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Notes on non-discrete subgroups of $\hat{U}(1, n; F)$

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1. Introduction

Let $\boldsymbol{F}$ denote the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers, or the division ring of real quaternions $\boldsymbol{K}$. Let $V=V^{1,n}(\boldsymbol{F})$ denote the (right) vector space $\boldsymbol{F}^{n+1}$, together with the unitary structure defined by the $\boldsymbol{F}$-Hermitian form

$$\Phi(z, w) = -\overline{z_0}w_0 + \overline{z_1}w_1 + \cdots + \overline{z_n}w_n$$

for $z=(z_0, z_1,\dots, z_n)$ and $w=(w_0, w_1,\dots, w_n)$. An automorphism g of V, that is, an $\boldsymbol{F}$-linear bijection of V onto V such that $\Phi(g(z), g(w))=\Phi(z, w)$ for $z, w \in V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, n; \boldsymbol{F})$. Let $\{e_0, e_1,\dots, e_n\}$ be the standard basis in V, and set $\hat{e}_0=(e_0-e_1)(1/\sqrt{2})$, $\hat{e}_1=(e_0+e_1)(1/\sqrt{2})$ and $\hat{e}_k=e_k$ for $2\leqq k\leqq n$. Let D be the matrix which changes the basis $\{e_0, e_1,\dots, e_n\}$ into the basis $\{\hat{e}_0, \hat{e}_1,\dots, \hat{e}_n\}$. Let $\hat{U}(1, n; \boldsymbol{F})=D^{-1}U(1, n; \boldsymbol{F})D$. $\hat{U}(1, n; \boldsymbol{F})$ is the automorphism group of the Hermitian form

$$\tilde{\Phi}(z, w) = -(\overline{z_0}w_1+\overline{z_1}w_0) + \overline{z_2}w_2 + \cdots + \overline{z_n}w_n$$

for $z, w \in V$.

In the study of kleinian groups one is concerned with sufficient conditions for subgroups of Möbius transformations to be non-discrete (cf. [2]). Our purpose here is to give similar conditions for subgroups of $\hat{U}(1, n; \boldsymbol{F})$ to be non-discrete.

2. Preliminaries

Let $V_-=\{z\in V: \Phi(z, z)<0\}$ and $\hat{V}_-=D^{-1}(V_-)$. Obviously $\hat{V}_-$ is invariant under $\hat{U}(1, n; \boldsymbol{F})$. Let $P(V)$ be the projective space obtained from V, that is, the quotient space $V-\{0\}$ with respect to the equivalence relation: $u\sim v$ if there exists $\lambda\in \boldsymbol{F}-\{0\}$ such that $u=v\lambda$. Let P: $V-\{0\}\to P(V)$ denote the projection map. We denote $P(\hat{V}_-)$ by Σ. Let $\bar{\Sigma}$ be the closure of Σ in the projective space. We shall view that each element of $\hat{U}(1, n; \boldsymbol{F})$ operates in $\bar{\Sigma}$. Let $G_0=\{g\in \hat{U}(1, n; \boldsymbol{F})$: $g(P(\hat{e}_0))=P(\hat{e}_0)\}$, $G_\infty=\{g\in \hat{U}(1, n; \boldsymbol{F})$: $g(P(\hat{e}_1))=P(\hat{e}_1)\}$ and $G_{0,\infty}=G_0\cap G_\infty$. The general form of elements in G_∞ is shown in [1; Lemma

3.3.1]. In the same manner we obtain

PROPOSITION 2.1. *Let $g \in G_0$. Then (with respect to the basis $\{\hat{e}_0, \hat{e}_1, \dots, \hat{e}_n\}$)*

$$g = \begin{bmatrix} \xi & t & \beta \\ 0 & \eta & 0 \\ 0 & \alpha & A \end{bmatrix},$$

where t, ξ, $\eta \in \boldsymbol{F}$, while α, β and A are $(n-1)\times 1$, $1\times(n-1)$ and *$(n-1)\times(n-1)$ matrices, respectively. Furthermore, $\xi\eta = 1$, $\mathrm{Re}(\bar{t}\eta) = (1/2)|\alpha|^2$ (where $|\alpha|$ is the Euclidean norm of α), $\beta = \xi\bar{\alpha}^T A$ (where T denotes the transpose), and $A \in U(n-1; \boldsymbol{F})$.*

By [1; Lemma 3.3.1] and Proposition 2.1, we have

PROPOSITION 2.2. *An element in $G_{0,\infty}$ has the form*

$$g = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $\bar{\mu}\lambda = 1$ and *$A \in U(n-1; \boldsymbol{F})$.*

REMARK 1 (cf. [1; p. 73]). $G_{0,\infty} = \hat{U}(1; \boldsymbol{F}) \times \hat{U}_0(1, 1; \boldsymbol{R}) \times \hat{U}(n-1; \boldsymbol{F})$.

