}_{-\ell-1,-1}(\mathbf{R}^n_+)$, such that

$$\forall \boldsymbol{v} \in \boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+), \qquad \langle \boldsymbol{z}, \boldsymbol{v} \rangle = \int_{\mathbf{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\mathbf{R}^n_+} \boldsymbol{g} \cdot \mathrm{L}\boldsymbol{v} \, \mathrm{d}x. \tag{46}$$

According to the Hahn-Banach theorem, it suffices to show that any $\boldsymbol{z}$ which vanishes on $\mathscr{D}(\overline{\mathbf{R}^n_+})$ is actually zero on $\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)$. Let us suppose that $\boldsymbol{z} = \mathbf{0}$ on $\mathscr{D}(\overline{\mathbf{R}^n_+})$, thus on $\mathscr{D}(\mathbf{R}^n_+)$. Then we can deduce from (46) that

$$\boldsymbol{f} + \mathrm{L}\boldsymbol{g} = \mathbf{0} \qquad \text{in } \mathbf{R}^n_+,$$

hence $\mathrm{L}\boldsymbol{g} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbf{R}^n_+)$. Let $\tilde{\boldsymbol{f}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbf{R}^n)$ and $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,p'}_{-\ell-1,-1}(\mathbf{R}^n)$ be respectively the extensions by $\mathbf{0}$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ to $\mathbf{R}^n$. Thanks to (46), it is clear that $\tilde{\boldsymbol{f}} + \mathrm{L}\tilde{\boldsymbol{g}} = \mathbf{0}$ in $\mathbf{R}^n$, and thus $\mathrm{L}\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbf{R}^n)$. Hence, according to Theorem 2.6—for equation (15)—, we can deduce that $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n)$, under hypothesis (45). Since $\tilde{\boldsymbol{g}}$ is an extension by $\mathbf{0}$, it follows that we have $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$. Then, by density of $\mathscr{D}(\mathbf{R}^n_+)$ in $\mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$, there exists a sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbf{N}} \subset \mathscr{D}(\mathbf{R}^n_+)$ such that $\boldsymbol{\varphi}_k \to \boldsymbol{g}$ in $\mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$. Thus we have, for any $\boldsymbol{v} \in \boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)$,

$$\begin{aligned}
\langle \boldsymbol{z}, \boldsymbol{v}\rangle &= -\int_{\mathbf{R}^n_+} \mathrm{L}\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\mathbf{R}^n_+} \boldsymbol{g}\cdot\mathrm{L}\boldsymbol{v}\,\mathrm{d}x \\
&= \lim_{k\to\infty}\left\{-\int_{\mathbf{R}^n_+} \mathrm{L}\boldsymbol{\varphi}_k\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\mathbf{R}^n_+} \boldsymbol{\varphi}_k\cdot\mathrm{L}\boldsymbol{v}\,\mathrm{d}x\right\} \\
&= 0,
\end{aligned}$$

i.e., $\boldsymbol{z}$ is identically zero. □

Thanks to this density lemma, we can prove the following result of traces:

Lemma 4.2. *Let* $\ell \in \mathbf{Z}$ *and assume* (45).

(i) *If* $n/p' \notin \{\ell, \ell+1\}$, *then the mapping*

$$(\gamma_{e_n}, \gamma'_1) : \mathscr{D}(\overline{\mathbf{R}^n_+}) \to \mathscr{D}(\mathbf{R}^{n-1})$$
$$\boldsymbol{v} \mapsto (v_n|_\Gamma, \partial_n \boldsymbol{v}'|_\Gamma),$$

can be extended to a linear continuous mapping

$$(\gamma_{e_n}, \gamma'_1) : \boldsymbol{U}_\ell(\mathbf{R}^n_+) \to W^{-1/p,p}_{\ell-1}(\Gamma) \times \boldsymbol{W}^{-1-1/p,p}_{\ell-1}(\Gamma).$$

