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Exterior Functions and Strictly Continuous Homomorphisms in the Algebra of Bounded Analytic Functions

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Let G be a bounded region in the complex plane, and let $\beta(G)$ be the algebra of bounded analytic functions on G, in the strict topology. The strict topology was introduced by Buck (see [B]) — for a survey of this and related matters see [R]. Suffice it to say that the strict topology is the strongest topology on the bounded holomorphic functions for which a sequence is convergent if and only if it is uniformly bounded and pointwise convergent to its limit (see [RR], Corollary, p. 172). A function $f \in \beta(G)$ is called exterior (see [RS], p. 72) when the principal ideal generated by f is dense in $\beta(G)$. Examples can be found (we give one later) of bounded regions G with $0 \in \partial G$ such that $f(z)=z$ is not exterior. On the other hand, if ∂G consisted of isolated Jordan curves, say, then this function z would certainly be exterior (see [RS], Theorem 5.17). In this paper, we consider $f(z)=z-\lambda$, $\lambda \in C$, and discuss the principal ideal that it generates. We prove that aside from the trivial case where $\lambda \notin \bar{G}$, so that $(z-\lambda)$ is a unit in $\beta(G)$, there are exactly three possibilities. We conclude by proving that $(z-\lambda)$ is exterior if and only if there is no strictly continuous multiplicative linear functional in the fiber over λ. We rely heavily in our exposition on results in Gamelin and Garnett's paper [GG] which appeared at about the time our work in this area was being done.

DEFINITION. A point $\lambda \in \partial G$ is called an essential boundary point of G if there is an $f \in \beta(G)$ such that for no region W that contains λ does there exist an extension of f in $\beta(G \cup W)$. (See [RUD], p. 333.)

THEOREM. *Let G be a bounded region, all of whose boundary points are essential. Let $\beta(G)$ be the algebra of all bounded analytic functions on G, in the strict topology. For $\lambda \in \bar{G}$, let*

$$I(\lambda) = \{(z - \lambda)f : f \in \beta(G)\}.$$

Then exactly one of the following possibilities holds:

1) *$I(\lambda)$ is dense in $\beta(G)$; that is $(z-\lambda)$ is exterior.*
2) *$I(\lambda)$ is closed in $\beta(G)$ and has codimension 1 in $\beta(G)$.*

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3) *$I(\lambda)$ is not closed in $\beta(G)$ and has codimension* 1.
Further, there exist a region G and three points λ_1, λ_2, $\lambda_3 \in \bar{G}$ for which possibility 1), 2), 3) *holds, respectively. Finally,* 2) *holds if and only if $\lambda \in G$. In the remaining case ($\lambda \in \partial G$),* 1) *holds if and only if the condition $*$ of Proposition* 1 *holds.*

REMARK 1. We have made the restriction $\lambda \in \bar{G}$ since otherwise $I(\lambda) = \beta(G)$.

PROOF OF THEOREM. First, if $\lambda \in G$, then $I(\lambda) = \{f \in \beta(G): f(\lambda) = 0\}$, so that $I(\lambda)$ clearly has codimension 1, and is easily seen to be strictly closed. Hence if $\lambda \in G$, then 2) holds.

To handle the case $\lambda \in \partial G$, we first recall the definition of analytic capacity (see [Z], p. 11). The remainder of our proof will be presented in a series of separate results, which, when taken together, complete the proof of the theorem.

DEFINITION. Let K be a compact subset of C. The analytic capacity of K is denoted by $\gamma(K)$ and is defined by

$$\gamma(K) = \sup\{|f'(\infty)|: f \in \beta(\hat{C}\backslash K),\ |f(z)| \leq 1 \quad \text{for all } z \in \hat{C}\backslash K, \text{ and } f(\infty) = 0\},$$

where $\hat{C}$ is the extended complex plane.

