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On the trace mappings in the space $B_{1,\mu}(R^N)$

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Let μ be a temperate weight function on $\Xi^N=(R^N)'$, that is, a positive valued continuous function on Ξ^N such that

$$\mu(\xi+\eta) \leqq C(1+|\xi|^k)\mu(\eta), \quad \xi, \eta \in \Xi^N$$

with positive constants k and C[4, p. 7]. By $B_{p,\mu}(R^N)$, $1 \leqq p \leqq \infty$, we denote the set of all temperate distributions $u \in \mathscr{S}'(R^N)$ such that its Fourier transform $\hat{u}$ is a locally summable function and

$$\|u\|_{p,\mu}^p = (2\pi)^{-N} \int_{\Xi^N} |\hat{u}(\xi)|^p \mu^p(\xi) d\xi < \infty,$$

and when $p=\infty$ we shall interpret $\|u\|_{\infty,\mu}$ as ess. sup $|\hat{u}(\xi)\mu(\xi)|$ [1, p. 36].

In our previous papers [2, 3] we have investigated the trace mappings in the space $B_{p,\mu}(R^N)$ with $1<p<\infty$. The purpose of this paper is to develop the analogues of the theorems in [3] for the space $B_{1,\mu}(R^N)$.

Let $N=n+m$. We shall use the notations: $x=(x', t) \in R^N$, $x'=(x'_1,\dots, x'_n)$, $t=(t_1,\dots, t_m)$ and $\xi=(\xi', \tau) \in \Xi^N$, $\xi'=(\xi'_1,\dots, \xi'_n)$, $\tau=(\tau_1,\dots, \tau_m)$. For a polynomial $P(\xi)=\Sigma a_\alpha \xi^\alpha$ in ξ, we put $\bar{P}(\xi)=\Sigma \bar{a}_\alpha \xi^\alpha$ and $P(D)=\Sigma a_\alpha D^\alpha$ with $D=(D_1,\dots, D_N)$, $D_j=-i\partial/\partial_j$. $P^{(\alpha)}$ means $i^{|\alpha|}D^\alpha P$. Let μ_1 and μ_2 be temperate weight functions on Ξ^N. Then $\mu_1+\mu_2$, $\mu_1\mu_2$ and $1/\mu_1$ are temperate weight functions on Ξ^N.

If μ is a positive valued function on Ξ^N satisfying the inequality

$$\mu(\xi+\eta) \leqq (1+C|\xi|)^k \mu(\eta), \quad \xi, \eta \in \Xi^N$$

with positive constants k and C, then we have

$$(1+C|\xi|)^{-k} \leqq \mu(\xi+\eta)/\mu(\eta) \leqq (1+C|\xi|)^k,$$

which implies the continuity of μ[1, p. 34]. Putting $\nu(\xi')=\sup_\tau \mu(\xi', \tau)$, we have $\nu(\xi'+\eta') \leqq (1+C|\xi'|)^k \nu(\eta')$ for any $\xi', \eta' \in \Xi^n$.

Let μ be the function defined on Ξ by $\mu(\xi)=1$ for $\xi \leqq 0$, $\mu(\xi)=1+(2\xi-\xi^2)^{1/2}$ for $0<\xi<1$ and $\mu(\xi)=2$ for $\xi \geqq 1$. Then μ is a temperate weight function but it does not satisfy the inequality $\mu(\xi+\eta) \leqq (1+C|\xi|)^k \mu(\eta)$ with positive constants k and C. If $\mu(\xi)=1+\arg(\xi'+ie^\tau)$ on Ξ^2, then μ is a temperate weight function but $\nu(\xi')=\sup_\tau \mu(\xi', \tau)$ is not continuous.

According to L. Hörmander [1, p. 36] we shall first prove

PROPOSITION 1. *Let μ be a positive valued function on Ξ^N satisfying the inequality*

$$\mu(\xi + \eta) \leqq C(1 + |\xi|^k)\mu(\eta), \quad \xi, \eta \in \Xi^N$$

with positive constants k and C. For any $\delta>0$ if we put

$$\mu_\delta(\xi) = \sup_{\zeta\in\Xi^N} e^{-\delta|\zeta|}\mu(\xi-\zeta),$$

then $\mu_\delta(\xi)$ is a temperate weight function on Ξ^N and there are positive constants C', C_δ such that

$$\mu_\delta(\xi + \eta) \leqq (1 + C'|\xi|)^k\mu_\delta(\eta), \quad \xi, \eta \in \Xi^N$$

and

$$1 \leqq \mu_\delta(\xi)/\mu(\xi) \leqq C_\delta, \quad \xi \in \Xi^N.$$

PROOF. From the relations

$$\mu(\xi) \leqq \mu_\delta(\xi) \leqq C\mu(\xi) \sup_{\zeta\in\Xi^N} e^{-\delta|\zeta|}(1 + |\zeta|^k)$$