In addition, we have the Green's formula

$$\forall \boldsymbol{v} \in \boldsymbol{U}_\ell(\mathbf{R}^n_+), \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+) \text{ such that } (\varphi_n, \partial_n\boldsymbol{\varphi}') = \mathbf{0} \text{ on } \Gamma,$$

$$\begin{aligned}
&\langle \mathrm{L}\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,p}_{\ell+1}(\mathbf{R}^n_+)\times\boldsymbol{W}^{0,p'}_{-\ell-1}(\mathbf{R}^n_+)} - \langle \boldsymbol{v}, \mathrm{L}\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+)\times\boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbf{R}^n_+)} \\
&\quad = -\nu\langle v_n, \partial_n\varphi_n\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1-1/p',p'}_{-\ell+1}(\Gamma)} + \nu\langle \partial_n\boldsymbol{v}', \boldsymbol{\varphi}'\rangle_{\boldsymbol{W}^{-1-1/p,p}_{\ell-1}(\Gamma)\times\boldsymbol{W}^{2-1/p',p'}_{-\ell+1}(\Gamma)} \\
&\qquad - \left(\mu + \frac{1}{\lambda}\right)\langle v_n, \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1-1/p',p'}_{-\ell+1}(\Gamma)}.
\end{aligned} \tag{47}$$

(ii) *If* $n/p' \in \{\ell, \ell+1\}$, *the same result holds with* $\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)$ *instead of* $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$, *where* $\langle \mathrm{L}\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,p}_{\ell+1,1}(\mathbf{R}^n_+)\times\boldsymbol{W}^{0,p'}_{-\ell-1,-1}(\mathbf{R}^n_+)}$ *replaces the first duality pairing in the Green's formula.*

Proof. (i) Case $n/p' \notin \{\ell, \ell+1\}$. So, we have $W^{2,p'}_{-\ell+1}(\mathbf{R}^n_+) \hookrightarrow W^{0,p'}_{-\ell-1}(\mathbf{R}^n_+)$, hence the following Green's formula:

$$\forall \boldsymbol{v} \in \mathscr{D}(\overline{\mathbf{R}^n_+}), \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+) \text{ such that } (\varphi_n, \partial_n \boldsymbol{\varphi}') = \mathbf{0} \text{ on } \Gamma,$$

$$\int_{\mathbf{R}^n_+} \boldsymbol{\varphi} \cdot \mathrm{L}\boldsymbol{v} \, \mathrm{d}x - \int_{\mathbf{R}^n_+} \boldsymbol{v} \cdot \mathrm{L}\boldsymbol{\varphi} \, \mathrm{d}x$$
$$= -\nu \int_\Gamma v_n \partial_n \varphi_n \, \mathrm{d}x' + \nu \int_\Gamma \partial_n \boldsymbol{v}' \cdot \boldsymbol{\varphi}' \, \mathrm{d}x' - \left(\mu + \frac{1}{\lambda}\right) \int_\Gamma v_n \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x'. \tag{48}$$

In particular, if $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$ is such that $\boldsymbol{\varphi} = \mathbf{0}$ and $\partial_n \boldsymbol{\varphi}' = \mathbf{0}$ on Γ, we have

$$\left| \int_\Gamma v_n \partial_n \varphi_n \, \mathrm{d}x' \right| \leq \frac{\lambda}{1 + \lambda(\mu + \nu)} \|\boldsymbol{v}\|_{\boldsymbol{U}_\ell(\mathbf{R}^n_+)} \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)}.$$

Let $g \in W^{1-1/p',p'}_{-\ell+1}(\Gamma)$. By Lemma 1.1, there exists a lifting function $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$ such that $\boldsymbol{\varphi} = \mathbf{0}$, $\partial_n \boldsymbol{\varphi}' = \mathbf{0}$ and $\partial_n \varphi_n = g$ on Γ, satisfying moreover

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)} \leq C \|g\|_{W^{1-1/p',p'}_{-\ell+1}(\Gamma)},$$

where C is a constant independent of $\boldsymbol{\varphi}$ and g. Then we can deduce that

$$\left| \int_\Gamma v_n g \, \mathrm{d}x' \right| \leq C \|\boldsymbol{v}\|_{\boldsymbol{U}_\ell(\mathbf{R}^n_+)} \|g\|_{W^{1-1/p',p'}_{-\ell+1}(\Gamma)},$$

and thus

$$\|v_n\|_{W^{-1/p,p}_{\ell-1}(\Gamma)} \leq C \|\boldsymbol{v}\|_{\boldsymbol{U}_\ell(\mathbf{R}^n_+)}.$$

Hence we can deduce that $\gamma_{e_n} : \boldsymbol{v} \mapsto v_n|_\Gamma$ defined on $\mathscr{D}(\overline{\mathbf{R}^n_+})$ is continuous for the norm of $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$. Since $\mathscr{D}(\overline{\mathbf{R}^n_+})$ is dense in $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$, the mapping γ_{e_n} can be extended by continuity to $\gamma_{e_n} \in L(\boldsymbol{U}_\ell(\mathbf{R}^n_+); W^{-1/p,p}_{\ell-1}(\Gamma))$.

