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Boundary limits of Green potentials of order α

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1. Introduction

In the half space $D=\{x=(x_1,\dots, x_n);\ x_n>0\}$, $n\geqq 2$, the Green potential of order α, $0<\alpha<n$, of a non-negative measurable function f on D is defined by

$$G_\alpha^f(x) = \int_D G_\alpha(x, y) f(y) dy,$$

where $G_\alpha(x, y)=|x-y|^{\alpha-n}-|\bar{x}-y|^{\alpha-n}$, $\bar{x}=(x_1,\dots, x_{n-1}, -x_n)$ for $x=(x_1,\dots, x_{n-1}, x_n)$. Our aim in this note is to study the existence of boundary limits of G_α^f. One of our results is as follows:

Let $p>1$, $\gamma<2p-1$ and f satisfy $G_\alpha^f \not\equiv \infty$ and

$$\int_G f(y)^p y_n^\gamma dy < \infty \qquad \text{for any bounded open set} \quad G \subset D.$$

Then there exists a set $E\subset\partial D$ with $H_{n-\alpha p+\gamma}(E)=0$ such that to each $\xi\in\partial D-E$, there corresponds a set $E_\xi\subset S_+=\{x\in D;\ |x|=1\}$ with the properties:

a) $B_{\alpha,p}(E_\xi) = 0$; b) $\lim_{r\downarrow 0} G_\alpha^f(\xi+r\zeta) = 0$ *for every* $\zeta\in S_+-E_\xi$,

where H_ℓ denotes the ℓ-dimensional Hausdorff measure and $B_{\alpha,p}$ denotes the Bessel capacity of index (α, p) (see [3]).

In case $\alpha=2$, according to Wu [8; Theorem 1], the exceptional set E_ξ has Hausdorff dimension at most $n-2p$; this is a consequence of our result in view of Fuglede [2].

Moreover, non-tangential limits, fine limits, mean continuous limits and perpendicular limits will be considered.

2. Preliminaries

Let us begin with the following lemma, which can be proved by elementary calculation.

LEMMA 1. *There exist positive constants c_1 and c_2 such that*

$$c_1 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2} \leqq G_\alpha(x, y) \leqq c_2 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2}$$

for $x=(x_1,\dots, x_n)$ and $y=(y_1,\dots, y_n)$ in D.

By Lemma 1, we can prove the next lemma.

LEMMA 2. *Let f be a non-negative measurable function on D. Then $G_\alpha^f \not\equiv \infty$ if and only if $\int_D (1+|y|)^{\alpha-n-2} y_n f(y) dy < \infty$.*

For $x \in R^n$ and $r>0$, denote by $B(x, r)$ the open ball with center at x and radius r, and set $B_+(x, r) = B(x, r) \cap D$.

LEMMA 3. *Let $\xi \in \partial D$, $c>0$ and f be a non-negative measurable function on D such that $G_\alpha^f \not\equiv \infty$. Set*

$$u(x) = \int_{\{y \in D; |x-y| \geqq c|x-\xi|\}} G_\alpha(x, y) f(y) dy.$$

Then $\lim_{x \to \xi, x \in D} u(x) = 0$ if and only if $\xi \in \partial D - A$, where

$$A = \left\{\xi \in \partial D;\ \limsup_{r \downarrow 0} r^{\alpha-n-1} \int_{B_+(\xi, r)} f(y) y_n dy > 0\right\}.$$

PROOF. Let $\delta > 0$. Then by Lemma 1,

$$\lim_{x \to \xi, x \in D} \int_{D - B(\xi, \delta)} G_\alpha(x, y) f(y) dy = 0$$

since $G_\alpha^f \not\equiv \infty$. Set $C_\delta = \sup\left\{ r^{\alpha-n-1} \int_{B_+(\xi, \delta)} f(y) y_n dy;\ 0 < r \leqq \delta \right\}$. We have again by Lemma 1,

$$\begin{aligned}
&\int_{\{y \in B_+(\xi, \delta); |x-y| \geqq c|x-\xi|\}} G_\alpha(x, y) f(y) dy \\
&\leqq M_1 x_n \int_{B_+(\xi, \delta)} (|x-\xi| + |y-\xi|)^{\alpha-n-2} f(y) y_n dy \\
&\leqq M_2 x_n \Big((|x-\xi| + \delta)^{\alpha-n-2} \int_{B_+(\xi, \delta)} f(y) y_n dy \\
&\quad + \int_0^\delta (|x-\xi| + r)^{\alpha-n-3} \left\{ \int_{B_+(\xi, r)} f(y) y_n dy \right\} dr \Big) \leqq M_3 C_\delta,
\end{aligned}$$

where M_1, M_2 and M_3 are positive constants independent of x and δ. It follows that $\limsup_{x \to \xi, x \in D} u(x) \leqq M_3 C_\delta$, which proves the "if" part. Lemma 1 also gives

$$u(\xi + r(0, \dots, 0, 1)) \geqq M_4 r^{\alpha-n-1} \int_{\{y \in D; (1+c)r < |y-\xi| < 2(1+c)r\}} f(y) y_n dy$$

for some positive constant M_4 independent of r, from which the "only if" part follows.

