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Admissible null controllability and optimal time control

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1. Introduction

This paper is concerned with a class of control problems where the control (or input) $f(t)$ and the output (or trajectory) $u(t)$ are related by the differential equation

$$(1.1) \qquad du(t)/dt = Au(t) + Bf(t).$$

Here A is the infinitesimal generator of a C_0 semigroup of bounded linear operators $U(t)$, $t \geqq 0$, on a Banach space X, and B is a bounded linear operator from a Banach space Y to X.

For any $u_0 \in X$ and Y-valued locally summable function f, we define

$$(1.2) \qquad u(t) = U(t)u_0 + \int_0^t U(t-s)Bf(s)ds$$

to be a mild solution of (1.1) with the initial state $u(0) = u_0$. It is well known that if $f(t)$ is continuously differentiable in $t > 0$ and u_0 is in the domain of A, then $u(t)$ defined by (1.2) is a genuine solution of (1.1) with $u(0) = u_0$.

When a subspace D of X, which is called a controlled space, is given, the usual controllability problem is as follows.

For any u_0 and u_1 in D, is there at all a control f which steers the initial state u_0 to the final state u_1?

In this paper, unlike the usual controllability problem, we require that the controls are constrained in a prescribed set, which is called a constraint set. When a constraint set is given, as are posed by Fattorini [5], [6], the following three questions arise naturally.

(a) For any u_0 and u_1 in D, is there at all a control f in the constraint set which steers the initial state u_0 to the final state u_1?

(b) Assuming the answer to (a) is affirmative, does there exist a control f_0 that does the transfer in minimum time? When there exists such an f_0, it is called an optimal time control.

(c) If there exists an optimal time control, is it unique? What additional properties does it have?

In case the dimensions of X and Y are finite and the constraint set is $\{f(t) \mid f(t) \in W$ almost everywhere in $t\}$, where W is a compact set in Y, some necessary and sufficient conditions for the controllability in the constraint set have been obtained by many authors, e.g. Lee and Markus [9], Saperstone [14], Saperstone and Yorke [15], Brammer [3], etc. When W is a unit ball in Y, Bellman, Glicksberg and Gross [2] showed that (b) is affirmative, and gave an answer to (c) in terms of the so-called "maximal" principle or "bang-bang" principle.

In case the dimensions of X and Y are infinite, the answer to the question (b) is by now well known ([1], [4]). As for the problems (a) and (c), a few results have been obtained for special control systems. Fattorini [4] considered the problem (c) in the case where $D=X=Y$ and B is the identity map. Further in [5] he considered the problems (a) and (c) for a control system described by the wave equation. The control system considered by Fattorini is given by

$$\partial^2 u/\partial t^2 - \sum \partial^2 u/\partial x_i^2 = f(x, t) \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.3}$$

$$u(x, t) = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty) \tag{1.4}$$

with constraint

$$\int_\Omega |f(x, t)|^2 dx \leqq 1 \quad \text{almost everywhere in} \quad t, \tag{1.5}$$

and the controlled space $D = H_0^1(\Omega) \times L^2(\Omega)$. Here Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$ and $H_0^1(\Omega)$ is the usual Sobolev space. The system (1.3), (1.4) can be reduced to a first order equation in the usual way: Set $V(t) = {}^t[u(t), (\partial u/\partial t)(t)]$ and write (1.3) in the form

$$dV(t)/dt = AV(t) + Bf(t) \tag{1.6}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ \triangle & 0 \end{bmatrix}$$

and B is the projection to the second coordinate. However, Fattorini did not treat the system in the form (1.6) because B is not the identity operator ([5], Footnote 3).

In this paper we consider the equation (1.1) where B is not the identity operator and show results similar to those in the case the dimensions of X and Y are finite.

As for problem (c), Fattorini [6] and Schmidt [16] considered a system described by the heat equation with boundary control. But here we do not refer to that control system.

2. Basic notations and definitions

Let X and Y be Banch spaces. For $1 \leqq p \leqq \infty$, $L^p(0, T; Y)$ denotes the space of all strongly measurable, Y-valued functions $f(t)$ defined in $0 \leqq t \leqq T$ with

$$\|f\|_p = \left(\int_0^T \|f(t)\|^p dt\right)^{1/p} < \infty$$

endowed with the norm $\|\cdot\|_p$ (the definition is modified in the usual way when $p=\infty$). We take the constraint set of the controls as

$$\mathscr{F}_\eta^p = \cup_{T>0} \{f \in L^p(0, T; Y) \mid \|f\|_p \leqq \eta\} \qquad \text{for} \quad \eta > 0. \tag{2.1}$$

As is stated in the introduction, A is the infinitesimal generator of a C_0 semigroup $U(t)$, $t \geqq 0$, on X and B is a bounded linear operator from Y to X. Furthermore we define the attainable set K_T^p by

$$K_T^p = \left\{\int_0^T U(T-s)Bf(s)ds \,\middle|\, f \in L^p(0, T; Y)\right\}. \tag{2.2}$$

To state admissible controllability, we recall some definitions.

Definition 1. (1) A subspace D of X is said to be *controllable in* $L^p(0, T; Y)$ if $K_T^p \supset D$.

(2) A subspace D of X is said to be *null controllable in* $L^p(0, T; Y)$ if for each $u_0 \in D$ there exists $f \in L^p(0, T; Y)$ such that

$$U(T)u_0 + \int_0^T U(T-s)Bf(s)ds = 0.$$

(3) The control system (1.1) is said to be *exactly controllable* and *exactly null controllable in* $L^p(0, T; Y)$ if the whole space X is controllable and null controllable in $L^p(0, T; Y)$ respectively.