we have $1\leqq\mu_\delta(\xi)/\mu(\xi)\leqq C_\delta$, where $C_\delta = C\sup_{\zeta\in\Xi^N} e^{-\delta|\zeta|}(1+|\zeta|^k)$. For any $\xi, \eta \in \Xi^N$ we have

$$\begin{aligned}\mu_\delta(\xi + \eta) &= \sup_\zeta e^{-\delta|\zeta|}\mu(\xi + \eta - \zeta)\\ &\leqq C(1 + |\xi|^k)\sup_\zeta e^{-\delta|\zeta|}\mu(\eta - \zeta) = C(1 + |\xi|^k)\mu_\delta(\eta)\end{aligned}$$

and

$$\mu_\delta(\xi + \eta) = \sup_\zeta e^{-\delta|\xi+\eta-\zeta|}\mu(\zeta) \leqq \sup_\zeta e^{\delta|\xi|}e^{-\delta|\eta-\zeta|}\mu(\zeta) = e^{\delta|\xi|}\mu_\delta(\eta).$$

If $|\xi|\geqq 1$, then we have

$$C(1 + |\xi|^k) \leqq 2C|\xi|^k < (1 + (2C)^{1/k}|\xi|)^k$$

and if $|\xi|<1$, then we have

$$e^{\delta|\xi|} \leqq 1 + (e^\delta - 1)|\xi| \leqq \begin{cases} (1 + (e^\delta - 1)|\xi|)^k & (k \geqq 1)\\ (1 + (e^{\delta/k} - 1)|\xi|)^k & (k < 1).\end{cases}$$

Thus there exists a positive constant C' such that

$$\mu_\delta(\xi + \eta) \leqq (1 + C'|\xi|)^k\mu_\delta(\eta), \quad \xi, \eta \in \Xi^N,$$

which completes the proof.

Hereafter, by $\tilde{\mu}$ we denote μ_1 defined in the above proposition for a temperate weight function μ. Then $B_{p,\mu}(R^N) = B_{p,\tilde{\mu}}(R^N)$.

LEMMA 1. *Let P be a non-trivial polynomial on Ξ^N. Then the function $\tilde{P}_\infty$ defined by $\tilde{P}_\infty(\xi)=\max_{|\alpha|\geqq 0}|P^{(\alpha)}(\xi)|$ is a temperate weight function on Ξ^N and there exist positive constants C, M such that*

$$\tilde{P}_\infty(\xi+\eta)\leqq(1+C|\xi|)^M\tilde{P}_\infty(\eta),\quad \xi,\eta\in\Xi^N.$$

PROOF. Clearly $\tilde{P}_\infty>0$. From Taylor's formula $P^{(\alpha)}(\xi+\eta)=\sum_{|\beta|\geqq 0}(\beta!)^{-1}\cdot\xi^\beta P^{(\alpha+\beta)}(\eta)$, we have the inequality

$$|P^{(\alpha)}(\xi+\eta)|\leqq\tilde{P}_\infty(\eta)(1+C|\xi|)^M$$

with positive constants C and M. Thus we have $\tilde{P}_\infty(\xi+\eta)\leqq(1+C|\xi|)^M\tilde{P}_\infty(\eta)$.

Let μ be a temperate weight function on Ξ^N. Then $B_{1,\mu}(R^N)$ is a Banach space with the norm $\|\cdot\|_{1,\mu}$ and $\mathscr{S}(R^N)\subset B_{1,\mu}(R^N)\subset\mathscr{S}'(R^N)$ in the topological sense.

Let us consider the trace mappings in the space $B_{1,\mu}(R^N)$. For any $u(x',t)\in\mathscr{D}(R^N)$, the trace $u(x',0)$ on R^n belongs to the space $\mathscr{D}(R^n)\subset\mathscr{D}'(R^n)$. $\mathscr{D}(R^N)$ is dense in $B_{1,\mu}(R^N)$. If the mapping $\mathscr{D}(R^N)\ni u\to u(x',0)\in\mathscr{D}'(R^n)$ can be continuously extended from $B_{1,\mu}(R^N)$ into $\mathscr{D}'(R^n)$, then the extended mapping is called the trace mapping on R^n and the image of $u\in B_{1,\mu}(R^N)$ is called the trace of u and denoted by $u(x',0)$.

Since the strong dual of $B_{1,\mu}(R^N)$ is $B_{\infty,1/\mu}(R^N)$, the trace mapping is defined if and only if $\phi\otimes\delta\in B_{\infty,1/\mu}(R^N)$ for any $\phi\in\mathscr{D}(R^n)$, where δ is the Dirac measure in R_t^m.