REMARK. If $\alpha<2$ and $f(y)=y_n^{-\alpha}$, then $G_\alpha^f \not\equiv \infty$ by Lemma 2 but $A=\partial D$. On the other hand, if $\alpha \geqq 2$, then $G_\alpha^f \not\equiv \infty$ implies $H_{n-\alpha+1}(A)=0$ (cf. [4; p. 165]), so that $A \neq \partial D$.

First we give the following result.

THEOREM 1. *Let $\alpha p>n$, $\gamma<2p-1$ and $n-\alpha p+\gamma<0$. Let f be a non-negative measurable function on D such that $G_\alpha^f \not\equiv \infty$, and let $\xi \in \partial D$. If $\int_{B_+(\xi,\rho)} f(y)^p y_n^\gamma dy<\infty$ for some $\rho>0$, then G_α^f has limit zero at ξ.*

PROOF. Let $\delta>0$. First consider the case $\gamma \leqq 0$. We have by Hölder's inequality and Lemma 1,

$$
\begin{aligned}
&\int_{B_+(\xi,\delta)} G_\alpha(x,y)f(y)dy \\
&\leqq \left\{\int_{B_+(\xi,\delta)} [c_2|x-y|^{\alpha-n}]^{p'} dy\right\}^{1/p'} \left\{\int_{B_+(\xi,\delta)} f(y)^p dy\right\}^{1/p} \\
&\leqq \text{const.}\left\{\delta^{\alpha p-\gamma-n} \int_{B_+(\xi,\delta)} f(y)^p y_n^\gamma dy\right\}^{1/p},
\end{aligned}
$$

where $1/p+1/p'=1$. In case $0<\gamma \leqq p$, we have

$$
\begin{aligned}
&\int_{B_+(\xi,\delta)} G_\alpha(x,y)f(y)dy \\
&\leqq \left\{\int_{B_+(\xi,\delta)} [c_2 x_n y_n^{1-\gamma/p}|x-y|^{\alpha-n}|\bar{x}-y|^{-2}]^{p'} dy\right\}^{1/p'} \left\{\int_{B_+(\xi,\delta)} f(y)^p y_n^\gamma dy\right\}^{1/p} \\
&\leqq \left\{\int_{B_+(\xi,\delta)} [c_2|x-y|^{\alpha-\gamma/p-n}]^{p'} dy\right\}^{1/p'} \left\{\int_{B_+(\xi,\delta)} f(y)^p y_n^\gamma dy\right\}^{1/p} \\
&\leqq \text{const.}\left\{\delta^{\alpha p-\gamma-n} \int_{B_+(\xi,\delta)} f(y)^p y_n^\gamma dy\right\}^{1/p}.
\end{aligned}
$$

Let $\gamma>p$. Then,

$$
\begin{aligned}
&\int_{B_+(\xi,\delta)} G_\alpha(x,y)f(y)dy \\
&\leqq \left\{\int_{B_+(\xi,\delta)} [c_2|x-y|^{\alpha-n-1} y_n^{1-\gamma/p}]^{p'} dy\right\}^{1/p'} \left\{\int_{B_+(\xi,\delta)} f(y)^p y_n^\gamma dy\right\}^{1/p}.
\end{aligned}
$$

Letting $I(y)=|x-y|^{\alpha-n-1} y_n^{1-\gamma/p}$, we note

$$
\begin{aligned}
\int_{\{y\in B_+(\xi,\delta);\, y_n \geqq x_n/2\}} I(y)^{p'} dy &\leqq \int_{B(\xi,\delta)} [|x-y|^{\alpha-n-1}|x_n-y_n|^{1-\gamma/p}]^{p'} dy \\
&\leqq \int_{B(x,\delta)} [|x-y|^{\alpha-n-1}|x_n-y_n|^{1-\gamma/p}]^{p'} dy \\
&= \text{const.}\, \delta^{p'(\alpha p-\gamma-n)/p};
\end{aligned}
$$

$$\int_{\{y \in B_+(\xi,\delta);\, y_n < x_n/2\}} I(y)^{p'} dy \leqq \int_{B_+(\xi,\delta)} [|(x',0)-y|^{\alpha-n-1} y_n^{1-\gamma/p}]^{p'} dy$$
$$= \text{const.}\ \delta^{p'(\alpha p-\gamma-n)/p},$$

where $x=(x', x_n)$. In all cases,

$$\int_{B_+(\xi,\delta)} G_\alpha(x, y) f(y) dy \leqq \text{const.} \left\{\delta^{\alpha p-\gamma-n} \int_{B_+(\xi,\delta)} f(y)^p y_n^\gamma dy\right\}^{1/p}.$$