PROPOSITION 1. *If $\lambda \in \partial G$ and $0 < a < 1$, let*

$$S(n, a, \lambda) = \{z: a^{n+1} \leq |z - \lambda| \leq a^n\}, \qquad n = 1, 2, \ldots.$$

If

$$* \qquad \sum a^{-n} \gamma(CG \cap S(n, a, \lambda)) = \infty,$$

then $(z - \lambda)$ is exterior in $\beta(G)$.

PROOF. Following P. C. Curtis [C], Proof of Theorem 3.5, pp. 42–44, we choose a sequence $\{f_n\}$, $n = 1, 2, \ldots$ in $\beta(G)$ with the following property: for each n, there is a simply-connected region W_n with $\lambda \in W_n$ and a function $f_n \in \beta(G \cup W_n)$ such that $f_n(\lambda) = 1$ for all n, $\sup\{|f_n(z)|: z \in G \cup W_n\} \leq 13$ for every n, and $\{f_n\}$ converges uniformly to zero on each compact subset of G.

Now let

$$g_n(z) = \frac{f_n(z) - f_n(\lambda)}{z - \lambda}$$

so that $g_n \in \beta(G \cup W_n)$ and $\sup\{|(z - \lambda)g_n(z)|: z \in G\} \leq 15$. If $z \in G$ then $\lim_{n \to \infty}(z - \lambda)g_n(z) = \lim_{n \to \infty}(f_n(z) - f_n(\lambda)) = -1$. By the above-mentioned property of the strict topology, $-(z - \lambda)g_n(z)$ converges strictly to 1 in G so that $(z - \lambda)$ is exterior, and the proposition is proved.

DEFINITION. If G is a bounded region and if $\lambda \in \partial G$, then the fiber over λ is defined as the family of all multiplicative linear functionals L on $\beta(G)$ for which $L(\hat{z})=\lambda$, where $\hat{z}$ is the identity function $\hat{z}(\zeta)=\zeta$ for all $\zeta \in G$.

The following is well-known and not hard to prove. A proof can be given using Corollary 2.2 on p. 458 of [GG].

REMARK 2. If $\lambda \in \partial G$ and if $f \in \beta(G)$ is such that $\lim_{n\to\infty} f(z_n)=a$ for each sequence $\{z_n\}$ in G with $\lim_{n\to\infty} z_n=\lambda$, then for any L in the fiber over λ, $L(f)=a$ must hold.

In [RS], Theorem 3.4, p. 245, Rubel and Shields showed that the dual space of $\beta(G)$ may be represented as $M'(G)=M(G)/N(G)$, where $M(G)$ is the space of all complex Borel measures that live in G, and $N(G)=\{\mu \in M(G): \int_G f\,d\mu=0$ for all $f \in \beta(G)\}$. For $\mu \in M(G)$ and $f \in \beta(G)$, set $L_\mu(f)=\int f\,d\mu$.

LEMMA 1. *Suppose G is a bounded region and that the fiber over the point $\lambda \in \partial G$ contains a strictly continuous multiplicative linear functional L. If $\{\mu\}$ is chosen in $M'(G)$ so that $L(f)=\int_G f\,d\mu$ for all $f \in \beta(G)$, then the closure of $I(\lambda)$ is equal to $I(\lambda)^-=\{f \in \beta(G): \int_G f\,d\mu=0\}$.*

PROOF. By the Hahn-Banach theorem, it is enough to show that if $\{\nu\} \in M'(G)$ has the property that if $\int_G f\,d\nu=0$ for every $f \in I(\lambda)$ then $\nu=c\mu$ for some $c \in C$. Define $H(G' \mid G)$ as the set of those $f \in \beta(G)$ that extend to lie in $\beta(G \cup W)$ for some region W that contains λ. Since $H(G' \mid G)$ is strictly sequentially dense in $\beta(G)$ (see [GG], Corollary 2.2), we need only show that ν and $c\mu$ have the same action on functions in $H(G' \mid G)$.