PROPOSITION 2. *Let P be a non-trivial polynomial on Ξ^N. Then a necessary and sufficient condition that the trace mapping $u\to[P(D)u](x',0)$ from $B_{1,\mu}(R^N)$ into $\mathscr{D}'(R^n)$ may be defined, is that one of the following equivalent conditions is satisfied*:

(1) $\sup_\tau\tilde{P}_\infty(\xi',\tau)/\mu(\xi',\tau)<\infty$ *for some point* $\xi'\in\Xi^n$.

(2) $\sup_\tau|P(\xi',\tau)|/\mu(\xi',\tau)<\infty$ *for every point* $\xi'\in\Xi^n$.

In this case, $[P(D)u](x',0)\in B_{1,\mu_{\tilde{P},\infty}}(R^n)$ *with* $\mu_{\tilde{P},\infty}(\xi')=\inf_\tau\tilde{\mu}(\xi',\tau)/\tilde{P}_\infty(\xi',\tau)$.

PROOF. Suppose the trace mapping $u\to[P(D)u](x',0)$ from $B_{1,\mu}(R^N)$ into $\mathscr{D}'(R^n)$ may be defined. For any $\eta\in\Xi^N$ the map $u\to e^{i\langle x,\eta\rangle}u$ is continuous from $B_{1,\mu}(R^N)$ into itself and $P(D)e^{i\langle x,\eta\rangle}u=e^{i\langle x,\eta\rangle}P(D+\eta)u$. For any $\phi\in\mathscr{D}(R^n)$ the map

$$u\longrightarrow\langle[P(D+\eta)u](x',0),\phi\rangle=\langle u,\bar{P}(D+\eta)(\phi\otimes\delta)\rangle$$

is a continuous linear form on $B_{1,\mu}(R^N)$ and therefore

$$\bar{P}(D+\eta)(\phi\otimes\delta)\in(B_{1,\mu}(R^N))'=B_{\infty,1/\mu}(R^N)$$

for any $\eta\in\Xi^N$ and $\phi\in\mathscr{D}(R^n)$. Namely,

$$\bar{P}(\xi+\eta)\hat{\phi}(\xi')/\mu(\xi)=\hat{\phi}(\xi')\Sigma(\alpha!)^{-1}\eta^{\alpha}\bar{P}^{(\alpha)}(\xi)/\mu(\xi)\in L^{\infty}(\Xi^N),$$

which implies

$$\hat{\phi}(\xi')\bar{P}^{(\alpha)}(\xi)/\mu(\xi)\in L^{\infty}(\Xi^N).$$

As a result,

$$\hat{\phi}(\xi')\sup_{\tau}|\tilde{P}_{\infty}(\xi',\tau)|/\mu(\xi',\tau)<\infty \quad \text{for a.e.} \quad \xi'\in\Xi^n.$$

Since $\hat{\phi}(\xi'-\xi'_0)$ is the Fourier transform of $e^{i\langle x',\xi'_0\rangle}\phi(x')$, we have

$$\sup_{\tau}|\tilde{P}_{\infty}(\xi',\tau)|/\mu(\xi',\tau)<\infty \quad \text{for a.e.} \quad \xi'\in\Xi^n.$$

Put $\kappa=\tilde{P}_{\infty}/\tilde{\mu}$ and assume $\sup_{\tau}\kappa(\xi'_0,\tau)<\infty$ for a point $\xi'_0\in\Xi^n$. Then κ is a temperate weight function and we have

$$\sup_{\tau}\kappa(\xi',\tau)\leqq(1+C|\xi'-\xi'_0|)^k\sup\kappa(\xi'_0,\tau)$$

with positive constants k and C. Thus $\sup_{\tau}\kappa(\xi',\tau)$ is finite for every $\xi'\in\Xi^n$ and $\sup_{\tau}\kappa(\xi',\tau)=1/\mu_{\bar{P},\infty}(\xi')$ is a temperate weight function on Ξ^n.

The implication (1)⇒(2) is trivial.

Suppose (2) holds true. Let $u\in\mathscr{D}(R^N)$. For any $\phi\in\mathscr{D}(R^n)$ we have

$$\begin{aligned}|\langle[P(D)u](x',0),\bar{\phi}\rangle|&=(2\pi)^{-N}\left|\int P(\xi)\hat{u}(\xi)\bar{\hat{\phi}}(\xi')d\xi\right|\\&\leqq\{\sup_{\xi',\tau}(|\hat{\phi}(\xi')|\,|P(\xi',\tau)|/\mu(\xi',\tau))\}\,\|u\|_{1,\mu}.\end{aligned}$$