Since $G_\alpha^f \not\equiv \infty$, it follows that

$$\limsup_{x\to\xi, x\in D} G_\alpha^f(x) = \limsup_{x\to\xi, x\in D} \int_{B_+(\xi,\delta)} G_\alpha(x, y) f(y) dy$$
$$\leqq \text{const.} \left\{\delta^{\alpha p-\gamma-n} \int_{B_+(\xi,\delta)} f(y)^p y_n^\gamma dy\right\}^{1/p},$$

which implies that $\lim_{x\to\xi, x\in D} G_\alpha^f(x)=0$, and hence our theorem is established.

To study the case $\alpha p \leqq n$ or $n-\alpha p+\gamma \geqq 0$, it is important to note the following lemma.

LEMMA 4. *Let $p \geqq 1$ and f be a non-negative measurable function on D satisfying*

(1) $$\int_G f(y)^p y_n^\gamma dy < \infty \qquad \textit{for any bounded open set} \quad G \subset D,$$

and set

$$A_{p,\beta} = \left\{\xi \in \partial D;\ \int_{B_+(\xi,\rho)} |\xi-y|^{\alpha p-\beta-n} f(y)^p y_n^\beta dy = \infty \ \textit{for any}\ \rho>0\right\}.$$

If $\beta>\gamma$, then $H_{n-\alpha p+\gamma}(A_{p,\beta})=0$; in case $n-\alpha p+\gamma \leqq 0$, $A_{p,\beta}$ is empty.

PROOF. If $n-\alpha p+\gamma \leqq 0$, then

$$\int_{B_+(\xi,1)} |\xi-y|^{\alpha p-\beta-n} f(y)^p y_n^\beta dy \leqq \int_{B_+(\xi,1)} f(y)^p y_n^\gamma dy < \infty,$$

which implies that $A_{p,\beta}$ is empty. Let $n-\alpha p+\gamma>0$, and suppose $H_{n-\alpha p+\gamma}(A_{p,\beta}) >0$. Then by [1; Theorems 1 and 3 in § II], there exists a positive measure μ with compact support in $A_{p,\beta}$ such that

$$\mu(B(x, r)) \leqq r^{n-\alpha p+\gamma} \qquad \text{for every} \quad x \in R^n \quad \text{and} \quad r > 0.$$

Note that

$$\int |\xi-y|^{\alpha p-\beta-n} d\mu(\xi) \leqq \text{const.}\ y_n^{\gamma-\beta}, \quad y \in D.$$

Taking $N>0$ such that the support of μ is included in $B(O, N)$, we obtain

$$\begin{aligned}\infty &= \int\left\{\int_{B_+(\xi,1)} |\xi-y|^{\alpha p-\beta-n} f(y)^p y_n^\beta dy\right\} d\mu(\xi)\\ &\leqq \int_{B_+(O,N+1)} \left\{\int |\xi-y|^{\alpha p-\beta-n} d\mu(\xi)\right\} f(y)^p y_n^\beta dy\\ &\leqq \text{const.} \int_{B_+(O,N+1)} f(y)^p y_n^\gamma dy < \infty,\end{aligned}$$

which is a contradiction. Thus $H_{n-\alpha p+\gamma}(A_{p,\beta})=0$, and our lemma is proved.

LEMMA 5. *If $p>1$, $\gamma<2p-1$ and f is a non-negative measurable function on D, then $A\subset A_{p,\beta}$, where A is the set given in Lemma 3.*

COROLLARY. *If $p\geqq 1$, $\gamma<2p-1$ and f is a non-negative measurable function on D satisfying (1), then $H_{n-\alpha p+\gamma}(A)=0$.*

REMARK. The function

$$f(y) = y_n^{-2}(\log(1+y_n^{-1}))^{-1}$$

satisfies (1) when $p>1$ and $\gamma=2p-1$, but $A=\partial D$ for this function f, so that $H_{n-\alpha p+\gamma}(A)=\infty$ if $\alpha\geqq 2$.

PROOF OF LEMMA 5. By Hölder's inequality, we have

$$\begin{aligned}&r^{\alpha-n-1}\int_{B_+(\xi,r)} f(y) y_n dy\\ &\leqq r^{\alpha-n-1}\left\{\int_{B_+(\xi,r)} [|\xi-y|^{(n-\alpha+\beta)/p} y_n^{1-\beta/p}]^{p'} dy\right\}^{1/p'}\\ &\quad\times\left\{\int_{B_+(\xi,r)} |\xi-y|^{\alpha p-\beta-n} f(y)^p y_n^\beta dy\right\}^{1/p}\\ &= \text{const.}\left\{\int_{B_+(\xi,r)} |\xi-y|^{\alpha p-\beta-n} f(y)^p y_n^\beta dy\right\}^{1/p},\end{aligned}$$

which implies that $A\subset A_{p,\beta}$.