To define the trace γ_1' on $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$, we consider now $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$ such that $\varphi_n = 0$, $\partial_n \boldsymbol{\varphi}' = \mathbf{0}$ and $\nu \partial_n \varphi_n + \left(\mu + \frac{1}{\lambda}\right) \operatorname{div} \boldsymbol{\varphi} = 0$ on Γ. Then, we have

$$\left| \int_\Gamma \partial_n \boldsymbol{v}' \cdot \boldsymbol{\varphi}' \, \mathrm{d}x' \right| \leq \frac{1}{\nu} \|\boldsymbol{v}\|_{\boldsymbol{U}_\ell(\mathbf{R}^n_+)} \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)}.$$

Let $\boldsymbol{g}' \in \boldsymbol{W}^{2-1/p',p'}_{-\ell+1}(\Gamma)$. According to Lemma 1.1, there exists $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)$ such that $\boldsymbol{\varphi}' = \boldsymbol{g}'$, $\varphi_n = 0$, $\partial_n \boldsymbol{\varphi}' = \mathbf{0}$ and $\partial_n \varphi_n = -\frac{\kappa_1}{\kappa_2} \operatorname{div}' \boldsymbol{g}'$ on Γ—see Proposition 3.2 for constants κ_1 and κ_2—, satisfying moreover

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)} \leq C \|\boldsymbol{g}\|_{W^{2-1/p',p'}_{-\ell+1}(\Gamma)},$$

where C is a constant independent of $\boldsymbol{\varphi}$ and $\boldsymbol{g}'$. Then we can deduce that

$$\left| \int_\Gamma \partial_n \boldsymbol{v}' \cdot \boldsymbol{g}' \, \mathrm{d}x' \right| \leq C \|\boldsymbol{v}\|_{\boldsymbol{U}_\ell(\mathbf{R}^n_+)} \|\boldsymbol{g}'\|_{W^{2-1/p',p'}_{-\ell+1}(\Gamma)},$$

and thus

$$\|\boldsymbol{v}'\|_{W^{1-1/p,p}_{\ell-1}(\Gamma)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{U}_\ell(\mathbf{R}^n_+)}.$$

Hence we can deduce that $\gamma'_1 : \boldsymbol{v} \mapsto \partial_n \boldsymbol{v}'|_\Gamma$ defined on $\mathscr{D}(\overline{\mathbf{R}^n_+})$ is continuous for the norm of $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$. Since $\mathscr{D}(\overline{\mathbf{R}^n_+})$ is dense in $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$, the mapping γ'_1 can be extended by continuity to $\gamma'_1 \in L(\boldsymbol{U}_\ell(\mathbf{R}^n_+); \boldsymbol{W}^{-1-1/p,p}_{\ell-1}(\Gamma))$.

To finish, we also can deduce the formula (47) from (48) by density of $\mathscr{D}(\overline{\mathbf{R}^n_+})$ in $\boldsymbol{U}_\ell(\mathbf{R}^n_+)$.

(ii) Case $n/p' \in \{\ell, \ell+1\}$. The imbedding $W^{2,p'}_{-\ell+1}(\mathbf{R}^n_+) \hookrightarrow W^{0,p'}_{-\ell-1}(\mathbf{R}^n_+)$ fails, but we have $W^{2,p'}_{-\ell+1}(\mathbf{R}^n_+) \hookrightarrow W^{0,p'}_{-\ell-1,-1}(\mathbf{R}^n_+)$. To avoid these two supplementary critical values with respect to hypothesis (45), we define the space $\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)$ with a logarithmic factor in the weight to replace the first term in (48) by the suitable duality pairing $\langle \mathrm{L}\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,p}_{\ell+1,1}(\mathbf{R}^n_+)\times \boldsymbol{W}^{0,p'}_{-\ell-1,-1}(\mathbf{R}^n_+)}$. Then, the proof is the same as (i). □

Through this lemma, our goal was to establish a suitable Green's formula in order to get a variational formulation of the homogeneous linear elasticity system with very singular boundary conditions, that is more precisely

$$\begin{aligned} \mathrm{L}\boldsymbol{u} = \mathbf{0} \qquad &\text{in } \mathbf{R}^n_+, \\ u_n = g_n \qquad \text{and} \qquad \partial_n \boldsymbol{u}' = \boldsymbol{g}' \qquad &\text{on } \Gamma, \end{aligned} \tag{49}$$

with $\boldsymbol{g} = (\boldsymbol{g}', g_n) \in \boldsymbol{W}^{-1-1/p,p}_{\ell-1}(\Gamma) \times W^{-1/p,p}_{\ell-1}(\Gamma)$.

