HIROSHIMA MATH. J.
16 (1986), 413–416

Irreducible decompositions of spinor representations of the Virasoro algebra

Minoru WAKIMOTO

(Received September 11, 1985)

0. Introduction

In our previous joint work [10], we discussed the structure of Fock representations of the Virasoro algebra. In this note, we treat the Fermion version and show that the physical states corresponding to suitable Maya diagrams give the irreducible decomposition of the spinor representations of the Virasoro algebra.

The author is grateful to Professor T. Miwa for his kind interest and suggestions on the translation invariant property of our operators.

1. Spinor representations of the Virasoro algebra

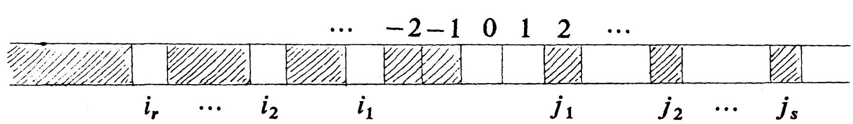
Let A be a Clifford algebra generated by 1 and $\{\psi_n, \psi_n^*\}_{n\in\mathbf{Z}}$ satisfying the following anti-commuting relations:

$$\psi_m\psi_n + \psi_n\psi_m = 0, \quad \psi_m^*\psi_n^* + \psi_n^*\psi_m^* = 0,$$
$$\psi_m\psi_n^* + \psi_n^*\psi_m = \delta_{m,n}.$$

Let B be the left ideal in A generated by $\{\psi_n\}_{n<0} \cup \{\psi_n^*\}_{n\geqq 0}$ and we set $V=A/B$. Each element in V is written as a linear combination of

$$f_{i_r,\dots,i_1;j_1,\dots,j_s} = \psi_{i_r}^*\cdots\psi_{i_1}^*\psi_{j_1}\cdots\psi_{j_s}$$

$(i_r<\cdots<i_1<0\leqq j_1<\cdots<j_s)$, which is assigned to a Maya diagram



The degree and the charge of a monomial $f=f_{i_r,\dots,i_1;j_1,\dots,j_s}$ are defined by

$$\deg f = -(i_1+\cdots+i_r) + (j_1+\cdots+j_s)$$

and

$$\chi(f) = s - r$$

respectively. For an integer m, let V_m be the linear subspace of V spanned by monomials of charge m.

Now fix any complex number μ and define a family of linear operators L_j ($j \in \boldsymbol{Z}$) acting on V by left multiplication as follows:

$$
\begin{aligned}
L_2 &= (1/2)\sum_{n\in\boldsymbol{Z}}(2n-1)\psi_{n-2}\psi_n^* + (\sqrt{2}\mu-\chi)\sum_{n\in\boldsymbol{Z}}\psi_{n-2}\psi_n^* + (1/2)[\sum_{n\in\boldsymbol{Z}}\psi_{n-1}\psi_n^*]^2 \\
L_1 &= \sum_{n\in\boldsymbol{Z}} n\psi_{n-1}\psi_n^* + (\sqrt{2}\mu-\chi)\sum_{n\in\boldsymbol{Z}}\psi_{n-1}\psi_n^* \\
L_0 &= (1/2)\sum_{n\in\boldsymbol{Z}}(2n+1):\psi_n\psi_n^*: + (1/2)(\mu^2-\chi^2) \\
L_{-1} &= \sum_{n\in\boldsymbol{Z}} n\psi_n\psi_{n-1}^* + (\mu/\sqrt{2}-\chi)\sum_{n\in\boldsymbol{Z}}\psi_n\psi_{n-1}^* \\
L_{-2} &= (1/2)\sum_{n\in\boldsymbol{Z}}(2n-1)\psi_n\psi_{n-2}^* + (\mu/\sqrt{2}-\chi)\sum_{n\in\boldsymbol{Z}}\psi_n\psi_{n-2}^* - (1/4)[\sum_{n\in\boldsymbol{Z}}\psi_n\psi_{n-1}^*]^2,
\end{aligned}
$$

and

$$
\begin{aligned}
L_{j+1} &= (j-1)^{-1}[L_j, L_1] && \text{for } j \geqq 2, \\
L_{-j-1} &= (1-j)^{-1}[L_{-j}, L_{-1}] && \text{for } j \leqq -2,
\end{aligned}
$$

where

$$
:\psi_n\psi_n^*: = \begin{cases} \psi_n\psi_n^* & (\text{if } n \geqq 0) \\ -\psi_n^*\psi_n & (\text{if } n < 0). \end{cases}
$$

Then one checks

$$
[L_j, L_k] = (j-k)L_{j+k} + (1/12)(j^3-j)\delta_{j+k,0}
$$

by induction on j and k, and so one sees that these operators generate the Virasoro algebra $\mathfrak{g}$.