Let $p>1$, $k(x, y)$ be a non-negative Borel measurable function on $R^n\times R^n$ and G be an open set in R^n. Following Meyers [3], we define the capacity

$$C_{k,p}(E;\, G) = \inf \|g\|_p^p, \quad E\subset R^n,$$

where the infimum is taken over all non-negative measurable functions g on R^n such that $g=0$ on R^n-G and

$$U_k^g(x) = \int k(x, y)g(y)dy \geqq 1 \qquad \text{for every} \quad x\in E.$$

In case $k(x, y)=|x-y|^{\ell-n}$, we write $C_{\ell,p}$ for $C_{k,p}$; in case k is the Bessel kernel of order ℓ (see [3]) and $G=R^n$, we write $B_{\ell,p}$ for $C_{k,p}(\cdot, R^n)$. We note the following properties (cf. [5; Lemma 1], [6; Sect. 2]).

(i) Let G and G' be bounded open sets in R^n, and K be a compact subset of $G \cap G'$. Then there exists $M>0$ such that

$$M^{-1}C_{\ell,p}(E; G) \leqq C_{\ell,p}(E; G') \leqq MC_{\ell,p}(E; G) \quad \text{whenever} \quad E \subset K.$$

(ii) For $r>0$, let $T_r x = rx$, $x \in R^n$. Then

$$C_{\ell,p}(T_r E; T_r G) = r^{n-\ell p} C_{\ell,p}(E; G).$$

(iii) Let E^* denote the projection of a set E to the hyperplane ∂D. Then

$$C_{\ell,p}(E^*; B(\xi, r)) \leqq C_{\ell,p}(E; B(\xi, r))$$

for $\xi \in \partial D$ and $r>0$.

(iv) Let $\tilde{E}$ denote the radial projection of a set E to the surface S_+, i.e., $\tilde{E} = \{\zeta \in S_+;\ r\zeta \in E \text{ for some } r>0\}$. Then there exists $M>0$ such that

$$C_{\ell,p}(\tilde{E}; B(O, 3)) \leqq MC_{\ell,p}(E; B(O, 3))$$

for $E \subset B(O, 2) - B(O, 1)$.

(v) If $C_{\ell,p}(E; G)=0$ for some bounded open set $G \subset R^n$, then $B_{\ell,p}(E)=0$.

(vi) $B_{\ell,p}(E)=0$ if and only if $C_{\ell,p}(E \cap G; G)=0$ for any bounded open set $G \subset R^n$.

3. Non-tangential and fine limits

A function u on D is said to have non-tangential limit zero at $\xi \in \partial D$ if

$$\lim_{x \to \xi, x \in \Gamma(\xi,a)} u(x) = 0 \qquad \text{for any} \quad a > 1,$$

where $\Gamma(\xi, a) = \{x=(x_1,\dots, x_n);\ |x-\xi| < ax_n\}$.

THEOREM 2. *Let f be a non-negative measurable function on D such that $G_\alpha^f \not\equiv \infty$. If $\xi \in \partial D - (A \cup A_{p,\beta})$ for some numbers β and p with $\alpha p > n$, then G_α^f has non-tangential limit zero at ξ.*

REMARK. Let f satisfy the additional assumption (1), and define $B = A \cup (\bigcap_{\beta > \gamma} A_{p,\beta})$. Then the conclusion of Theorem 2 holds for $\xi \in \partial D$ except for the set B. In general, $H_{n-\alpha+1}(B)=0$, and if $\gamma < 2p-1$, then $H_{n-\alpha p+\gamma}(B)=0$ on account of Lemmas 4 and 5.

PROOF OF THEOREM 2. Write $G_\alpha^f = u+v$, where

$$u(x) = \int_{\{y \in D; |x-y| \geqq x_n/2\}} G_\alpha(x, y) f(y) dy,$$

$$v(x) = \int_{\{y \in D; |x-y| < x_n/2\}} G_\alpha(x, y) f(y) dy.$$

By Lemma 3, u has non-tangential limit zero at ξ. If $x \in \Gamma(\xi, a)$, then Hölder's inequality yields