Let $P(\xi)=\Sigma(\alpha'!)^{-1}\xi'^{\alpha'}P^{(\alpha')}(0,\tau)$. From the inequality $\mu(0,\tau)\leqq C(1+|\xi'|^k)\mu(\xi)$ we have

$$\sup_{\tau}|P^{(\alpha')}(0,\tau)|/\mu(\xi',\tau)\leqq C(1+|\xi'|^k)\sup_{\tau}|P^{(\alpha')}(0,\tau)|/\mu(0,\tau)$$

and therefore $\sup_{\tau}|P(\xi',\tau)|/\mu(\xi',\tau)$ is a slowly increasing function of ξ'. Since $\hat{\phi}$ belongs to the space $\mathscr{S}(\Xi^n)$, the trace mapping $u\to[P(D)u](x',0)$ from $B_{1,\mu}(R^N)$ into $\mathscr{D}'(R^n)$ is defined and we have

$$\begin{aligned}\|[P(D)u](x',0)\|_{1,\mu_{\bar{P},\infty}}&=(2\pi)^{-n-m}\int\mu_{\bar{P},\infty}(\xi')\left|\int P(\xi)\hat{u}(\xi)d\tau\right|d\xi'\\&\leqq(2\pi)^{-N}\int\mu_{\bar{P},\infty}(\xi')(\sup_{\tau}|P(\xi',\tau)|/\mu(\xi',\tau))\left(\int|\hat{u}(\xi)|\mu(\xi)d\tau\right)d\xi'\\&\leqq\|u\|_{1,\mu}.\end{aligned}$$

Since $\mathscr{D}(R^N)$ is dense in $B_{1,\mu}(R^N)$ we see that $[P(D)u](x',0)\in B_{1,\mu_{\bar{P},\infty}}(R^n)$ for every $u\in B_{1,\mu}(R^N)$.

THEOREM 1. *Suppose* $\mu_{\bar{P},\infty}(\xi')=\inf_{\tau}\tilde{\mu}(\xi',\tau)/\tilde{P}_{\infty}(\xi',\tau)>0$. *Then each of the following conditions is necessary and sufficient in order that the trace mapping*

$$\mathscr{T}\colon B_{1,\mu}(R^N)\ni u\longrightarrow [P(D)u](x',0)\in B_{1,\mu_{\bar{P},\infty}}(R^n)$$

may be an epimorphism:

(1) *The range of the transposed map* ${}^t\mathscr{T}$ *is closed in* $B_{\infty,1/\mu}(R^N)$.

(2) $\mu_{\bar{P},\infty}$ *is equivalent to* ν_∞, *where* $1/\nu_\infty(\xi')=\sup_\tau|P(\xi',\tau)|/\mu(\xi',\tau)$: *Namely*, $C_1\nu_\infty\leqq\mu_{\bar{P},\infty}\leqq C_2\nu_\infty$ *with positive constants* C_1 *and* C_2.

(3) *If* $f(\xi')\bar{P}(\xi)/\mu(\xi)\in L^\infty(\Xi^N)$ *with* $f(\xi')\in L^1_{loc}(\Xi^n)$, *then* $f/\mu_{\bar{P},\infty}\in L^\infty(\Xi^n)$.

PROOF. For any $v\in B_{\infty,1/\mu_{\bar{P},\infty}}(R^n)$ and $f\in\mathscr{D}(R^N)$ we have

$$\begin{aligned}\langle\mathscr{T}f,\bar{v}\rangle &= (2\pi)^{-n}\int_{\Xi^n}([P(D)f](x',0))^\wedge(\xi')\overline{\hat{v}(\xi')}d\xi'\\ &= (2\pi)^{-N}\int_{\Xi^N}P(\xi)\hat{f}(\xi)\overline{\hat{v}(\xi')}d\xi\end{aligned}$$

and

$$\langle\overline{{}^t\mathscr{T}v},f\rangle = (2\pi)^{-N}\int_{\Xi^N}\overline{\widehat{{}^t\mathscr{T}v}(\xi)}\hat{f}(\xi)d\xi,$$

and therefore $\widehat{{}^t\mathscr{T}v}(\xi)=\hat{v}(\xi')\bar{P}(\xi)$.

If ${}^t\mathscr{T}v=0$, then

$$\text{ess. sup}_{\xi'}\,(|\hat{v}(\xi')|\sup_\tau|P(\xi',\tau)|/\mu(\xi',\tau))=0.$$

Since the polynomial $P(\xi',\tau)$ is non-trivial, $\sup_\tau|P(\xi',\tau)|/\mu(\xi',\tau)$ does not identically vanish in any relatively compact open subset of Ξ^n. Thus $\hat{v}(\xi')=0$ a.e. in Ξ^n, which implies $v=0$.

Thus $\mathscr{T}$ is an epimorphism if and only if the range of ${}^t\mathscr{T}$ is closed in $B_{\infty,1/\mu}(R^N)$.

Suppose (1) holds. Then we have

$$\|v\|_{\infty,1/\mu_{\bar{P},\infty}}\leqq C\|{}^t\mathscr{T}v\|_{\infty,1/\mu}$$

with a positive constant C for any $v\in B_{\infty,1/\mu_{\bar{P},\infty}}(R^N)$; namely,

$$\text{ess. sup}_{\xi'}|\hat{v}(\xi')|/\mu_{\bar{P},\infty}(\xi')\leqq C\ \text{ess. sup}_{\xi',\tau}|\hat{v}(\xi')P(\xi',\tau)|/\mu(\xi',\tau),$$

which implies $\nu_\infty\sim\mu_{\bar{P},\infty}$.

