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An elementary proof of the rationality of the moduli space for rank 2 vector bundles on P^2

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0. Introduction. Let k be an algebraically closed field of characteristic zero and let $M(0, n)$ be the moduli space of stable vector bundles of rank 2 with chern classes $c_1 = 0$ and $c_2 = n$ on the projective plane P_k^2 over k. W. Barth showed that the function field of $M(0, n)$ over k is rational (= purely transcendental) of dimension $2n$ over a certain field F which is rational of dimension $2n - 3$ over k and hence $M(0, n)$ is a rational variety of dimension $4n - 3$ over k for all $n \geq 2$ [1]. However, M. Maruyama pointed out later that there was a gap in his proof of the rationality of the field F [4]. For an odd integer n, the rationality of $M(0, n)$ is proved by G. Ellingsrud and S. Strømme by a different method [2]. As for an even integer n, I. Naruki showed that when $n = 4$, the field F is rational over k and hence $M(0, 4)$ is a rational variety over k [5]. For even integer $n \geq 6$, many people have pointed out that the rationality of F is reduced to that of the moduli space $M_{g,hy}$ of hyperelliptic curves of genus $g = (n - 2)/2$ [3] by the descent theory of vector bundles.

However, in this paper we shall give an elementary proof of the rationality of the field F for all integers $n \geq 3$.

The author heartily thanks Professor M. Maruyama for introducing him to this subject.

1. Now we shall explain the above field F. Let $K = k(x_1, \dots, x_n, y_1, \dots, y_n)$ be a field of $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$ and let W_n be the group of semi-direct product of S_n and $H_n = \bigoplus^n (\boldsymbol{Z}/2\boldsymbol{Z})$:

$$1 \to H_n \to W_n \to S_n \to 1 ,$$

where S_n is the symmetric group of degree n which acts on H_n as permutations of direct factos.

Let $G = SL(2, k) \times W_n$ act on K as follows:

$$x_i^g = \alpha x_i + \beta y_i , \qquad y_i^g = \gamma x_i + \delta y_i \qquad \text{for} \quad g = \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix} \in SL(2, k) ,$$

$$x_i^\varepsilon = \varepsilon_i x_i , \qquad y_i^\varepsilon = \varepsilon_i y_i \qquad \text{for} \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in H_n \quad (\varepsilon_i = \pm 1) ,$$

$$x_i^\sigma = x_{\sigma(i)} , \qquad y_i^\sigma = y_{\sigma(i)} \qquad \text{for} \quad \sigma \in S_n .$$

Then we put F to be the fixed field K^G by the above action [1, 4]. We shall prove that $F = K^G$ *is rational of dimension* $2n-3$ *over* k.

2. We see that

$$K^{H_n} = k(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n),$$

where $\xi_i = y_i/x_i$ and $\eta_i = x_i^2$ $(1 \le i \le n)$. We shall find a system of generators of $K^{W_n} = k(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n)^{S_n}$. Let a_i $(1 \le i \le n)$ be the elementary symmetric polynomial of degree i in $\xi_1, \dots, \xi_n$ and for every integer m,

$$b_m = \textstyle\sum_{i=1}^n \xi_i^{m+1} \eta_i .$$

Putting $L = k(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n)$, we have the following diagram:

$$\begin{tikzcd}
L \arrow[r, dash] \arrow[d, dash] & k(\xi_1, \dots, \xi_n, b_0, \dots, b_{n-1}) \arrow[d, dash] \\
L^{S_n} \arrow[r, dash] & k(a_1, \dots, a_n, b_0, \dots, b_{n-1})
\end{tikzcd}$$

LEMMA 1. $L^{S_n} = k(a_1, \dots, a_n, b_0, \dots, b_{n-1})$.

PROOF. Let A be the $n \times n$-matrix $(\xi_j^i)_{1 \le i,j \le n}$. By definition of $b_0, \dots, b_{n-1}$, we have ${}^t(b_0, b_1, \dots, b_{n-1}) = A \cdot {}^t(\eta_1, \eta_2, \dots, \eta_n)$. Since $\det A = \xi_1 \xi_2 \cdots \xi_n \cdot \prod_{i<j} (\xi_i - \xi_j)$ is non-zero, $\eta_1, \dots, \eta_n$ are contained in the field $k(\xi_1, \dots, \xi_n, b_0, \dots, b_{n-1})$, and hence $L = k(\xi_1, \dots, \xi_n, b_0, \dots, b_{n-1})$. On the other hand,

$$n! = [L : L^{S_n}] = [k(\xi_1, \dots, \xi_n, b_0, \dots, b_{n-1}) : k(a_1, \dots, a_n, b_0, \dots, b_{n-1})] .$$

Thus we see that $L^{S_n} = k(a_1, \dots, a_n, b_0, \dots, b_{n-1})$.