$$\begin{aligned} v(x) &\leqq c_2 \int_{\{y \in D; |x-y| < x_n/2\}} |x-y|^{\alpha-n} f(y) dy \\ &\leqq c_2 \left\{ \int_{\{y \in D; |x-y| < x_n/2\}} |x-y|^{p'(\alpha-n)} dy \right\}^{1/p'} \left\{ \int_{\{y \in D; |x-y| < x_n/2\}} f(y)^p dy \right\}^{1/p} \\ &= \text{const.} \left\{ x_n^{\alpha p-n} \int_{\{y \in D; |x-y| < x_n/2\}} f(y)^p dy \right\}^{1/p} \\ &\leqq \text{const.} \left\{ \int_{B_+(\xi, 2ax_n)} |\xi - y|^{\alpha p-\beta-n} f(y)^p y_n^\beta dy \right\}^{1/p}, \end{aligned}$$

which implies that $\lim_{x \to \xi, x \in \Gamma(\xi,a)} v(x) = 0$ if $\xi \in \partial D - A_{p,\beta}$. Thus the theorem is proved.

Let $k_{\alpha,\beta}(x, y) = |x-y|^{\alpha-n} |y_n|^{-\beta/p}$. Then we can easily prove the following result.

LEMMA 6. (a) $C_{k_{\alpha,\beta},p}(T_r E; T_r G) = r^{n-\alpha p+\beta} C_{k_{\alpha,\beta},p}(E; G)$.

(b) *For $a > 1$, there exists a positive constant M (which depends on β) such that*

$$M^{-1} C_{\alpha,p}(E; B(\xi, 3)) \leqq C_{k_{\alpha,\beta},p}(E; B(\xi, 3)) \leqq M C_{\alpha,p}(E; B(\xi, 3))$$

whenever $E \subset \Gamma(\xi, a) \cap B(\xi, 2) - B(\xi, 1)$, $\xi \in \partial D$.

Following Meyers [4], we say that a set $E \subset R^n$ is (α, p)-thin at $\xi \in \partial D$ if

$$\int_0^1 [r^{\alpha p-n} C_{\alpha,p}(E \cap B(\xi, r) - B(\xi, r/2); B(\xi, 2r))]^{1/(p-1)} \frac{dr}{r} < \infty.$$

Further we say that E is $(k_{\alpha,\beta}, p)$-thin at ξ if

$$\int_0^1 \left(r^{\alpha p-\beta-n} C_{k_{\alpha,\beta},p}(E \cap B(\xi, r) - B(\xi, r/2); B(\xi, 2r)) \right)^{1/(p-1)} \frac{dr}{r} < \infty.$$

By Lemma 6, we obtain the next result.

LEMMA 7. *If E is $(k_{\alpha,\beta}, p)$-thin at ξ, then $E \cap \Gamma(\xi, a)$ is (α, p)-thin at ξ for any $a > 1$.*

For a non-negative measurable function f on D, we set

$$A'_{p,\beta} = \left\{\xi \in \partial D;\ \int_0^\rho \left(r^{\alpha p-\beta-n}\int_{B+(\xi,r)} f(y)^p y_n^\beta dy\right)^{1/(p-1)} \frac{dr}{r} = \infty \text{ for any } \rho>0\right\}.$$

THEOREM 3. *Let f be a non-negative measurable function on D such that $G_\alpha^f \not\equiv \infty$. If $\xi \in \partial D-(A \cup A'_{p,\beta})$ for some numbers $p>1$ and β, then there exists $E_\xi \subset D$ such that E_ξ is $(k_{\alpha,\beta},\ p)$-thin at ξ and*

$$\lim_{x\to\xi, x\in D-E_\xi} G_\alpha^f(x) = 0.$$

REMARK 1. If f satisfies (1), then $B_{\alpha-\gamma/p,p}(A'_{p,\beta})=0$ for $\beta>\gamma$ on account of [4; Theorem 2.1]. We do not know whether $H_{n-\alpha p+\gamma}(A'_{p,\beta})=0$ or not in case $\beta>\gamma$.

REMARK 2. By Lemma 7, $E_\xi \cap \Gamma(\xi, a)$ is (α, p)-thin at ξ for any $a>1$.

PROOF OF THEOREM 3. Write $G_\alpha^f = u+v$, where

$$u(x) = \int_{\{y\in D;\,|x-y|\geqq|x-\xi|/2\}} G_\alpha(x, y)f(y)dy,$$

$$v(x) = \int_{\{y\in D;\,|x-y|<|x-\xi|/2\}} G_\alpha(x, y)f(y)dy.$$

If $\xi \in \partial D-A$, then Lemma 3 shows that $\lim_{x\to\xi,x\in D} u(x)=0$.