REMARK 2 (cf. [2; Theorem 1E in Chap. 1]). Even if two elements in $\hat{U}(1, n; \boldsymbol{F})$ have the same set of fixed points on $\partial\Sigma = \bar{\Sigma} - \Sigma$, they are not necessarily commutative.

3. Sufficient conditions for subgroups of $\hat{U}(1, n; \boldsymbol{F})$ to be non-discrete

In this section we prove two theorems. We shall call $g \in \hat{U}(1, n; \boldsymbol{F})$ loxodromic if it has exactly two fixed points in $\bar{\Sigma}$ and these belong to $\partial\Sigma$.

THEOREM 3.1 (cf. [2; Theorem 2I in Chap. 1]). *Let $g \in \hat{U}(1, n; \boldsymbol{F})$ be loxodromic. Let f be an element in $\hat{U}(1, n; \boldsymbol{F})$ which has one and only one fixed point in common with g. Then the group generated by f and g is not discrete.*

PROOF. By [1; Proposition 2.1.3], we may assume that the fixed points of g are $P(\hat{e}_0)$ and $P(\hat{e}_1)$ and the latter is the common fixed point of f and g. Thus $f \in G_\infty$ and $g \in G_{0,\infty}$. By Proposition 2.2, we have

$$g = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $\bar{\mu}\lambda=1$ and $A \in U(n-1; \boldsymbol{F})$. By Proposition 2.1, we see that f is of the form

$$f=\begin{bmatrix} \xi & 0 & 0 \\ s & \eta & b \\ a & 0 & B \end{bmatrix},$$

where ξ, η, $s \in \boldsymbol{F}$ and a, b and B are $(n-1)\times 1$, $1\times(n-1)$ and $(n-1)\times(n-1)$ matrices, respectively, such that $\bar{\xi}\eta=1$, $\operatorname{Re}(\bar{\xi}s)=(1/2)|a|^2$, $b=\eta\bar{a}^T B$ and $B \in U(n-1; \boldsymbol{F})$. The commutator of f and g^n is

$$fg^n f^{-1} g^{-n} = h_n = \begin{bmatrix} \alpha_{11}^{(n)} & 0 & 0 \\ a_{21}^{(n)} & \alpha_{22}^{(n)} & A_n \\ B_n & 0 & C_n \end{bmatrix},$$

where

$$\begin{aligned}
\alpha_{11}^{(n)} &= \xi\mu^n\xi^{-1}\mu^{-n}, \\
\alpha_{21}^{(n)} &= s\mu^n\xi^{-1}\mu^{-n} - \eta\lambda^n\eta^{-1}s\xi^{-1}\mu^{-n} + \eta\lambda^n\eta^{-1}bB^{-1}a\xi^{-1}\mu^{-n} \\
&\qquad - bA^nB^{-1}a\xi^{-1}\mu^{-n}, \\
\alpha_{22}^{(n)} &= \eta\lambda^n\eta^{-1}\lambda^{-n}, \\
A_n &= -\eta\lambda^n\eta^{-1}bB^{-1}A^{-n} + bA^nB^{-1}A^{-n}, \\
B_n &= a\mu^n\xi^{-1}\mu^{-n} - BA^nB^{-1}a\xi^{-1}\mu^{-n}
\end{aligned}$$

and

$$C_n = BA^nB^{-1}A^{-n}.$$

We shall show that $h_1, h_2, \ldots$ are distinct. Suppose that $h_k=h_m$ for some $k \neq m$, and put $n=m-k$. Then $fg^nf^{-1}g^{-n}=id$. It follows that

$$\begin{aligned}
\xi\mu^n\xi^{-1}\mu^{-n} &= 1, \\
\eta\lambda^n\eta^{-1}\lambda^{-n} &= 1
\end{aligned}$$

and

$$a\mu^n\xi^{-1}\mu^{-n} - BA^nB^{-1}a\xi^{-1}\mu^{-n} = 0.$$

From these we see that

$$\alpha_{21}^{(n)} = s\xi^{-1} - \bar{\mu}^{-n}s\xi^{-1}\mu^{-n} + \bar{\mu}^{-n}bB^{-1}a\xi^{-1}\mu^{-n} - bB^{-1}a\xi^{-1} = 0.$$