Definition 2. (1) A subspace D of X is said to be *admissibly controllable in the constraint set* $\mathscr{F}_\eta^p$ if for each u_0 and u_1 in D, there exists $f \in \mathscr{F}_\eta^p$ such that

$$u_1 = U(T)u_0 + \int_0^T U(T-s)Bf(s)ds.$$

(2) A subspace D of X is said to be *admissibly null controllable in the constraint set* $\mathscr{F}_\eta^p$ if for each $u_0 \in D$ there exists $f \in \mathscr{F}_\eta^p$ such that

$$U(T)u_0 + \int_0^T U(T-s)Bf(s)ds = 0.$$

(3) The control system (1.1) is said to be *admissibly controllable* and *admissibly null controllable in* $\mathscr{F}_\eta^p$ if the whole space X is admissibly controllable and admissibly null controllable in $\mathscr{F}_\eta^p$ respectively.

3. Admissible controllability

In this section we discuss problem (a), that is, the admissible controllability in the constraint set $\mathscr{F}^p_\eta$ $(p \neq 1)$.

THEOREM 1. *Let* $1 < p \leqq \infty$ *and assume the following* (1)$\sim$(4):

(1) *the controlled space* D *is a Banach space endowed with a norm* $\|\cdot\|_D$ *stronger than the norm* $\|\cdot\|_X$ *of* X, *that is, there exists some positive constant* γ *such that* $\|u\|_X \leqq \gamma \|u\|_D$ *for any* $u \in D$;

(2) *the controlled space* D *is invariant under* $U(t)$ *for all* $t \geqq 0$, *that is,* $U(t)D \subset D$ *for all* $t \geqq 0$;

(3) $U(t)$ *is contractive on* D, *that is,* $\|U(t)u\|_D \leqq \|u\|_D$ *for all* $t \geqq 0$;

(4) *the controlled space* D *is null controllable in* $L^p(0, T_0; Y)$ *for some* $T_0 > 0$.

Then the controlled space D *is admissibly null controllable in the constraint set* $\mathscr{F}^p_\eta$ *for any positive* η.

PROOF. Let $1 < p < \infty$. First we define a closed subspace $\mathscr{N}$ of $L^p(0, T_0; Y)$ as

$$\mathscr{N} = \left\{ f \in L^p(0, T_0; Y) \,\middle|\, \int_0^{T_0} U(T_0 - s)Bf(s)ds = 0 \right\}$$

and denote by $\mathscr{Z}$ the quotient space $L^p(0, T_0; Y)/\mathscr{N}$. By assumption (4), to each $u_0 \in D$ there corresponds $f_0 \in L^p(0, T_0; Y)$ satisfying

$$U(T_0)u_0 + \int_0^{T_0} U(T_0 - s)Bf_0(s)ds = 0.$$

Let F be the operator which maps u_0 to the equivalent class of f_0. By the boundedness of $U(T_0)$ on X and B from Y to X, and by assumption (1), it is easy to see that F is a closed operator from D to $\mathscr{Z}$. Hence by the closed graph theorem, F is a bounded operator from D to $\mathscr{Z}$.

For a positive number α, let us put

$$A_\alpha = \left\{ u_0 \in D \,\middle|\, \begin{array}{l} \text{there exists } f \in L^p(0, T_0; Y) \text{ such that } \|f\|_p^p \leqq \alpha \\ \text{and } U(T_0)u_0 + \int_0^{T_0} U(T_0 - s)Bf(s)ds = 0 \end{array} \right\}$$

For any given $u_0 \in D$, let $L = \|u_0\|_D$. Then by the boundedness of F, there exists a positive number M such that $B_L = \{u \in D \mid \|u\|_D \leqq L\}$ is contained in A_M.

Now we choose a sequence $\{u_k\}_{1 \leqq k \leqq n}$ in D so that

$$u_1 = u_0 - Lu_0/(n\|u_0\|_D) = (1 - 1/n)u_0, \tag{3.1}$$

$$u_k = U(T_0)u_{k-1} - LU(T_0)u_{k-1}/(n\|U(T_0)u_{k-1}\|_D) \tag{3.2}$$

for $2 \leqq k \leqq n$. By assumption (2), $\{u_k\}_{1 \leqq k \leqq n}$ is well defined. Furthermore by assumption (3),

$$\|u_k\|_D = \|U(T_0)u_{k-1}\|_D - L/n \leqq \|u_{k-1}\|_D - L/n$$

for $k \geqq 2$. Thus there exists an integer m such that

$$\|u_m\|_D \leqq L/n, \qquad 1 \leqq m \leqq n.$$

If $v \in D$ and $\|v\|_D \leqq L/n$, then $nv \in B_L$. Thus there exists $f \in L^p(0, T_0; Y)$ such that

$$\int_0^{T_0} \|f(t)\|^p dt \leqq M, \quad U(T_0)(nv) + \int_0^{T_0} U(T_0 - s)Bf(s)ds = 0.$$

Putting $g(t) = f(t)/n$, we have

$$\int_0^{T_0} \|g(t)\|^p dt \leqq M/n^p, \quad U(T_0)v + \int_0^{T_0} U(T_0 - s)Bg(s)ds = 0.$$