Let $\xi\in\partial D-A'_{p,\beta}$, and take a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i=\infty$ and

$$\sum_{i=j}^\infty \left\{a_i 2^{i(n-\alpha p+\beta)}\int_{B+(\xi,2^{-i+2})} f(y)^p y_n^\beta dy\right\}^{1/(p-1)} < \infty,$$

where j is a positive integer such that

$$\int_0^{2^{-j+2}} \left(r^{\alpha p-\beta-n}\int_{B+(\xi,r)} f(y)^p y_n^\beta dy\right)^{1/(p-1)} \frac{dr}{r} < \infty.$$

Consider the sets

$$E_i = \{x\in D;\ 2^{-i} \leqq |x-\xi| < 2^{-i+1},\ v(x) \geqq a_i^{-1/p}\}$$

for $i=j, j+1,\dots$. If $x\in E_i$ and $|x-y|<|x-\xi|/2$, then $|y-\xi|<2^{-i+2}$, so that

$$C_{k_{\alpha,\beta},p}(E_i;\ B(\xi,\ 2^{-i+2})) \leqq a_i \int_{B+(\xi,2^{-i+2})} f(y)^p y_n^\beta dy.$$

Consequently,

$$\sum_{i=j}^\infty \{2^{i(n-\alpha p+\beta)} C_{k_{\alpha,\beta},p}(E_i;\ B(\xi,\ 2^{-i+2}))\}^{1/(p-1)} < \infty,$$

which implies that $E=\bigcup_{i=j}^\infty E_i$ is $(k_{\alpha,\beta},\ p)$-thin at ξ. Clearly, $\lim_{x\to\xi,x\in D-E} v(x)=0$, and our theorem is proved.

4. Mean continuous limits

A function u on D is said to have mean continuous limit zero of order q (or simply mc_q-limit zero) at $\xi \in \partial D$ if

$$\lim_{r\downarrow 0} r^{-n}\int_{B_+(\xi,r)} |u(x)|^q dx = 0 \qquad \text{in case} \quad q<\infty,$$

$$\lim_{x\to\xi, x\in D} u(x) = 0 \qquad \text{in case} \quad q=\infty.$$

THEOREM 4. *Let f be a non-negative measurable function on D such that $G_\alpha^f \not\equiv \infty$. If $\xi \in \partial D-(A\cup A_{p,\beta})$ for some numbers $p>1$ and β, then G_α^f has mc_q-limit zero at ξ, where q is given as follows:*

i) $1/q = 1/p-(\alpha-\beta/p)/n$ *if* $0\leqq \beta < 2p-1$ *and* $0<\alpha p-\beta<n$;

ii) $1<q<\infty$ *if* $0\leqq \beta < 2p-1$ *and* $\alpha p-\beta=n$;

iii) $q=\infty$ *if* $0\leqq \beta < 2p-1$ *and* $\alpha p-\beta>n$;

iv) $1/q = 1/p-\alpha/n$ *if* $\beta<0$ *and* $\alpha p<n$;

v) $1<q<\infty$ *if* $\beta<0$ *and* $\alpha p=n$;

vi) $q=\infty$ *if* $\beta<0$ *and* $\alpha p>n$.

PROOF. As in the proof of Theorem 3, write $G_\alpha^f=u+v$. If $\xi\in\partial D-A$, then Lemma 3 shows that $\lim_{x\to\xi,x\in D} u(x)=0$. For v, we note the following estimates:

$$v(x) \leqq c_2\int_{\{y\in D;|x-y|<|x-\xi|/2\}} |x-y|^{\alpha-n} f(y)dy \qquad \text{in case} \quad \beta\leqq 0,$$

$$v(x) \leqq c_2\int_{\{y\in D;|x-y|<|x-\xi|/2\}} |x-y|^{\alpha-\beta/p-n}[f(y)y_n^{\beta/p}]dy \quad \text{in case} \quad 0<\beta\leqq p,$$

$$v(x) \leqq c_2\int_{\{y\in D;|x-y|<|x-\xi|/2\}} |x-y|^{\alpha-1-n}y_n^{1-\beta/p}[f(y)y_n^{\beta/p}]dy$$
$$\text{in case} \quad p<\beta<2p-1.$$

The remaining part of the proof can be carried out along the same lines as in the proof of [7; Theorem 6].

We say that a function u on D has non-tangential mean continuous limit zero of order q (or simply NT-mc_q-limit zero) at $\xi\in\partial D$ if

$$\lim_{r\downarrow 0} r^{-n}\int_{\Gamma(\xi,a,r)} |u(x)|^q dx = 0 \qquad \text{for all} \quad a>1,$$

where $\Gamma(\xi, a, r)=\Gamma(\xi, a)\cap B(\xi, r)$.

THEOREM 5. *Let f be a non-negative measurable function on D such that $G_\alpha^f \not\equiv \infty$. If $\xi \in \partial D - (A \cup A_{p,\beta})$ for some numbers $p > 1$ and β, then G_α^f has NT-mc$_q$-limit zero at ξ, where q is given as follows:*

i) $1/q = 1/p - \alpha/n$ *if* $\alpha p < n$;

ii) $1 < q < \infty$ *if* $\alpha p = n$.