The above proof shows that for any integer d, we have

$$L^{S_n} = k(a_1, \dots, a_n, b_{d+1}, b_{d+2}, \dots, b_{d+n}) . \tag{2.1}$$

We assume $n = 2s$ an even integer (for an odd n, see Remark in the final part of this paper) and let $d = -s$ in (2.1) to obtain

$$L^{S_n} = k(a_1, \dots, a_n, b_{-s+1}, \dots, b_s) = k(a_1, \dots, a_n, b_{-s}, b_{-s+1}, \dots, b_s) .$$

LEMMA 2. $\sum_{j=0}^n (-1)^j a_j b_{s-j} = 0$ $(a_0 = 1)$.

PROOF. Since a_j is the j-th elementary symmetric polynomial in $\xi_1, \dots, \xi_n$, we have the identity

$$\sum_{j=0}^{n} (-1)^j a_j \xi_i^{n-j} = a_0 \xi_i^n - a_1 \xi_i^{n-1} + \cdots + a_n = 0 .$$

Multiplying by $\xi_i^{s+1-n}\eta_i$ and summing them up from $i = 1$ to n, we obtain:

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \sum_{j=0}^{n} (-1)^j a_j \xi_i^{n-j} \xi_i^{s+1-n} \eta_i \\ &= \sum_{i=1}^{n} \sum_{j=0}^{n} (-1)^j a_j \xi_i^{s+1-j} \eta_i \\ &= \sum_{j=0}^{n} (-1)^j a_j \sum_{i=1}^{n} \xi_i^{s+1-j} \eta_i \\ &= \sum_{j=0}^{n} (-1)^j a_j b_{s-j} . \end{aligned}$$

3. Since

$$\begin{aligned} b_{-1} &= \sum_{i=1}^{n} \eta_i = \sum_{i=1}^{n} x_i^2 , \\ b_0 &= \sum_{i=1}^{n} \xi_i \eta_i = \sum_{i=1}^{n} x_i y_i , \\ b_1 &= \sum_{i=1}^{n} \xi_i^2 \eta_i = \sum_{i=1}^{n} y_i^2 , \end{aligned}$$

the action of $SL(2, k)$ on $\{b_{-1}, b_0, b_1\}$ is as follows:

$$(3.1) \qquad \begin{bmatrix} b_{-1}^g \\ b_0^g \\ b_1^g \end{bmatrix} = \begin{bmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{bmatrix} \begin{bmatrix} b_{-1} \\ b_0 \\ b_1 \end{bmatrix} \quad \text{for} \quad g = \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix} \in SL(2, k) .$$

Let N be the normalizer of the diagonal maximal torus T of $SL(2, k)$:

$$1 \to T \to N \to \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \to 1 .$$

LEMMA 3. *There is the following isomorphism*

(3.2)
$$k(a_1, \ldots, a_n, b_{-s+1}, \ldots, b_s)^{SL(2,k)} \cong k(a_1, \ldots, a_n, b_{-s+1}, \ldots, b_s)^N , \qquad b_{-1} = b_1 = 0 .$$

The meaning of this isomorphism is as follows: let A_k^{2n} be the $2n$-dimensional affine space with affine coordinates $a_1, \ldots, a_n, b_{-s+1}, \ldots, b_s$. Then the linear subvariety $Y\colon b_{-1} = b_1 = 0$ with codimension 2 is N-invariant. Our assertion is that in the commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & A^{2n} \\ \downarrow & & \downarrow \\ Y/N & \xrightarrow{\phi} & A^{2n}/SL(2, k) \end{array} ,$$

ϕ is birational, where $A^{2n}/SL(2, k)$ (resp. Y/N) is an algebraic variety over k whose function field is isomorphic to the left (resp. right) hand of (3.2).

PROOF OF LEMMA 3. We shall represent general points of Y/N and $A^{2n}/SL(2, k)$ by the orbit $O^N(y)$ and $O^{SL(2,k)}(x)$ of general points y of Y and x of A^{2n} respectively. Then ϕ is the rational map which sends $O^N(y)$ to $O^{SL(2,k)}(y)$. Since the orbit map $\gamma: SL(2, k) \times Y \to A^{2n}$, $\gamma(g, y) = g \cdot y$, is dominant, so is ϕ. We claim the following:

$$O^{SL(2,k)}(y) \cap Y = O^N(y) \qquad \text{for all} \quad y = (a, b) \in Y \text{ such that } b_0 \neq 0 . \tag{3.3}$$