Since $\|u_0 - u_1\|_D = \|U(T_0)u_{k-1} - u_k\|_D = L/n$ $(2 \leqq k \leqq m)$, there exists $f_k \in L^p(0, T_0; Y)$ satisfying

$$\int_0^{T_0} \|f_k(t)\|^p dt \leqq M/n^p \tag{3.3}$$

for $1 \leqq k \leqq m$, and

$$U(T_0)[u_0 - u_1] + \int_0^{T_0} U(T_0 - s)Bf_1(s)ds = 0, \tag{3.4}$$

$$U(T_0)[U(T_0)u_{k-1} - u_k] + \int_0^{T_0} U(T_0 - s)Bf_k(s)ds = 0 \tag{3.5}$$

for $2 \leqq k \leqq m$. Since

$$\|U(T_0)u_m\|_D \leqq \|u_m\|_D \leqq L/n,$$

there exists $f_{m+1} \in L^p(0, T_0; Y)$ such that

$$\int_0^{T_0} \|f_{m+1}(t)\|^p dt \leqq M/n^p \tag{3.6}$$

and

$$U(T_0)[U(T_0)u_m] + \int_0^{T_0} U(T_0 - s)Bf_{m+1}(s)ds = 0. \tag{3.7}$$

By the equations (3.4), (3.5) and (3.7), we obtain

$$\begin{aligned}
0 &= U(T_0)U(T_0)u_m + \int_0^{T_0} U(T_0-s)Bf_{m+1}(s)ds \\
&= U(T_0)[U(T_0)U(T_0)u_{m-1} \\
&\qquad + \int_0^{T_0} U(T_0-s)Bf_m(s)ds] + \int_0^{T_0} U(T_0-s)Bf_{m+1}(s)ds \\
&= U(2T_0)[U(T_0)U(T_0)u_{m-2} + \int_0^{T_0} U(T_0-s)Bf_{m-1}(s)ds] \\
&\qquad + \int_0^{T_0} U(2T_0-s)Bf_m(s)ds + \int_0^{T_0} U(T_0-s)Bf_{m+1}(s)ds \\
&= \cdots \\
&= U(mT_0)[U(T_0)u_0 + \int_0^{T_0} U(T_0-s)Bf_1(s)ds] \\
&\qquad + \textstyle\sum_{k=1}^{m} \displaystyle\int_0^{T_0} U(kT_0-s)Bf_{m+2-k}(s)ds \\
&= U((m+1)T_0)u_0 + \textstyle\sum_{k=1}^{m+1} \displaystyle\int_0^{T_0} U(kT_0-s)Bf_{m+2-k}(s)ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^{T_0} U(kT_0-s)Bf_{m+2-k}(s)ds \\
&\quad = \int_{(m+1-k)T_0}^{(m+2-k)T_0} U((m+1)T_0-s)Bf_{m+2-k}(s-(m+1-k)T_0)ds,
\end{aligned}$$

we have

$$U((m+1)T_0)u_0 + \int_0^{(m+1)T_0} U((m+1)T_0-s)Bg(s)ds = 0.$$

Here

$$g(t) = f_{m+2-k}(t-(m+1-k)T_0) \text{ for } (m+1-k)T_0 \leqq t < (m+2-k)T_0, \quad 1 \leqq k \leqq m+1.$$

By the inequalities (3.3) and (3.6),

$$\int_0^{(m+1)T_0} \|g(t)\|^p dt = \textstyle\sum_{k=1}^{m+1} \displaystyle\int_0^{T_0} \|f_{m+2-k}(t)\|^p dt \leqq (m+1)M/n^p \leqq (n+1)M/n^p.$$

If we choose n so large that

$$(n+1)M/n^p \leqq \eta^p,$$

then $g \in \mathscr{F}_\eta^p$. Thus we have admissible null controllability of the controlled space D in $\mathscr{F}_\eta^p$.

The proof for the case $p=\infty$ is similar.

When the whole space X is taken as the controlled space, assumptions (1), (2) are trivially satisfied and assumption (3) means that $U(t)$ is a contraction C_0 semigroup. Thus we have

COROLLARY 1. *Let $1<p\leqq\infty$ and A be the infinitesimal generator of a contraction C_0 semigroup. If the control system* (1.1) *is exactly null controllable in $L^p(0, T_0; Y)$, then the system* (1.1) *is admissibly null controllable in $\mathscr{F}^p_\eta$ for any positive η.*

If A is the generator of a contraction C_0 semigroup $U(t)$ and the domain of $(-A)^\alpha$ $(\alpha\geqq 0)$, $D((-A)^\alpha)$, is taken as the controlled space, then assumptions (1), (2) and (3) are satisfied. Here $D((-A)^\alpha)$ is endowed with the norm

$$\|w\|_{D((-A)^\alpha)} = \|w\|_X + \|(-A)^\alpha w\|_X \qquad \text{for} \quad w\in D((-A)^\alpha).$$

Thus we have

COROLLARY 2. *Let $1<p\leqq\infty$ and A be the infinitesimal generator of a contraction C_0 semigroup. For some positive α, if $D((-A)^\alpha)$ is null controllable in $L^p(0, T; Y)$, then $D((-A)^\alpha)$ is admissibly null controllable in $\mathscr{F}^p_\eta$ for any positive η.*

REMARK 1. We cannot obtain the same result in the case $p=1$. Here we show two simple examples which satisfy the assumptions of Theorem 1, but one is admissibly null controllable and the other is not admissibly null controllable in $\mathscr{F}^1_\eta$.

Let us consider the heat equation

$$[\partial u/\partial t](x, t) - \Delta u(x, t) = f(x, t) \quad \text{in} \quad \Omega\times(0, T) \tag{3.8}$$

with Dirichlet or Neumann boundary condition, that is,

$$u(x, t) = 0 \quad \text{on} \quad \partial\Omega\times(0, T) \tag{3.9}$$

or

$$[\partial u/\partial n](x, t) = 0 \quad \text{on} \quad \partial\Omega\times(0, T). \tag{3.10}$$

Here Ω is a bounded domain with smooth boundary $\partial\Omega$, $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$. Let us put $X=Y=L^2(\Omega)$, $A=\Delta$ with $D(A)=H^2(\Omega)\cap H^1_0(\Omega)$ or $\{u\in H^2(\Omega)\,|\,\partial u/\partial n=0\}$ according as the boundary condition is (3.9) or (3.10), where $H^1_0(\Omega)$ and $H^2(\Omega)$ denote the usual Sobolev spaces. Then the control system

$$du(t)/dt = Au(t) + f(t) \tag{3.11}$$

means (3.8) with (3.9) or (3.10). These control systems are exactly null controllable in $L^{\infty}(0, T; Y)$ at any positive time T. (See [7], [13].) Therefore these are null controllable in $L^1(0, T; Y)$.