The case $\alpha p > n$ was considered in Theorem 2.

PROOF OF THEOREM 5. We write $G_\alpha^f = u + v$ as in the proof of Theorem 2. Then Lemma 3 shows that u has non-tangential limit zero at $\xi \in \partial D - A$, so that u has NT-mc$_q$-limit zero at $\xi \in \partial D - A$ for all $q \geqq 1$.

Let $\xi \in \partial D - A_{p,\beta}$. First we consider the case $\alpha p < n$. For $r > 0$, set $\Delta(a, r) = \{x \in \Gamma(\xi, a);\ r/2 < |x - \xi| < 2r\}$. Then we have by Lemma 1 and Sobolev's inequality (cf. [7; Lemma 9]),

$$\begin{aligned}
& r^{-n} \int_{\Delta(a,r)} v(x)^q dx \\
& \leqq r^{-n} \int_{\Delta(a,r)} \left\{ c_2 \int_{\Delta(3a,r/2) \cup \Delta(3a,2r)} |x-y|^{\alpha-n} f(y) dy \right\}^q dx \\
& \leqq M_1 r^{-n} \left\{ \int_{\Delta(3a,r/2) \cup \Delta(3a,2r)} f(y)^p dy \right\}^{q/p} \\
& \leqq M_2 \left\{ \int_{\Gamma(\xi,3a,4r)} |\xi - y|^{\alpha p - \beta - n} f(y)^p y_n^\beta dy \right\}^{q/p},
\end{aligned}$$

where M_1 and M_2 are positive constants independent of r. Therefore v has NT-mc$_q$-limit zero at ξ. The case $\alpha p = n$ can be proved similarly.

5. Radial limits

Our aim in this section is to establish the following theorem.

THEOREM 6. *Let f be a non-negative measurable function on D such that $G_\alpha^f \not\equiv \infty$. If $\xi \in \partial D - (A \cup A_{p,\beta})$ for some numbers $p > 1$ and β, then there exists a set $E \subset S_+$ such that $B_{\alpha,p}(E) = 0$ and*

$$\lim_{r \downarrow 0} G_\alpha^f(\xi + r\zeta) = 0 \qquad \textit{for every} \quad \zeta \in S_+ - E.$$

To prove this, we need the next lemmas.

LEMMA 8. *Let*

$$v(x) = \int_{\{y \in D;\, |x-y| < |x-\xi|/2\}} G_\alpha(x, y) f(y) dy.$$

If $\xi \in \partial D - A_{p,\beta}$ for some numbers $p>1$ and β, then there exists a set $E \subset D$ with the properties:

(i) $\lim_{x \to \xi, x \in D - E} v(x) = 0$;

(ii) $\sum_{i=1}^{\infty} 2^{i(n-\alpha p+\beta)} C_{k_\alpha,\beta,p}(E^{(i)}; B(\xi, 2^{-i+2})) < \infty$,

where $E^{(i)} = \{x \in E; 2^{-i} \leqq |x-\xi| < 2^{-i+1}\}$.

PROOF. Let $\{a_i\}$ be a sequence of positive numbers such that $\lim_{i \to \infty} a_i = \infty$ and $\sum_{i=i_0}^{\infty} a_i b_i < \infty$, where

$$b_i = \int_{B_i} |\xi - y|^{\alpha p - \beta - n} f(y)^p y_n^\beta dy,$$

$$B_i = \{y \in D; 2^{-i-1} < |\xi - y| < 2^{-i+2}\},$$

and i_0 is a positive integer such that $\sum_{i=i_0}^{\infty} b_i < \infty$. Consider

$$E_i = \{x \in D; 2^{-i} \leqq |x - \xi| < 2^{-i+1},\ v(x) \geqq a_i^{-1/p}\}.$$

If $x \in E_i$ and $|x-y| < |x-\xi|/2$, then $y \in B_i$, so that

$$C_{k_\alpha,\beta,p}(E_i; B_i) \leqq c_2^p a_i \int_{B_i} f(y)^p y_n^\beta dy \leqq \text{const. } a_i b_i 2^{-i(n-\alpha p+\beta)}.$$

Thus the set $E = \cup_{i=i_0}^{\infty} E_i$ has the required properties.

LEMMA 9. *If E satisfies* (ii) *in Lemma* 8, *then*

$$B_{\alpha,p}(\cap_{j=1}^{\infty} (\cup_{i=j}^{\infty} E^{(i)})^{\sim}) = 0,$$

where $F^{\sim}$ in general denotes the set $\{\xi + \zeta; \zeta \in S_+$ and $\xi + r\zeta \in F$ for some $r>0\}$.