Suppose (2) holds. Let $f(\xi')\bar{P}(\xi)/\mu(\xi)\in L^\infty(\Xi^N)$ for any $f\in L^1_{loc}(\Xi^n)$. Then we have immediately $f/\mu_{\bar{P},\infty}\in L^\infty(\Xi^n)$.

Suppose (3) holds. We shall first note that

$$\sup_\tau|P(\xi',\tau)|/\mu(\xi',\tau)>0$$

for any $\xi'\in\Xi^n$. Let $\xi'_0\in\Xi^n$ and let B be a closed unit ball with center ξ'_0. Let E be the set of $f\in L^1_{loc}(\Xi^n)$ such that $\operatorname{supp} f\subset B$ and

$$\text{ess. sup}_{\xi',\tau}|f(\xi')P(\xi)|/\mu(\xi) < \infty.$$

Then E is a Banach space with the norm $\|f\|_E$:

$$\|f\|_E = \int_B |f(\xi')|d\xi' + \text{ess. sup}_{\xi'\in B,\tau\in\Xi^m}|f(\xi')P(\xi)|/\mu(\xi).$$

Let $f\in E$. Then $f/\mu_{\bar{P},\infty}\in L^\infty(\Xi^n)$ by (3). By the closed graph theorem, the map $f\to f/\mu_{\bar{P},\infty}$ is continuous from E into $L^\infty(\Xi^n)$ and there exists a positive constant C such that

$$\text{ess. sup}_{\xi'}|f(\xi')|/\mu_{\bar{P},\infty}(\xi') \leqq C\|f\|_E.$$

Taking the characteristic function $f=\chi_\varepsilon$ of a closed ball B_ε with center $\xi'_0\in\Xi^n$ and radius ε, $0<\varepsilon<1$, and passing to the limit $\varepsilon\to 0$, we have

$$0 < \sup_\tau \tilde{P}(\xi'_0, \tau)/\tilde{\mu}(\xi'_0, \tau) \leqq C \sup_\tau |P(\xi'_0, \tau)|/\mu(\xi'_0, \tau).$$

Let $\{v^j\}$ be any sequence in $B_{\infty,1/\mu_{\bar{P},\infty}}(R^n)$ such that ${}^t\mathcal{C}v^j$ tends to u in $B_{\infty,1/\mu}(R^N)$. Namely, $\hat{v}^j(\xi')\bar{P}(\xi)/\mu(\xi)$ tends to $\hat{u}/\mu$ in $L^\infty(\Xi^N)$. Then $\hat{v}^j(\xi')\cdot\sup_\tau|\bar{P}(\xi',\tau)|/\mu(\xi',\tau)$ is a Cauchy sequence in $L^\infty(\Xi^n)$. Since $\sup_\tau|\bar{P}(\xi',\tau)|/\mu(\xi',\tau)>0$ we see that $\hat{v}^j(\xi')$ converges in $L^1_{loc}(\Xi^n)$ to $f(\xi')$ and $\hat{u}=f(\xi')\bar{P}(\xi)$. By the condition (3) $f/\mu_{\bar{P},\infty}\in L^\infty(\Xi^n)$. Thus the range of ${}^t\mathcal{C}$ is closed in the space $B_{\infty,1/\mu}(R^N)$, which completes the proof.

COROLLARY. *If v_∞ is a temperate weight function, then $v_\infty\sim\mu_{\bar{P},\infty}$ and the trace mapping $u\to[P(D)u](x',0)$ from $B_{1,\mu}(R^N)$ into $B_{1,\mu_{\bar{P},\infty}}(R^n)$ is an epimorphism.*

PROOF. For any $\eta\in\Xi^N$ with $|\eta|\leqq 1$ we have

$$C_1/v_\infty(\xi') \geqq 1/v_\infty(\xi'+\eta') \geqq C\sup_\tau|P(\xi+\eta)|/\mu(\xi)$$

with positive constants C, C_1, and therefore

$$1/v_\infty(\xi') \geqq C'\sup_\tau|P^{(\alpha)}(\xi)|/\mu(\xi).$$

Thus we have $v_\infty\sim\mu_{\bar{P},\infty}$, which completes the proof.

EXAMPLE 1. Suppose $\mu_{\bar{P},\infty}(\xi')>0$. If the differential operator $P(D)$ is hypoelliptic, that is, $P^{(\alpha)}(\xi)/P(\xi)\to 0$ when $\xi\to\infty$ in R^N for $\alpha\neq 0$ [1, p. 100], then the trace mapping $u\to[P(D)u](x',0)$ from $B_{1,\mu}(R^N)$ into $B_{1,\mu_{\bar{P},\infty}}(R^n)$ is an epimorphism.