THEOREM 4.3. *Let $\ell \in \mathbf{Z}$ and assume* (45). *For any $\boldsymbol{g}' \in \boldsymbol{W}^{-1-1/p,p}_{\ell-1}(\Gamma)$ and $g_n \in W^{-1/p,p}_{\ell-1}(\Gamma)$, satisfying the compatibility condition* (25), *the linear elasticity system* (49) *has a solution $\boldsymbol{u} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+)$, unique up to an element of ${}^+\mathscr{L}_{[1-\ell-n/p]}$, with the estimate*

$$\inf_{\boldsymbol{\chi} \in {}^+\mathscr{L}_{[1-\ell-n/p]}} \|\boldsymbol{u} + \boldsymbol{\chi}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+)} \leq C(\|g_n\|_{W^{-1/p,p}_{\ell-1}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1-1/p,p}_{\ell-1}(\Gamma)}).$$

PROOF. (i) We can observe that problem (49) is equivalent to the following variational formulation: find $\boldsymbol{u} \in \boldsymbol{U}_\ell(\mathbf{R}^n_+)$—$\boldsymbol{U}_{\ell,1}(\mathbf{R}^n_+)$ if $\frac{n}{p'} \in \{\ell, \ell+1\}$—satisfying

$$\begin{aligned} &\forall \boldsymbol{v} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbf{R}^n_+) \text{ such that } (v_n, \partial_n \boldsymbol{v}') = \mathbf{0} \text{ on } \Gamma, \\ &\langle \boldsymbol{u}, \mathrm{L}\boldsymbol{v}\rangle_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbf{R}^n_+)\times \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbf{R}^n_+)} \\ &\quad = -\langle \boldsymbol{g}', \nu \boldsymbol{v}'\rangle_{\boldsymbol{W}^{-1-1/p,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{2-1/p',p'}_{-\ell+1}(\Gamma)} \\ &\qquad + \left\langle g_n, \nu\partial_n v_n + \left(\mu + \frac{1}{\lambda}\right) \operatorname{div} \boldsymbol{v}\right\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1-1/p',p'}_{-\ell+1}(\Gamma)}. \end{aligned} \tag{50}$$

Indeed the direct implication is straightforward. Conversely, if $\boldsymbol{u}$ satisfies (50), then we have for any $\boldsymbol{\varphi} \in \mathscr{D}(\mathbf{R}_+^n)$,

$$\langle \mathrm{L}\boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\mathscr{D}'(\mathbf{R}_+^n) \times \mathscr{D}(\mathbf{R}_+^n)} = \langle \boldsymbol{u}, \mathrm{L}\boldsymbol{\varphi} \rangle_{\mathscr{D}'(\mathbf{R}_+^n) \times \mathscr{D}(\mathbf{R}_+^n)} = 0,$$

thus $\mathrm{L}\boldsymbol{u} = 0$ in $\mathbf{R}_+^n$. Moreover, by the Green's formula (47), we have

$$\begin{aligned} &\forall \boldsymbol{v} \in \boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbf{R}_+^n) \text{ such that } (v_n, \partial_n \boldsymbol{v}') = \mathbf{0} \text{ on } \Gamma, \\ &\left\langle \boldsymbol{g}, \left(-\nu \boldsymbol{v}', \nu \partial_n v_n + \left(\mu + \frac{1}{\lambda} \right) \operatorname{div} \boldsymbol{v} \right) \right\rangle_\Gamma \\ &= \left\langle (\partial_n \boldsymbol{u}', u_n), \left(-\nu \boldsymbol{v}', \nu \partial_n v_n + \left(\mu + \frac{1}{\lambda} \right) \operatorname{div} \boldsymbol{v} \right) \right\rangle_\Gamma . \end{aligned}$$