PROOF. Let $F = \cap_{j=1}^{\infty} (\cup_{i=j}^{\infty} E^{(i)})^{\sim}$. Then Lemma 6 together with (iv) in Section 2 implies that $C_{\alpha,p}(F \cap \Gamma(\xi, a); B(\xi, 3)) = 0$ for $a > 1$, from which $B_{\alpha,p}(F) = 0$ follows.

PROOF OF THEOREM 6. As in the proof of Theorem 3, write $G_\alpha^f = u + v$. If $\xi \in \partial D - A$, then $\lim_{x \to \xi, x \in D} u(x) = 0$ by Lemma 3. Further, if $\xi \in \partial D - A_{p,\beta}$, then there exists a set $E \subset S_+$ such that $B_{\alpha,p}(E) = 0$ and

$$\lim_{r \downarrow 0} v(\xi + r\zeta) = 0 \qquad \text{for every} \quad \zeta \in S_+ - E,$$

because of Lemmas 8 and 9. Thus the proof of Theorem 6 is complete.

6. Perpendicular limits

Let $e=(0,\dots,0,1)\in S_+$.

THEOREM 7. *Let* $0\leqq\gamma<2p-1$, $p>1$ *and* f *be a non-negative measurable function on* D *satisfying* (1) *such that* $G_\alpha^f\not\equiv\infty$. *Then there exists a set* $E\subset\partial D$ *such that* $B_{\alpha-\gamma/p,p}(E)=0$ *and*

$$\lim_{r\downarrow 0} G_\alpha^f(\xi+re)=0 \qquad \text{for every} \quad \xi\in\partial D-E.$$

PROOF. As in the proof of Theorem 2, we write $G_\alpha^f=u+v$. First note that $\lim_{r\downarrow 0} u(\xi+re)=0$ for $\xi\in\partial D-A$ by Lemma 3. Since $H_{n-\alpha p+\gamma}(A)=0$ by the corollary to Lemma 5, $B_{\alpha-\gamma/p,p}(A)=0$ on account of [3; Theorem 21].

Let $r>0$, and consider the sets

$$E_i=\{x=(x_1,\dots,x_n);\ 2^{-i}\leqq x_n<2^{-i+1},\ v(x)\geqq a_{r,i}^{-1/p}\}$$

for $i=1,2,\dots$, where $\{a_{r,i}\}$ is a sequence of positive numbers such that $\lim_{i\to\infty} a_{r,i}=\infty$ but

$$\sum_{i=1}^\infty a_{r,i}\int_{\{y\in B_+(O,2r);\,2^{-i-1}<y_n<2^{-i+2}\}} f(y)^p y_n^\gamma dy<\infty.$$

Since $|x-y|<y_n$ if $|x-y|<x_n/2$, we see from Lemma 1 that

$$a_{r,i}^{-1/p}\leqq v(x)\leqq c_2\int_{\{y\in D;\,|x-y|<x_n/2\}}|x-y|^{\alpha-\gamma/p-n}y_n^{\gamma/p}f(y)dy$$

for $x\in E_i$. Hence it follows from the definition of $C_{\alpha-\gamma/p,p}$ that

$$C_{\alpha-\gamma/p,p}(E_i\cap B(O,r);\ B(O,2r))$$
$$\leqq c_2^p a_{r,i}\int_{\{y\in B_+(O,2r);\,2^{-i-1}<y_n<2^{-i+2}\}} f(y)^p y_n^\gamma dy,$$

which gives

$$\sum_{i=1}^\infty C_{\alpha-\gamma/p,p}(E_i\cap B(O,r);\ B(O,2r))<\infty.$$

Set $E(r)=\bigcap_{j=1}^\infty(\bigcup_{i=j}^\infty E_i\cap B(O,r))^*$. Then by properties (ii) and (iii) in Section 2, we have $C_{\alpha-\gamma/p,p}(E(r);\ B(O,2r))=0$, which implies that $B_{\alpha-\gamma/p,p}(E(r))=0$. Moreover, $\lim_{r\downarrow 0} v(\xi+re)=0$ for $\xi\in\partial D\cap B(O,r)-E(r)$. Thus $E=\bigcup_{r=1}^\infty E(r)$ has the required properties.

REMARK 1. In case $\gamma=0$, Theorem 7 is the best possible as to the size of the exceptional set; in fact, for $\gamma\leqq 0$ and a set $E\subset\partial D$ with $B_{\alpha,p}(E)=0$ we can find a non-negative measurable function f on D satisfying (1) such that $G_\alpha^f(\xi+i^{-1}e)=\infty$ for any $\xi\in E$ and any positive integer i.

REMARK 2. In case α is an integer and $0 \leqq \gamma < p-1$, Theorem 7 also follows from Theorem 3 in [7].

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