In fact, by the definition of hypoellipticity we see that there exist positive constants C and K such that

$$|P^{(\alpha)}(\xi)| \leqq C|P(\xi)| \qquad \text{for} \quad |\xi| > K.$$

Let $P(\xi)=\Sigma(\alpha''!)^{-1}\tau^{\alpha''}P^{(\alpha'')}(\xi',0)$. Even though P vanishes at ξ_0, there exists

α'' with $P^{(\alpha'')}(\xi_0', 0)\neq 0$ by hypoellipticity of P. Thus there exist positive constants C_0 and $\sigma_j\in\Xi^m$, $1\leqq j\leqq s$, such that for any $|\xi|\leqq K$

$$|P^{(\alpha)}(\xi)| \leqq C_0(|P(\xi)| + |P(\xi', \tau + \sigma_1)| +\cdots+|P(\xi', \tau + \sigma_s)|).$$

Consequently there exists a positive constant C_1 such that for any $\xi\in\Xi^N$

$$|P^{(\alpha)}(\xi)| \leqq C_1(|P(\xi)| + |P(\xi', \tau + \sigma_1)| +\cdots+|P(\xi', \tau + \sigma_s)|).$$

Since μ is a temperate weight function we have

$$\begin{aligned}\sup_\tau |P(\xi', \tau + \sigma_j)|/\mu(\xi', \tau) &= \sup_\tau |P(\xi', \tau)|/\mu(\xi', \tau - \sigma_j)\\ &\leqq C_j \sup_\tau |P(\xi', \tau)|/\mu(\xi', \tau)\end{aligned}$$

with a positive constant C_j and therefore

$$\sup_\tau |P^{(\alpha)}(\xi', \tau)|/\mu(\xi', \tau) \leqq C \sup_\tau |P(\xi', \tau)|/\mu(\xi', \tau)$$

with a positive constant C. Thus $\nu_\infty\sim\mu_{\mathfrak{p},\infty}$. By virtue of Theorem 1 the trace mapping $u\to[P(D)u](x', 0)$ from $B_{1,\mu}(R^N)$ into $B_{1,\mu_{\mathfrak{p},\infty}}(R^n)$ is an epimorphism.

REMARK. With the same notations as in [3] we can similarly show that if $\mu_{\mathfrak{p},p'}(\xi')>0$ with $1<p'<\infty$ and P is hypoelliptic, then the trace mapping $u\to [P(D)u](x', 0)$ from $B_{p,\mu}(R^N)$ into $B_{p,\mu_{\mathfrak{p},p'}}(R^n)$ is an epimorphism.

EXAMPLE 2. If $P(D)$ is a polynomial of D_t and

$$1/\nu_\infty(\xi') = \sup_\tau |P(\tau)|/\tilde{\mu}(\xi', \tau) < \infty,$$

then ν_∞ is a temperate weight function on Ξ^n and the trace mapping $u\to [P(D)u](x', 0)$ from $B_{1,\mu}(R^N)$ into $B_{1,\nu_\infty}(R^n)$ is an epimorphism.

In fact, from the relations

$$1/\nu_\infty(\xi' + \eta') = \sup_\tau |P(\tau)|/\tilde{\mu}(\xi' + \eta', \tau) \geqq (1 + C|\xi'|)^{-k} \sup_\tau |P(\tau)|/\tilde{\mu}(\eta', \tau)$$

with positive constants k and C, we have $\nu_\infty(\xi'+\eta')\leqq(1+C|\xi'|)^k\nu_\infty(\eta')$. Since $\nu_\infty(\xi')>0$, ν_∞ is a temperate weight function on Ξ^n. By virtue of the above corollary the trace mapping $u\to[P(D)u](x', 0)$ from $B_{1,\mu}(R^N)$ into $B_{1,\nu_\infty}(R^n)$ is an epimorphism.

Suppose that for some non-negative integer M

$$\inf_\tau |\tau|^{-M}\mu(\xi', \tau) > 0.$$

For any $k=(k_1,\dots, k_m)$, k_j being non-negative integers, such that $|k|\leqq M$ we put

$$\nu_{k,\infty}(\xi') = \inf_\tau |\tau^{-k}|\tilde{\mu}(\xi', \tau).$$

Then $\nu_{k,\infty}$ is a temperate weight function on Ξ^n. We consider the trace mapping $\mathscr{T}: u \to \{D_t^k u(x', 0)\}$ from $B_{1,\mu}(R^N)$ into $\prod_{|k|\leqq M} B_{1,\nu_{k,\infty}}(R^n)$.