Hence

$$\begin{aligned}
\operatorname{Re}(\alpha_{21}^{(n)}) &= \operatorname{Re}(s\bar{\xi})|\xi|^{-2} - \operatorname{Re}(s\bar{\xi})|\xi|^{-2}|\mu|^{-2n} \\
&\qquad - (1-|\mu|^{-2n})\operatorname{Re}(\eta\bar{a}^TBB^{-1}a\xi^{-1})
\end{aligned}$$

$$= (1/2)|a|^2|\xi|^{-2}(1-|\mu|^{-2n}) - |a|^2|\xi|^{-2}(1-|\mu|^{-2n})$$
$$= (-1/2)|a|^2|\xi|^{-2}(1-|\mu|^{-2n}) = 0.$$

Therefore $|\mu|=1$ or $a=0$. If $|\mu|=1$, then g would not be loxodromic. If $a=0$, then $\alpha_{21}^{(n)}=s\xi^{-1}-\bar{\mu}^{-n}s\xi^{-1}\mu^{-n}=0$. Since $|\mu|\neq 1$, it follows that $s=0$. Hence f would fix $P(\hat{e}_0)$. This is a contradiction. Thus $h_1, h_2,\dots$ are distinct.
Choosing subsequences, if necessary, we may assume that $\alpha_{11}^{(n)}\to\alpha$ and $\alpha_{22}^{(n)}\to\beta$ as $n\to\infty(|\mu|>1)$ or $n\to-\infty(|\mu|<1)$. Since $U(n-1;\boldsymbol{F})$ is compact, we may also assume that the sequence $\{C_n\}$ converges to some element U in $U(n-1;\boldsymbol{F})$ and from this fact we see that $\{A_n\}$ converges to $bB^{-1}U$. It is easy to show that $\alpha_{21}^{(n)}\to s\xi^{-1}\alpha$ and $B_n\to a\xi^{-1}\alpha$ as $n\to\infty$ or $n\to-\infty$. Hence

$$h_n \longrightarrow \begin{bmatrix} \alpha & 0 & 0 \\ s\xi^{-1}\alpha & \beta & bB^{-1}U \\ a\xi^{-1}\alpha & 0 & U \end{bmatrix}, \qquad (n\to\infty \text{ or } n\to-\infty).$$

Noting that the limit matrix belongs to $\hat{U}(1, n; \boldsymbol{F})$, we can conclude our assertion.

THEOREM 3.2 (cf. [2; Theorem 4J–1 in Chap. 1]). *Let* $f=(\alpha_{i,j})_{i,j=1,2,\dots,n+1}$ *be an element in* $\hat{U}(1, n; \boldsymbol{F})$ *and*

$$g = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & E_{n-1} \end{bmatrix},$$

where $s\neq 0$ *and* $\mathrm{Re}(s)=0$. *Then the group generated by f and g is not discrete if* $0<|\alpha_{1,2}|<1/|s|$.

PROOF. Let $f_0=f$ and for $k\geqq 0$, $k\in\boldsymbol{Z}$, set $f_{k+1}=f_k g f_k^{-1}$. We shall show that $f_{k+1}\to g$ as $k\to\infty$. We write $f_k=(\alpha_{i,j}^{(k)})_{i,j=1,2,\dots,n+1}$. Computing f_{k+1}, we have

$$\alpha_{1,1}^{(k+1)} = 1 + \alpha_{1,2}^{(k)} s\overline{\alpha_{2,2}^{(k)}}, \tag{1}$$
$$\alpha_{1,2}^{(k+1)} = \alpha_{1,2}^{(k)} s\overline{\alpha_{1,2}^{(k)}}, \tag{2}$$
$$\alpha_{2,1}^{(k+1)} = \alpha_{2,2}^{(k)} s\overline{\alpha_{2,2}^{(k)}}, \tag{3}$$
$$\alpha_{2,2}^{(k+1)} = 1 + \alpha_{2,2}^{(k)} s\overline{\alpha_{1,2}^{(k)}}, \tag{4}$$
$$\begin{cases} \alpha_{1,j}^{(k+1)} = -\alpha_{1,2}^{(k)} s\overline{\alpha_{j,2}^{(k)}} & \text{for } 3\leqq j\leqq n+1, \\ \alpha_{2,j}^{(k+1)} = -\alpha_{2,2}^{(k)} s\overline{\alpha_{j,2}^{(k)}} & \text{for } 3\leqq j\leqq n+1, \\ \alpha_{i,1}^{(k+1)} = \alpha_{i,2}^{(k)} s\overline{\alpha_{2,2}^{(k)}} & \text{for } 3\leqq i\leqq n+1, \end{cases} \tag{5}$$
$$\alpha_{i,2}^{(k+1)} = \alpha_{i,2}^{(k)} s\overline{\alpha_{1,2}^{(k)}} \qquad \text{for } 3\leqq i\leqq n+1, \tag{6}$$