Let $y = (a_1, \ldots, a_n, b_{-s+1}, \ldots, b_{-2}, 0, b_0, 0, b_2, \ldots, b_s) \in Y$ with $b_0 \neq 0$. We see from (3.1),

$$g \cdot y = (a'_1, \ldots, a'_n, b'_{-s+1}, \ldots, b'_{-2}, 2\alpha\beta b_0, (\alpha\delta + \beta\gamma)b_0, 2\gamma\delta b_0, b'_2, \ldots, b'_s)$$

for $g = \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix} \in SL(2, k)$. Hence if $g \cdot y$ is contained in Y, then $2\alpha\beta b_0$ and $2\gamma\delta b_0$ are equal to zero. The assumption $b_0 \neq 0$ implies that g is a member of N. Thus

$$O^{SL(2,k)}(y) \cap Y \subset O^N(y) .$$

Since the converse inclusion is clear, (3.3) is proved. Then, (3.3) means that $\phi^{-1}\phi O^N(y) = O^N(y)$ for such a point y of Y, which completes the proof of Lemma 3.

By Lemma 2 and Lemma 3, we have the following isomorphism:

$$\begin{aligned}
&k(a_1, \ldots, a_n, b_{-s+1}, \ldots, b_s)^{SL(2,k)} \\
&\cong k(a_1, \ldots, a_n, b_{-s+1}, \ldots, b_s)^N , \qquad b_{-1} = b_1 = 0 , \\
&\cong k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)^N , \qquad b_{-1} = b_1 = \textstyle\sum_{j=0}^{n} (-1)^j a_j b_{s-j} = 0 , \\
&\cong k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)^N , \qquad b_{-1} = b_0 = b_1 = 0 .
\end{aligned} \tag{3.4}$$

4. We look at the action of N on $\{a_i\}$ and $\{b_m\}$. Let

$$g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Then we have

$$x_i^g = t x_i , \qquad y_i^g = t^{-1} y_i , \qquad x_i^\tau = y_i , \qquad y_i^\tau = -x_i .$$

Thus

$$\begin{aligned}
\xi_i^g &= t^{-1} y_i / t x_i = t^{-2} \xi_i , \qquad & \xi_i^\tau &= -x_i / y_i = -1/\xi_i , \\
\eta_i^g &= (t x_i)^2 = t^2 \eta_i , \qquad & \eta_i^\tau &= y_i^2 = (y_i/x_i)^2 x_i^2 = \xi_i^2 \eta_i .
\end{aligned}$$

Therefore

$$
\begin{aligned}
& a_i^g = \left(\textstyle\sum_{j_1<\cdots<j_i} \xi_{j_1}\ldots\xi_{j_i}\right)^g = t^{-2i}a_i\,, \\
& a_i^\tau = \left(\textstyle\sum_{j_1<\cdots<j_i} \xi_{j_1}\ldots\xi_{j_i}\right)^\tau = \textstyle\sum_{j_1<\cdots<j_i} (-1/\xi_{j_1})\ldots(-1/\xi_{j_i}) \\
& \quad = (-1)^i \textstyle\sum_{q_1<\cdots<q_{n-i}} \xi_{q_1}\ldots\xi_{q_{n-i}}/\prod_{p=1}^n \xi_p = (-1)^i a_{n-i}/a_n\,, \\
& b_m^g = \left(\textstyle\sum_{i=1}^n \xi_i^{m+1}\eta_i\right)^g = \textstyle\sum_{i=1}^n (t^{-2}\xi_i)^{m+1}t^2\eta_i = t^{-2m}b_m\,, \\
& b_m^g = \left(\textstyle\sum_{i=1}^n \xi_i^{m+1}\eta_i\right) = \textstyle\sum_{i=1}^n (-1/\xi_i)^{m+1}\xi_i^2\eta_i \\
& \quad = (-1)^{m+1}\textstyle\sum_{i=1}^n \xi_i^{1-m}\eta_i = (-1)^{m+1}b_{-m}\,.
\end{aligned}
\tag{4.1}
$$

This shows that the action of N on the field $k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)$ with $b_{-1} = b_0 = b_1 = 0$, is as follows:

g acts on a_i and b_m diagonally.
τ transposes b_m with b_{-m} and transforms a_i to $(-1)^i a_{n-i}/a_n$.

Now it is not hard to prove the rationality of $k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)^N$. Hence $K^G = k(a_1, \ldots, a_n, b_{-s+1}, \ldots, b_s)^{SL(2,k)}$ is rational over k by (3.4).

REMARK. For an odd integer $n = 2s + 1$, we put $d = -s$ in (2.1) to obtain

$$L^{S_n} = k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)\,.$$

By the same proof as in Lemma 3 we have an isomorphism

$$k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)^{SL(2,k)} = k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)^N,\ b_{-1} = b_1 = 0.$$

The action of N on $k(a_1, \ldots, a_n, b_{-s}, \ldots, b_s)$ is not so complicated as in (4.1) and we see the rationality of the field.

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