First let us consider the control system (3.8) with Dirichlet boundary condition (3.9). Let $\{\lambda_k\}_{k=1,2,\dots}$, $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$, be the eigenvalues of $-\Delta$ with Dirichlet boundary condition and $\{\varphi_{kl}\}_{l=1,2,\dots,m_k}$ be the eigenfunctions for the eigenvalue λ_k, where m_k is the multiplicity, such that $\{\varphi_{kl}\}_{k=1,2,\dots;l=1,2,\dots,m_k}$ form an orthonormal basis for the space $L^2(\Omega)$. If the admissible null controllability holds, then for any $u_0 \in L^2(\Omega)$ there exists $T(>0)$ and $f(x, t) \in \mathscr{F}^1_\eta$ satisfying the equality

$$U(T)u_0 + \int_0^T U(T-s)f(s)ds = 0. \tag{3.12}$$

Let us expand $u_0(x)$ and $f(x, t)$ as

$$u_0(x) = \sum c_{kl}\varphi_{kl}(x), \tag{3.13}$$

$$f(x, t) = \sum f_{kl}(t)\varphi_{kl}(x). \tag{3.14}$$

Then

$$\sum |c_{kl}|^2 = \|u_0\|^2 \tag{3.15}$$

and

$$\sum |f_{kl}(t)|^2 = \|f(t)\|^2 \tag{3.16}$$

for almost every $t \in [0, T]$. The equation (3.12) is expressed as

$$\sum c_{kl}\varphi_{kl}(x) \exp(-\lambda_k T) + \sum \int_0^T f_{kl}(s)\varphi_{kl}(x) \exp(-\lambda_k(T-s))ds = 0. \tag{3.17}$$

Comparing the coefficients of φ_{kl}, we have

$$c_{kl} \exp(-\lambda_k T) + \int_0^T f_{kl}(s) \exp(-\lambda_k(T-s))ds = 0. \tag{3.18}$$

Now let

$$f_{kl}(s) = \lambda_k c_{kl}/[\exp(\lambda_k T) - 1].$$

Then it is easy to see that the equality (3.18) holds for each k, l. Furthermore we have

$$\sum |f_{kl}(t)|^2 \leqq \sup_k [\lambda_k/(\exp(\lambda_k T) - 1)]^2 \sum |c_{kl}|^2.$$

Hence by (3.15) and (3.16)

$$\int_0^T \|f(t)\|_{L^2(\Omega)}dt = \int_0^T (\sum |f_{kl}(t)|^2)^{1/2}dt \leqq \|u_0\| \sup_k |\lambda_k T/[\exp(\lambda_k T) - 1]|.$$

Since $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$,

$$\sup_k |\lambda_k T/[\exp(\lambda_k T)-1]| \to 0 \quad \text{as} \quad T\to\infty.$$

Thus we can take $T(>0)$ satisfying

$$\int_0^T \|f(t)\|_{L^2(\Omega)}dt \leqq \eta.$$

This means that the control system (3.8) with (3.9) is admissibly null controllable in $\mathscr{F}^1_\eta$.

Next we consider the control system (3.8) with Neumann boundary condition (3.10). We shall see that this system cannot be admissibly null controllable in $\mathscr{F}^1_\eta$. In fact, integrating the equation (3.8) over $\Omega\times(0, T)$ and using Green's formula, we obtain

$$\int_\Omega u(x, T)dx - \int_\Omega u(x, 0)dx = \int_0^T \int_\Omega f(x, t)dxdt \tag{3.19}$$

for the genuine solution of (3.11). Let us choose the sequences in $D(A)$ and $C^1(0, T; L^2(\Omega))$ (=the space of all $L^2(\Omega)$-valued continuously differentiable functions) which converge to u_0 in $L^2(\Omega)$ and to $f(x, t)$ in $L^1(0, T; L^2(\Omega))$ respectively. By taking limit, the equality (3.19) holds for the mild solution of (3.11) with initial state u_0. If the system is admissibly null controllable in $\mathscr{F}^1_\eta$, then there exist $T(>0)$ and $f\in\mathscr{F}^1_\eta$ which satisfy

$$-\int_\Omega u_0(x)dx = \int_0^T \int_\Omega f(x, t)dxdt. \tag{3.20}$$

Now let us take $u_0(x)=-\gamma$, where $\gamma>\eta|\Omega|^{-1/2}$. Then the left integral is $\gamma|\Omega|$. On the other hand

$$\left|\int_0^T \int_\Omega f(x, t)dxdt\right| \leqq \int_0^T \|f(t)\|_{L^2(\Omega)}dt|\Omega|^{1/2} \leqq \eta|\Omega|^{1/2}.$$

Since $\gamma>\eta|\Omega|^{-1/2}$, the equality (3.20) cannot hold for $u_0(x)=-\gamma$. Therefore the control system (3.8) with Neumann boundary condition (3.10) is not admissibly null controllable in $\mathscr{F}^1_\eta$ for any $\eta>0$.

REMARK 2. When $U(t)$ is a unitary group, any nonzero subspace D is not admissibly null controllable in $\mathscr{F}^1_\eta$. In fact, if the controlled space D is admissibly null controllable in $\mathscr{F}^1_\eta$, then for any $u_0\in D$ there exists $f\in\mathscr{F}^1_\eta$ satisfying

$$U(T)u_0 + \int_0^T U(T-s)Bf(s)ds = 0.$$

Since $U(T)$ is unitary,

$$\|U(T)u_0\| = \|u_0\| \qquad \text{for any} \quad u_0\in D.$$

On the other hand, any function $f(t)$ in $\mathscr{F}^1_\eta$ satisfies

$$\left\|\int_0^T U(T-s)Bf(s)ds\right\| \leqq \|B\| \int_0^T \|f(s)\|ds \leqq \|B\|\eta.$$

Hence $u_0 \in D$ which satisfies $\|u_0\| > \eta\|B\|$ cannot be steered to the zero state by $f(t)$ in $\mathscr{F}^1_\eta$. Thus any nonzero controlled space is not admissibly null controllable in $\mathscr{F}^1_\eta$.