THEOREM 2. *The trace mapping $\mathscr{T}$ is an epimorphism if and only if the range of the transposed map ${}^t\mathscr{T}$ is closed in $B_{\infty,1/\mu}(R^N)$.*

PROOF. Let $\tilde{v} = \{v_k\} \in \prod_{|k|\leqq M} B_{\infty,1/\nu_{k,\infty}}(R^n)$. In the same way as in the proof of Theorem 2 [3, p. 174] we have

$$\widehat{{}^t\mathscr{T}\tilde{v}}(\xi) = \sum_{|k|\leqq M} \hat{v}_k(\xi')\tau^k.$$

By this equation we see that the transposed map ${}^t\mathscr{T}$ is injective. Thus $\mathscr{T}$ is an epimorphism if and only if the range of ${}^t\mathscr{T}$ is closed in $B_{\infty,1/\mu}(R^N)$.

In the same way as in the proofs of Theorem 3 and its corollary in our previous paper [3] we can prove

THEOREM 3. *The following conditions are equivalent*:

(1) *If $u \in B_{\infty,1/\mu}(R^N)$ and $\hat{u}(\xi) = \sum_{|k|\leqq M} f_k(\xi')\tau^k$, then $f_k/\nu_{k,\infty} \in L^\infty(\Xi^n)$ for any k with $|k| \leqq M$.*

(2) *If $u \in B_{\infty,1/\mu}(R^N)$ and $\hat{u}(\xi) = \sum_{|k|\leqq M} f_k(\xi')\tau^k$, then*

$$\hat{u}(\xi', \tau_1, \ldots, \tau_{j-1}, 2^{-1}\tau_j, \tau_{j+1}, \ldots, \tau_m)/\mu \in L^\infty(\Xi^N)$$

for every $j = 1, 2, \ldots, m$.

(3) *If $u \in B_{\infty,1/\mu}(R^N)$ and $\hat{u}(\xi) = \sum_{|k|\leqq M} f_k(\xi')\tau^k$, then*

$$\hat{u}(\xi', 2^{-i_1}\tau_1, \ldots, 2^{-i_m}\tau_m) \in L^\infty(\Xi^N)$$

for any non-negative integers i_j.

In this case, the trace mapping $u \to \{D_t^k u(x', 0)\}$ from $B_{1,\mu}(R^N)$ into $\prod_{|k|\leqq M} B_{1,\nu_{k,\infty}}(R^n)$ is an epimorphism.

COROLLARY. *If $\mu(\xi', \tau_1, \ldots, \tau_{j-1}, 2\tau_j, \tau_{j+1}, \ldots, \tau_m) \geqq C\mu(\xi)$ with a positive constant C for $j = 1, 2, \ldots, m$, then the trace mapping $\mathscr{T}: u \to \{D_t^k u(x', 0)\}$ from $B_{1,\mu}(R^N)$ into $\prod_{|k|\leqq M} B_{1,\nu_{k,\infty}}(R^n)$ is an epimorphism.*

PROPOSITION 3. *Let $\{f_k\}$ be an arbitrary element of $\prod_{|k|\leqq M} B_{1,\nu_{k,\infty}}(R^n)$ and suppose for each k there exist a positive valued continuous function λ_k on Ξ^n and a slowly increasing continuous function Φ_k on Ξ^m such that*

$$\mu(\xi', \lambda_k(\xi')\tau) \leqq \lambda_k^{|k|}(\xi')\nu_{k,\infty}(\xi')\Phi_k(\tau).$$

Let $\psi \in \mathscr{D}(R^m)$ satisfy $\psi = 1$ in a neighbourhood of 0. *If we put*

$$\hat{u}_{x'}(\xi', t) = \sum_{|k|\leqq M} (k!)^{-1}\hat{f}_k(\xi')(it)^k\psi(\lambda_k(\xi')t),$$

then $u \in B_{1,\mu}(R^N)$ *and* $D_\tau^k u(x', 0) = f_k(x')$ *for* $|k| \leqq M$.

PROOF. From the equation

$$\begin{aligned}\hat{u}(\xi) &= \sum_{|k|\leqq M} (-i)^{|k|}(k!)^{-1}\hat{f}_k(\xi')D_\tau^k \int_{\Xi^m} \psi(\lambda_k t)e^{-i\langle t,\tau\rangle}dt \\ &= \sum_{|k|\leqq M} (-1)^{|k|}(k!)^{-1}\hat{f}_k(\xi')\lambda_k^{-|k|-m}D_\tau^k\hat{\psi}(\lambda_k^{-1}\tau)\end{aligned}$$

we have

$$\begin{aligned}\int_{\Xi^N} |\hat{u}|\mu d\xi &\leqq \sum_{|k|\leqq M} (k!)^{-1}\int_{\Xi^n} \lambda_k^{-|k|}|\hat{f}_k(\xi')| \int_{\Xi^m} |D_\tau^k\hat{\psi}(\tau)|\mu(\xi', \lambda_k(\xi')\tau)d\xi' d\tau \\ &\leqq \sum_{|k|\leqq M} (k!)^{-1}\int_{\Xi^n} |\hat{f}_k(\xi')|\nu_{k,\infty}(\xi')d\xi' \int_{\Xi^m} |D_\tau^k\hat{\psi}(\tau)|\Phi_k(\tau)d\tau < \infty.\end{aligned}$$

Thus $u \in B_{1,\mu}(R^N)$ and clearly $D_\tau^k u(x', 0) = f_k(x')$ for $|k| \leqq M$.