Suppose now that $f \in \beta(G \cup W)$ for a region W that contains λ. By a result of Arens (see [A], Theorem 2.7, p. 646), we know that there exists a sequence $\{h_n\}$ in $\beta(G \cup W)$ such that the sequence $\{r_n\}$ has zero as its uniform limit on $G \cup W$, where $r_n(z)=f(z)-f(\lambda)-(z-\lambda)h_n(z)$. This implies that $\{r_n\}$ converges strictly to zero, so that

$$\int_G [f-f(\lambda)]d\nu = \lim_{n\to\infty} \int_G [(z-\lambda)h_n]d\nu = 0.$$

Hence $\int_G f\,d\nu=f(\lambda)\int_G 1\,d\nu$ for all $f \in H(G' \mid G)$. But since $f(\lambda)=\int_G f\,d\mu$ for all such f, the result follows.

PROPOSITION 2. *Suppose G is a bounded region with every point on ∂G an essential boundary point. If $\lambda \in \partial G$ and if $(z-\lambda)$ is not exterior on G, then $I(\lambda)$ is properly contained in $I(\lambda)^-$ and $I(\lambda)^-$ has codimension 1 in $\beta(G)$.*

PROOF. If $(z-\lambda)$ is not exterior on G, then Proposition 1 implies that $\sum a^{-n}\gamma(CG \cap S(n, a, \lambda)) < \infty$. By [GG, Theorem 3.2], we know that the fiber over λ contains a strictly continuous multiplicative linear functional. By Lemma 1, we conclude that $I(\lambda)^{-}$ has codimension 1 in $\beta(G)$.

Now suppose that for each $f \in \beta(G)$, there is a constant c and a function $h \in \beta(G)$ such that $f = c + (z-\lambda)h$. In this event, the cluster set of f at λ is just the singleton $\{c\}$. This contradicts a result of Rudin (see [RUD], Theorem 14) which says that if λ is an essential boundary point of G, then there is an $f \in \beta(G)$ whose cluster set at λ is the whole closed unit disk. Hence $I(\lambda) \neq I(\lambda)^{-}$ and the Proposition is proved. This completes the proof of the theorem, except for the example of G and λ_1, λ_2, λ_3. But to make 1) hold for $\lambda = \lambda_1$, we need only be sure that ∂G contains an isolated arc on which λ_1 lies. Then 2) holds for any $\lambda_2 \in G$. Finally to make 3) hold for $\lambda = \lambda_3 \in \partial G$, we need only further construct G so that $*$ fails. Since $\lambda \notin G$, we must then have either 1) or 3), and the failure of $*$ rules out 1), so that 3) must hold for this λ. Such an example may be found in [Z], pp. 57–58, and consists of the unit disc with a sequence of small discs removed that converge to $\lambda = 0$.

REMARK 3. If G is a bounded simply-connected region and $\lambda \in \partial G$, then $(z-\lambda)$ is exterior on G.

PROOF. It is enough, by Proposition 1, to prove that $*$ holds. Now ∂G is a continuum and so for all large j, $S(j, a, \lambda) \cap \partial G$ contains a continuum that meets both the inner and outer boundaries of the annulus $S(j, a, \lambda)$. Hence the diameter of $S(j, a, \lambda) \cap \partial G$ is no smaller than $a^j(1-a)$. Thus (see for example p. 13 of Zalcman's notes [Z]) for each such j,

$$\gamma(CG \cap S(j, a, \lambda)) \geq \gamma(\partial G \cap S(j, a, \lambda)) \geq \frac{a^j(1-a)}{4}.$$

So the above series diverges, and the result is proved.

REMARK 4. The function $(z-\lambda)$ is exterior on G if and only if the fiber over λ contains no strictly continuous multiplicative linear functional.

PROOF. Compare our Theorem with the Theorem on p. 456 of [GG].

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