The wave equation considered by Fattorini [5], which is stated in the introduction, generates a unitary group. Therefore the control system (1.3) with (1.4) is not admissibly null controllable in $\mathscr{F}^1_\eta$.

REMARK 3. Null controllability and admissible null controllability can also be considered when the controls are applied on the boundary. For example, let us consider the wave equation

$$\partial^2 u/\partial t^2 - \sum \partial^2 u/\partial x_i^2 = 0 \quad \text{in} \quad \Omega \times (0, T) \tag{3.21}$$

with boundary condition

$$\alpha u(x, t) + \beta(\partial u/\partial n)(x, t) = f(x, t) \quad \text{on} \quad \partial\Omega \times (0, T). \tag{3.22}$$

Here α and β are constants satisfying $\alpha^2 + \beta^2 \neq 0$. Let a control function $f(x, t)$ be defined on $\partial\Omega \times (0, T)$.

By Russell [11], for any given $[u_0, u_1] \in H^2(\Omega) \times H^1(\Omega)$ and $[v_0, v_1] \in H^2(\Omega) \times H^1(\Omega)$, there exist a positive time T and a control $f(x, t)$ in $L^\infty(0, T; H^s(\partial\Omega))$ such that the solution $u(t)$ of (3.21) with (3.22) satisfies

$$[u(0), (\partial u/\partial t)(0)] = [u_0, u_1]$$

and

$$[u(T), (\partial u/\partial t)(T)] = [v_0, v_1].$$

Here

$$s = \begin{cases} 1/2 & \text{if} \quad \beta \neq 0, \\ 3/2 & \text{if} \quad \beta = 0. \end{cases}$$

As is stated in the introduction, if $f = 0$, (3.21) with (3.22) can be reduced to a first order equation of the form (1.6) and the operator with the domain

$$\{[u, v] \in H^2(\Omega) \times H^1(\Omega) \,|\, \alpha u + \beta(\partial u/\partial n) = 0 \text{ on } \partial\Omega\}$$

when $\beta \neq 0$ and $(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ when $\beta = 0$, generates a unitary C_0 semigroup $U(t)$ on X, where X is $H^1(\Omega) \times L^2(\Omega)$ when $\beta \neq 0$ and $H^1_0(\Omega) \times L^2(\Omega)$ when $\beta = 0$. If we take the domain of A as the controlled space D, then D is admissibly controllable in $\mathscr{F}^p_\eta$ for any $p > 1$ and $\eta > 0$. Here

$$\mathscr{F}_\eta^p = \cup_{T>0} \left\{ f \in L^p(0,\, T;\, H^s(\partial\Omega)) \middle| \left(\int_0^T \|f(t)\|^p_{H^s(\partial\Omega)} dt \right)^{1/p} \leqq \eta \right\}.$$

In fact, noting that D is controllable in $L^\infty(0, T; H^s(\partial\Omega))$, $U(t)D \subset D$ and

$$\left\| U(t) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right\|_{H^2(\Omega)\times H^1(\Omega)} \leqq \left\| \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right\|_{H^2(\Omega)\times H^1(\Omega)}$$

for any $[u_0, u_1] \in D$, we obtain the admissible null controllability of D in $\mathscr{F}_\eta^p$ as in the proof of Theorem 1. Since (3.21) and (3.22) are invariant under time reversal, the controlled space D is admissibly controllable in $\mathscr{F}_\eta^p$.

By Graham and Russell [8], when the domain Ω is a sphere and $\alpha=0$, the whole space $H^1(\Omega) \times L^2(\Omega)$ is controllable in $L^2(0, T; L^2(\partial\Omega)) = L^2((0, T) \times \partial\Omega)$ for positive T greater than diam Ω. Then, when the domain Ω is a sphere and $\alpha=0$, the control system (3.21) with (3.22) is admissibly controllable in $\mathscr{F}_\eta^2$.

For simplicity we have considered the case when the controls are applied on the whole boundary. But if the region where the controls are applied is limited to a subset of the boundary which satisfies the "star-complemented" condition, we obtain similar results by the same arguments. For details see Russell [12].

For the heat equation with boundary control we can also state similar results. As for the control systems described by the heat equation, see Fattorini and Russell [7], Russell [11], [12], Seidman [17], [18].

As an example of Corollary 1, let us consider a vibrating string. The forced motion of a string with density $\rho(x)$ and modulus of elasticity $c(x)$ is described by the equation

$$(3.23) \qquad \rho(x)[\partial^2 u/\partial t^2] - (\partial/\partial x)[c(x)(\partial u/\partial x)] = \gamma(x) f(t), \quad 0 < x < 1,\ t > 0.$$

By means of transformations we obtain a simplified equation

$$(3.24) \qquad \partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 - r(x)u = g(x)f(t),\ 0 < x < L,\ t > 0,$$

where $r(x)$ is a continuous function on $[0, L]$ and $g(x)$ is a function in $L^2(0, L)$. The function $g(x)$ is called the force distribution function. Let $Y=\boldsymbol{R}^1$ and control space be $L^2(0, T; Y) = L^2(0, T)$. For simplicity let both the end-points be fixed, that is,

$$(3.25) \qquad u(0, t) = u(L, t) = 0, \quad t > 0.$$

Then as is stated in the introduction, (3.24) can be reduced to a first order equation on the Hilbert space $X = H_0^1(0, L) \times L^2(0, L)$. According to Russell [10], under some assumption on $g(x)$, any initial state in $(H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)$ can be steered to the zero state by a control $f(t)$ in $L^2(0, 2L)$. This means that the controlled space $D=(H^2(0, L)\cap H_0^1(0, L))\times H_0^1(0, L)$ is null controllable in $L^2(0, 2L)$.