By Lemma 1.1, for any $\boldsymbol{\zeta} = (\boldsymbol{\zeta}', \zeta_n) \in \boldsymbol{W}_{-\ell+1}^{2-1/p',p'}(\Gamma) \times W_{-\ell+1}^{1-1/p',p'}(\Gamma)$, there exists $\boldsymbol{v} \in \boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbf{R}_+^n)$ such that on the one hand $(\partial_n \boldsymbol{v}', v_n) = \mathbf{0}$ on Γ and on the other hand $\left(-\nu \boldsymbol{v}', \nu \partial_n v_n + \left(\mu + \frac{1}{\lambda}\right) \operatorname{div} \boldsymbol{v}\right) = \boldsymbol{\zeta}$ on Γ. Consequently,

$$\langle (\partial_n \boldsymbol{u}', u_n) - \boldsymbol{g}, \boldsymbol{\zeta} \rangle_{\boldsymbol{W}_{\ell-1}^{-1-1/p,p}(\Gamma) \times W_{\ell-1}^{-1/p,p}(\Gamma), \boldsymbol{W}_{-\ell+1}^{2-1/p',p'}(\Gamma) \times W_{-\ell+1}^{1-1/p',p'}(\Gamma)} = 0,$$

that is $\partial_n \boldsymbol{u}' = \boldsymbol{g}'$ and $u_n = g_n$ on Γ.

(ii) Now, let us solve problem (50). According to Theorem 3.4, we know that under hypothesis (45), for all $\boldsymbol{F} \in \boldsymbol{W}_{-\ell+1}^{0,p'}(\mathbf{R}_+^n) \perp {}^+\mathscr{L}_{[1-\ell-n/p]}$, there exists a unique $\boldsymbol{v} \in \boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbf{R}_+^n)/{}^+\mathscr{L}_{[1+\ell-n/p']}$, solution to

$$\begin{gathered} \mathrm{L}\boldsymbol{v} = \boldsymbol{F} \qquad \text{in } \mathbf{R}_+^n, \\ v_n = 0 \qquad \text{and} \qquad \partial_n \boldsymbol{v}' = \mathbf{0} \qquad \text{on } \Gamma, \end{gathered}$$

with the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbf{R}_+^n)/{}^+\mathscr{L}_{[1+\ell-n/p']}} \leq C \|\boldsymbol{F}\|_{\boldsymbol{W}_{-\ell+1}^{0,p'}(\mathbf{R}_+^n)}.$$

Now, consider the linear form

$$\Xi : \boldsymbol{F} \mapsto \left\langle \boldsymbol{g}, \left(-\nu \boldsymbol{v}', \nu \partial_n v_n + \left(\mu + \frac{1}{\lambda} \right) \operatorname{div} \boldsymbol{v} \right) \right\rangle_\Gamma,$$

defined on $\boldsymbol{W}_{-\ell+1}^{0,p'}(\mathbf{R}_+^n) \perp {}^+\mathscr{L}_{[1-\ell-n/p]}$. According to (25), that we also can write

$$\forall \boldsymbol{\chi} \in {}^+\mathscr{L}_{[1+\ell-n/p']}, \qquad \left\langle \boldsymbol{g}, \left(-\nu \boldsymbol{\chi}', \nu \partial_n \chi_n + \left(\mu + \frac{1}{\lambda} \right) \operatorname{div} \boldsymbol{\chi} \right) \right\rangle_\Gamma = 0$$

—see (21)—, we have

$$|\Xi F| \le C\|v\|_{W^{2,p'}_{-\ell+1}(\mathbf{R}^n_+)/{}^+\mathscr{L}_{[1+\ell-n/p']}} \|g\|_{W^{-1-1/p,p}_{\ell-1}(\Gamma)\times W^{-1/p,p}_{\ell-1}(\Gamma)}$$

$$\le C\|F\|_{W^{0,p'}_{-\ell+1}(\mathbf{R}^n_+)} \|g\|_{W^{-1-1/p,p}_{\ell-1}(\Gamma)\times W^{-1/p,p}_{\ell-1}(\Gamma)}.$$

Hence Ξ is continuous on $W^{0,p'}_{-\ell+1}(\mathbf{R}^n_+) \perp {}^+\mathscr{L}_{[1-\ell-n/p]}$, and thanks to the Riesz representation theorem, there exists a unique u in the dual space, that is $u \in W^{0,p}_{\ell-1}(\mathbf{R}^n_+)/{}^+\mathscr{L}_{[1-\ell-n/p]}$, such that

$$\forall F \in W^{0,p'}_{-\ell+1}(\mathbf{R}^n_+) \perp {}^+\mathscr{L}_{[1-\ell-n/p]}, \qquad \Xi F = \langle u, F\rangle_{W^{0,p}_{\ell-1}(\mathbf{R}^n_+)\times W^{0,p'}_{-\ell+1}(\mathbf{R}^n_+)},$$

i.e., u satisfies (50). □

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