and

(7) $$\alpha_{i,j}^{(k+1)} = \delta_{i,j} - \alpha_{j,2}^{(k)} s \overline{\alpha_{i,2}^{(k)}} \qquad \text{for} \quad 3 \leqq i, j \leqq n+1.$$

Form (2), it follows that $f_1, f_2, \dots$ are distinct and that

(8) $$|s|\,|\alpha_{1,2}^{(k)}| = (|s|\,|\alpha_{1,2}^{(0)}|)^{2^k}, \quad \text{so that} \quad \lim_{k \to \infty} |\alpha_{1,2}^{(k)}| = 0.$$

Choose r so that $1/(1-|s||\alpha_{1,2}|) < r$ and $|\alpha_{2,2}| < r$. If $|\alpha_{2,2}^{(k)}| < r$, then

$$\begin{aligned} |\alpha_{2,2}^{(k+1)}| &\leqq 1 + |s||\alpha_{2,2}^{(k)}||\alpha_{1,2}^{(k)}| \\ &< 1 + |s| r |\alpha_{1,2}^{(k)}| \\ &= 1 + r(|s||\alpha_{1,2}^{(0)}|)^{2^k} \\ &\leqq 1 + r|s||\alpha_{1,2}| < r. \end{aligned}$$

Thus, by induction $|\alpha_{2,2}^{(k)}| < r$ for all k. From (4) and (8) we conclude that

(9) $$\lim_{k \to \infty} \alpha_{2,2}^{(k)} = 1.$$

Then by (3), we have

$$\lim_{k \to \infty} \alpha_{2,1}^{(k)} = s.$$

The equalities (1), (8) and (9) imply that

$$\lim_{k \to \infty} \alpha_{1,1}^{(k)} = 1.$$

Next we consider $\alpha_{i,2}^{(k)}$ for $i \geqq 3$. By (8) there exist $\delta > 0$ and $N > 0$ such that $|s||\alpha_{1,2}^{(k)}| < \delta < 1$ for any $k \geqq N$. Then by (6)

$$|\alpha_{i,2}^{(k+1)}| < \delta |\alpha_{i,2}^{(k)}| < \delta^{k+1} |\alpha_{i,2}|.$$

This shows that

(10) $$\lim_{k \to \infty} \alpha_{i,2}^{(k)} = 0.$$

It follows from (5), (7), (9) and (10) that

$$\lim_{k \to \infty} \alpha_{i,j}^{(k)} = \delta_{i,j}$$

except the case $(i, j) = (2, 1)$. Thus $f_k \to g$ as $k \to \infty$.

COROLLARY 3.3. *Let $f = (\alpha_{i,j})_{i,j=1,2,\dots,n+1}$ be an element in $\hat{U}(1, n; \boldsymbol{F})$. Let*

$$g_1 = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & \bar{a}^T \\ a & 0 & E_{n-1} \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & \bar{b}^T \\ b & 0 & E_{n-1} \end{bmatrix},$$

where $\mathrm{Re}(s)=(1/2)|a|^2$ *and* $\mathrm{Re}(t)=(1/2)|b|^2$. *If* $\bar{a}^T b \neq \bar{b}^T a$ *and* $0<|\alpha_{1,2}|<1/|\bar{a}^T b-\bar{b}^T a|$, *then the group generated by* f, g_1 *and* g_2 *is not discrete.*

PROOF. The commutator of g_1 and g_2 is

$$g_1 g_2 g_1^{-1} g_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \bar{a}^T b-\bar{b}^T a & 1 & 0 \\ 0 & 0 & E_{n-1} \end{bmatrix}.$$

We easily see that $\mathrm{Re}(\bar{a}^T b-\bar{b}^T a)=0$. Thus Theorem 3.2 leads immediately to our conclusion.

References

[1] S. S. Chen and L. Greenberg, Hyperbolic spaces, Contributions to Analysis, Academic Press, New York, (1974), 49–87.
[2] J. Lehner, A short course in automorphic functions, Holt, Rinehart and Winston, Inc., New York, 1966.

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