Let $\{\lambda_k\}$ and $\{\varphi_k\}$ be the eigenvalues and eigenfunctions respectively of $-d^2/dx^2-r(x)$ with Dirichlet boundary condition. Furthermore let $\{\varphi_k\}$ form an orthonormal basis in $L^2(0, L)$. Then the assumption on $g(x)$ is as follows: If $g(x)$ is expanded as

$$g(x) = \sum g_k\varphi_k(x),$$

then

$$g_k \neq 0, \quad k = 0, 1, 2,\dots,$$

and

$$\liminf_{k\to\infty} k|g_k| > 0.$$

Since the controlled space D is a domain of

$$A = \begin{bmatrix} 0 & 1 \\ d^2/dx^2+r(x) & 0 \end{bmatrix},$$

which generates a contraction C_0 semigroup, we can apply Corollary 1. Thus the controlled space $D=(H^2(0, L)\cap H_0^1(0, L))\times H_0^1(0, L)$ is admissibly null controllable in $\mathscr{F}_\eta^2$. Since (3.24) and (3.25) are invariant under time reversal, the controlled space D is admissibly controllable in $\mathscr{F}_\eta^2$.

Now let us assume the hypotheses in Theorem 1 and $u_1\in X$ is represented in the form

$$u_1 = \int_0^{T_1} U(T_1-s)Bf_1(s)ds, \tag{3.26}$$

where $f_1(t)\in L^p(0, T_1; Y)$ $(1<p<\infty)$ and

$$\left(\int_0^{T_1} \|f_1(t)\|^p dt\right)^{1/p} < \eta.$$

By Theorem 1, for any $u_0\in D$ there exist some positive number T_0 and $f_0(t)\in L^p(0, T_0; Y)$ satisfying

$$U(T_0)u_0 + \int_0^{T_0} U(T_0-s)Bf_0(s)ds = 0$$

and

$$\left(\int_0^{T_0} \|f_0(t)\|^p dt\right)^{1/p} < \eta - \left(\int_0^{T_1} \|f_1(t)\|^p dt\right)^{1/p}.$$

If we put

$$f(t) = \begin{cases} f_0(t), & 0 \leqq t < T_0, \\ f_1(t-T_0), & T_0 \leqq t \leqq T_0+T_1, \end{cases}$$

then we have

$$u_1 = U(T_0+T_1)u_0 + \int_0^{T_0+T_1} U(T_0+T_1-s)Bf(s)ds$$

and

$$\left(\int_0^{T_0+T_1} \|f(t)\|^p dt\right)^{1/p} \leqq \left(\int_0^{T_0} \|f_0(t)\|^p dt\right)^{1/p} + \left(\int_0^{T_1} \|f_1(t)\|^p dt\right)^{1/p} \leqq \eta.$$

Thus there exists $f(t)$ in $\mathscr{F}_\eta^p$ which steers u_0 to u_1. In the case of $p=\infty$, in a similar way, any $u_0 \in D$ can be steered in $\mathscr{F}_\eta^\infty$ to $u_1 \in X$ of the form (3.26) with $\|f_1(t)\| \leqq \eta$ almost everywhere on $[0, T_1]$.

Now let us define

$$X_\eta^p = \cup_{T>0} \left\{\int_0^T U(T-s)Bf(s)ds \,\middle|\, \|f\|_p \leqq \eta\right\}$$

and

$$\mathring{X}_\eta^p = \cup_{T>0} \left\{\int_0^T U(T-s)Bf(s)ds \,\middle|\, \|f\|_p < \eta\right\}.$$

Then we have

COROLLARY 3. *Let the control system* (1.1) *and the controlled space D satisfy the hypotheses of Theorem* 1. *Then for any $u_0 \in D$ and $u_1 \in \mathring{X}_\eta^p$, $1<p<\infty$, or $u_1 \in X_\eta^\infty$, there exists $f(t) \in \mathscr{F}_\eta^p$ or $\mathscr{F}_\eta^\infty$ which steers u_0 to u_1, that is,*

$$u_1 = U(T)u_0 + \int_0^T U(T-s)Bf(s)ds.$$

4. Optimal time control and extremum principle

Suppose for given u_0 and u_1 in X there exists a control $f(t)$ in $\mathscr{F}_\eta^p$ which steers u_0 to u_1, that is, for some positive number T

$$u_1 = U(T)u_0 + \int_0^T U(T-s)Bf(s)ds.$$

Then the time T is called a transition time and the infimum of transition times, when the control varies in the constraint set $\mathscr{F}_\eta^p$, is called an optimal time. If there is a control $f_0(t)$ in $\mathscr{F}_\eta^p$ which steers u_0 to u_1 with the optimal time, then it is called an optimal time control.

The existence of the optimal time control is by now well known.

THEOREM 2. *Assume* $L^p(0, T; Y)=L^q(0, T; Y^*)^*$, $1/p+1/q=1$, *where* $*$ *denotes the adjoint space. Assume, further, that* u_0, $u_1 \in X$ *are such that there exists an admissible control in* $\mathscr{F}_\eta^p$ *transferring* u_0 *to* u_1. *Then there exists an optimal time control.*

For the proof see [1], [4].

Now we show a characterization of optimal time control, i.e., the extremum principle, for $1<p<\infty$.