EXAMPLE 3. Let μ_1, μ_2 be temperate weight functions defined on Ξ^n such that $\mu_2 \leqq C\mu_1$ with a positive constant C and put $\mu(\xi) = \mu_1(\xi') + |\tau|^a\mu_2(\xi')$ with a positive real number a. Then μ is a temperate weight function on Ξ^N and

$$\nu_{k,\infty} \sim \mu_1^{1-|k|/a}\mu_2^{|k|/a} \qquad \text{for} \quad |k| \leqq a.$$

If we take $\lambda_k = (\mu_1/\mu_2)^{1/a}$ and $\Phi_k(\tau) = 1 + |\tau|^a$ for $|k| \leqq a$, then

$$\mu(\xi', \lambda_k(\xi')\tau) \leqq C\lambda_k^{|k|}(\xi')\nu_{k,\infty}(\xi')\Phi_k(\tau).$$

In fact, from the relations

$$\begin{aligned}\nu_{k,\infty}(\xi') &\sim \inf_\tau |\tau^k|^{-1}(\mu_1(\xi') + |\tau|^a\mu_2(\xi')) \\ &= \mu_1^{1-|k|/a}\mu_2^{|k|/a} \inf_\tau |\tau^k|^{-1}(1 + |\tau|^a),\end{aligned}$$

we have $\nu_{k,\infty} \sim \mu_1^{1-|k|/a}\mu_2^{|k|/a}$ and therefore

$$\begin{aligned}\lambda_k^{|k|}(\xi')\nu_{k,\infty}(\xi')\Phi_k(\tau) &\sim (\mu_1/\mu_2)^{|k|/a}\mu_1^{1-|k|/a}\mu_2^{|k|/a}(1 + |\tau|^a) \\ &= \mu_1(\xi')(1 + |\tau|^a).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mu(\xi', \lambda_k(\xi')\tau) &= \mu_1(\xi') + |\lambda_k(\xi')\tau|^a\mu_2(\xi') \\ &= \mu_1(\xi') + |\tau|^a\mu_1(\xi') = \mu_1(\xi')(1 + |\tau|^a).\end{aligned}$$

Thus Proposition 3 is applicable to this case.

In Section 5 of our previous paper [3], we have investigated the relation between the trace mappings and other notions in the space $B_{p,\mu}(R^N)$, $1 < p < \infty$. With necessary modifications, our treatments will also hold for the space $B_{1,\mu}(R^N)$.

Under the same notations and terminologies as in the paper [3] we have

THEOREM 4. *For the space* $B_{1,\mu}(R^N)$ *the following statements are equivalent:*

(1) *The trace mapping* $B_{1,\mu}(R^N) \ni u \to u(x', 0) \in \mathscr{D}'(R^n)$ *is defined.*

(2) *The section for* $t=0$ *exists for every* $u \in B_{1,\mu}(R^N)$.

(2)′ *The condition* (2) *holds in the strict sense.*

(3) *The partial product* δu *exists for every* $u \in B_{1,\mu}(R^N)$, *where* δ *is the Dirac measure in* R^m_t.

(3)′ *The partial product* $\delta \cdot u$ *exists for every* $u \in B_{1,\mu}(R^N)$.

(4) *The distributional limit* $\lim_{j \to \infty} (1 \otimes \delta)(u * \rho_j)$ *exists for a fixed restricted* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R^N)$, *for every* $u \in B_{1,\mu}(R^N)$.

(5) *The distributional limit* $\lim_{j \to \infty} \rho_j u$ *exists for a fixed* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathscr{D}(R^m_t)$, *for every* $u \in B_{1,\mu}(R^N)$.

Let μ be a temperate weight function on Ξ^N and suppose $\inf_\tau \mu(0, \tau) > 0$. If we put $\nu_\infty(\xi') = \inf_\tau \tilde{\mu}(\xi', \tau)$, then ν_∞ is a temperate weight function on Ξ^n. Let $t_0 \in R^n$ and $u \in \mathscr{D}(R^N)$. In the proof of Proposition 2 we have shown

$$\|u(\cdot, t_0)\|_{1,\nu_\infty} \leq \|u\|_{1,\mu}.$$

Thus the trace $u(\cdot, t_0)$ on $t = t_0$ belongs to the space $B_{1,\nu_\infty}(R^n)$ for any $u \in B_{1,\mu}(R^N)$. Furthermore $u(\cdot, t)$ may be considered as a $B_{1,\nu_\infty}(R^n)$-valued continuous function $\boldsymbol{u}(t)$ of t.

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