THEOREM 3. *Let* $1<p<\infty$ *and the control system* (1.1) *satisfy the hypotheses of Theorem* 2. *Let us assume that the control system* (1.1) *is exactly null controllable in* $L^p(0, T_0; Y)$. *Further let* $f_0(t)$ *be an optimal time control in* $\mathscr{F}_\eta^p$ *and* T *be its optimal time. If* $T>T_0$, *then* $f_0(t)$ *satisfies the extremum principle, that is,*

$$\left(\int_0^T \|f_0(t)\|^p dt\right)^{1/p} = \eta.$$

PROOF. Let $f_0(t)$ be an optimal time control which steers u_0 to u_1 at the optimal time T, that is,

$$u_1 = U(T)u_0 + \int_0^T U(T-s)Bf_0(s)ds.$$

Suppose that

$$M = \left(\int_0^T \|f_0(t)\|^p dt\right)^{1/p} < \eta$$

and $T>T_0$. For any $0<\varepsilon<T$, we have

$$\begin{aligned} u_1 = {} & U(T-\varepsilon)u_0 + [U(T)-U(T-\varepsilon)]u_0 \\ & + \int_0^\varepsilon U(T-s)Bf_0(s)ds + \int_\varepsilon^T U(T-s)Bf_0(s)ds. \end{aligned} \tag{4.1}$$

As we showed in the proof of Theorem 1, for any positive constant γ there exists some positive constant δ such that for any u contained in the δ-neighborhood of the origin in X there exists $g(t)$ in $L^p(0, T_0; Y)$ which satisfies

$$U(T_0)u + \int_0^{T_0} U(T_0-s)Bg(s)ds = 0$$

and

$$\left(\int_0^{T_0} \|g(t)\|^p dt\right)^{1/p} \leqq \gamma.$$

Let $\gamma=(\eta-M)/3$ and ε be so small that

$$T-T_0 \geqq \varepsilon > 0, \quad \|[U(\varepsilon)-I]u_0\| \leqq \delta$$

and

$$\left\|\int_0^\varepsilon U(\varepsilon-s)Bf_0(s)ds\right\| \leqq \delta.$$

Then we can choose $g_1(t)$ and $g_2(t)$ in $L^p(0, T_0; Y)$ which satisfy

$$U(T_0)[U(\varepsilon)-I]u_0+\int_0^{T_0} U(T_0-s)Bg_1(s)ds = 0, \tag{4.2}$$

$$U(T_0)\left[\int_0^\varepsilon U(\varepsilon-s)Bf_0(s)ds\right]+\int_0^{T_0} U(T_0-s)Bg_2(s)ds = 0 \tag{4.3}$$

and

$$\left(\int_0^{T_0} \|g_i(t)\|^p dt\right)^{1/p} \leqq \gamma \quad (i=1, 2). \tag{4.4}$$

Operating $U(T-T_0-\varepsilon)$ on (4.2) and (4.3), we have

$$[U(T)-U(T-\varepsilon)]u_0 = -\int_0^{T_0} U(T-\varepsilon-s)Bg_1(s)ds \tag{4.5}$$

and

$$\int_0^\varepsilon U(T-s)Bf_0(s)ds = -\int_0^{T_0} U(T-\varepsilon-s)Bg_2(s)ds. \tag{4.6}$$

Now putting

$$f(t)=\begin{cases} f_0(t+\varepsilon)-g_1(t)-g_2(t), & 0 \leqq t \leqq T_0, \\ f_0(t+\varepsilon), & T_0 < t \leqq T-\varepsilon, \end{cases}$$

and using (4.1), (4.4), (4.5) and (4.6), we have

$$u_1 = U(T-\varepsilon)u_0+\int_0^{T-\varepsilon} U(T-\varepsilon-s)Bf(s)ds$$

and

$$\left(\int_0^{T-\varepsilon} \|f(t)\|^p dt\right)^{1/p} \leqq M+2(\eta-M)/3 < \eta.$$

Thus the control $f(t)$ belongs to $\mathscr{F}_\eta^p$ and steers u_0 to u_1 at the time $T-\varepsilon$. This contradicts that T is an optimal time.

It is well known that in general an optimal time control does not satisfy the "extremum" principle or "bang-bang" principle. Under some assumptions we shall classify X in two parts: the initial datum which are steered to the zero state by optimal time controls satisfying the extremum principle and the others.

Let us assume that there is some T_u such that

$$\int_0^{T_u} U(T_u - s)Bf(s)ds = 0 \tag{4.7}$$

implies $f=0$. If u_1 is represented in the form

$$u_1 = \int_0^{T_u} U(T_u - s)Bf(s)ds \tag{4.8}$$

where

$$\left(\int_0^{T_u} \|f(t)\|^p dt\right)^{1/p} < \eta,$$

then any optimal time control which steers the zero state to u_1 does not satisfy the extremum principle. Indeed if $f_0(t)$ is an optimal time control and T_0 is the optimal time, then $T_0 \leqq T_u$ and

$$u_1 = \int_0^{T_0} U(T_0 - s)Bf_0(s)ds = \int_0^{T_u} U(T_u - s)Bg(s)ds$$

where

$$g(t) = \begin{cases} 0, & 0 \leqq t \leqq T_u - T_0, \\ f_0(t-(T_u - T_0)), & T_u - T_0 < t \leqq T_u. \end{cases}$$

By (4.7) and (4.8), $f(t)=g(t)$. Thus

$$\left(\int_0^{T_0} \|f_0(t)\|^p dt\right)^{1/p} = \left(\int_0^{T} \|f(t)\|^p dt\right)^{1/p} < \eta.$$

Hence $f_0(t)$ does not satisfy the extremum principle. Clearly if

$$\left(\int_0^{T_u} \|f(t)\|^p dt\right)^{1/p} = \eta,$$

then an optimal time control, which steers the zero state to u_1, satisfies the extremum principle. By similar arguments we obtain the same results for an optimal time control which steers u_0 to the zero state.

Now we define the critical time T_c as the infimum of the time T for which the control system is exactly null controllable in $L^p(0, T; Y)$. Let us put

$$N_0 = \left\{ u_0 \in X \;\middle|\; \begin{array}{l} \text{there exists } f(t) \text{ such that} \\ U(T_c)u_0 + \int_0^{T_c} U(T_c - s)Bf(s)ds = 0 \text{ and} \\ \left(\int_0^{T_c} \|f(t)\|^p dt\right)^{1/p} < \eta \end{array} \right\}$$

and

$$N_1 = \left\{ u_1 \in X \;\middle|\; \begin{array}{l} \text{there exists } f(t) \text{ such that} \\ u_1 = \int_0^{T_c} U(T_c - s)Bf(s)ds \text{ and} \\ \left(\int_0^{T_c} \| f(t)\|^p dt\right)^{1/p} < \eta. \end{array} \right\}.$$

Then we have shown

THEOREM 4. *Let the control system* (1.1) *satisfy the hypotheses of Corollary* 1 *and Theorem* 2. *Further let us assume that*

$$\int_0^{T_c} U(T_c - s)Bf(s)ds = 0$$

implies $f=0$. *Then any optimal time control* $f_0(t)$, *which steers any* u_0 *in* N_0 *to the zero state, or the zero state to any* u_1 *in* N_1, *does not satisfy the extremum principle. An optimal time control, which steers any* u_0 *in* $X - N_0$ *to the zero state, or the zero state to any* u_1 *in* $X_\eta^p - N_1$, *satisfies the extremum principle.*

Let $f_1(t), f_2(t) \in \mathscr{F}_\eta^p$ be optimal time controls which steer u_0 to u_1, then $[f_1(t)+f_2(t)]/2$ is also an optimal time control. Therefore if the optimal time is greater than T_0, then

$$\|f_1\|_p = \|f_2\|_p = \|(f_1+f_2)/2\|_p = \eta.$$

Hence if $L^p(0, T; Y)$ is strictly convex, then $f_1 = f_2$. Thus we have

COROLLARY 4. *Let us assume that* Y *is strictly convex and let* T_u *be a time such that*

$$\int_0^{T_u} U(T_u - s)Bf(s)ds = 0$$

implies $f=0$. *Then the optimal time control whose optimal time is greater than* T_c *or smaller than* T_u *is unique.*

Now as an example for Corollary 4, we consider a control system which is slightly different from the one considered in section 3, that is,

$$\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 - r(x)u = f(x, t), \quad 0 < x < L, \; t > 0, \tag{4.9}$$

$$u(0, t) = u(L, t) = 0, \quad t > 0. \tag{4.10}$$

Let an external force be applied only on the limited subset E of the string, that is, $f \in L^p(0, \infty; L^2(0, L))$ and the support of $f(x, t)$ is contained in $E \times (0, \infty)$. Here E is a measurable subset with positive measure. The control system can be reduced to a first order equation on the Hilbert space $X = H_0^1(0, L) \times L^2(0, L)$. The control space is taken as $Y = \{w \in L^2(0, L) \mid \operatorname{supp} w \subset E\}$. By solving the moment problem, we obtain the controllability of this control system as follows. Let the initial state $[u_0, u_1] \in H_0^1(0, L) \times L^2(0, L)$ be expanded as

$$u_0(x) = \sum \alpha_k \varphi_k(x), \quad u_1(x) = \sum \beta_k \varphi_k(x),$$

where $\varphi_k(x)$, $k=1, 2,\dots$, are the eigenfunctions of $-d^2/dx^2 - r(x)$ which form an orthonormal basis in $L^2(0, L)$. Let $\{\omega_k\}$ be square roots of the eigenvalues. Then a necessary and sufficient condition that $f(x, t)$ steers $[u_0, u_1]$ to the zero state is

$$\beta_k = -\int_0^T \int_E \cos(\omega_k t)\varphi_k(x) f(x, t) dx dt, \tag{4.11}$$

$$\alpha_k = \int_0^T \int_E \omega_k^{-1} \sin(\omega_k t)\varphi_k(x) f(x, t) dx dt, \quad k = 1, 2,\dots. \tag{4.12}$$

For $T \geqq 2L$ there exists a biorthogonal system $\{p_k(t), q_k(t)\}$ for $\{\cos(\omega_k t), \sin(\omega_k t)\}$ in $L^2(0, T)$, that is,

$$\int_0^T \cos(\omega_k t) p_l(t) dt = \delta_{kl}, \quad \int_0^T \cos(\omega_k t) q_l(t) dt = 0,$$

$$\int_0^T \sin(\omega_k t) q_l(t) dt = \delta_{kl}, \quad \int_0^T \sin(\omega_k t) p_l(t) dt = 0.$$

Putting

$$\begin{aligned} f(x, t) = & - \sum \beta_k p_k(t)\varphi_k(x)\left[\int_E |\varphi_k(x)|^2 dx\right]^{-1} \\ & + \sum \alpha_k \omega_k q_k(t)\varphi_k(x)\left[\int_E |\varphi_k(x)|^2 dx\right]^{-1} \qquad \text{for} \quad x \in E, \\ f(x, t) = 0 & \qquad \text{for} \quad x \in (0, L) - E, \end{aligned}$$

we have $f(x, t) \in L^2((0, L) \times (0, T)) = L^2(0, T; L^2(0, L))$,

$$\operatorname{supp} f(x, t) \subset E \times (0, T)$$

and the equalities (4.11) and (4.12).

Thus this control system is exactly null controllable in $L^2(0, T; Y)$ for $T \geqq 2L$. Hence by Corollary 1, we have the admissible null controllability in $\mathscr{F}_\eta^2$. The admissible null controllability and the invariance under time reversal imply $\mathring{X}_\eta^2 = X$. Hence by Corollary 3, this system is admissibly controllable in $\mathscr{F}_\eta^2$. By the finite propagation speed of the support, it is clear that the system is not null controllable in a short time. Hence $T_c > 0$. But we do not know the exact values of T_c and T_u.

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