We set

$$
f^{(m)} = \begin{cases} \psi_0\psi_1\cdots\psi_{m-1} & (\text{for } m > 0) \\ 1 & (\text{for } m = 0) \\ \psi_m^*\cdots\psi_{-2}^*\psi_{-1}^* & (\text{for } m < 0), \end{cases}
$$

which will be called the (normalized) ground state vector with the charge m. For each $m \in \boldsymbol{Z}$, the space V_m is stable under the action of $\mathfrak{g}$, and $f^{(m)}$ satisfies

$$
L_0 f^{(m)} = (\mu^2/2) f^{(m)}, \quad L_j f^{(m)} = 0 \qquad \text{for } j > 0.
$$

Since $\deg f^{(m)} = m(m-1)/2$, the formal character of V_m is equal to $q^{m(m-1)/2}/\varphi(q)$ for every $m \in \boldsymbol{Z}$, where $\varphi(q) = \prod_{n\geqq 1}(1-q^n)$ is the Euler's function. Owing to the results in [1] or [5], one sees that the $\mathfrak{g}$-module V_m is irreducible if and only if $\sqrt{2}\mu$ is not an integer.

Let σ be the linear automorphism of V defined by

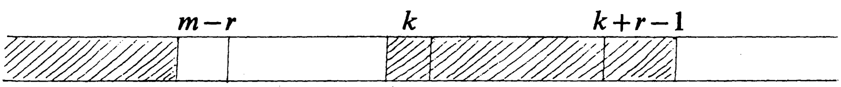
$$\sigma(f_{i_r,\dots,i_1;\,j_1,\dots,j_s}) = \begin{cases} f_{i_r+1,\dots,i_1+1;\,0,j_1+1,\dots,j_s+1} & (\text{if } i_1 < -1) \\ f_{i_r+1,\dots,i_2+1;\,j_1+1,\dots,j_s+1} & (\text{if } i_1 = -1). \end{cases}$$

Then one checks easily that σ maps V_m onto V_{m+1}, and that σ intertwines with the action of $\mathfrak{g}$. So one obtains

THEOREM 1. *Fix any* $\mu \in \boldsymbol{C}$. *Then the representations of* $\mathfrak{g}$ *on* V_m*'s are mutually equivalent.*

2. Irreducible components in V_m

In this section we consider the case of $\sqrt{2}\mu \in \boldsymbol{Z}$; i.e., $\mu = -n/\sqrt{2}$ $(n \in \boldsymbol{Z})$. For integers $r \geqq 1$ and $k \geqq m-r$, we denote by $M = M(m;\, k,\, r)$ the Maya diagram



with charge m, and by $f_M = f_{M(m;\,k,r)}$ the corresponding vector in V. Especially for $r=0$, $M(m;\, k,\, 0)$ is defined to be the Maya diagram corresponding to the ground state $f^{(m)}$.

By computing $L_j f_{M(m;\,k,r)}$ $(j=1,\, 2)$, one sees that f_M is killed by L_j with all positive j if and only if $k = m+n$. Moreover one has

$$L_0 f_M = (1/4)(n+2r)^2 f_M \quad \text{and} \quad \deg f_M = m(m-1)/2 + r(r+n)$$

for $M = M(m;\, m+n,\, r)$, and so, due to the Corollary in p. 444 of [5], the formal character of the irreducible submodule generated by f_M is given by

$$\begin{aligned} &q^{m(m-1)/2+r(r+n)}(1-q^{n+2r+1})/\varphi(q) \\ &= q^{m(m-1)/2}(q^{r(r+n)} - q^{(r+1)(r+1+n)})/\varphi(q). \end{aligned}$$

The sum of these characters over all non-negative integers r is equal to $q^{m(m-1)/2}/\varphi(q)$, which is just the formal character of V_m. Thus we have proved

THEOREM 2. *In case that* $\mu = -n/\sqrt{2}$ $(n \in \boldsymbol{Z})$, *the* $\mathfrak{g}$*-module* V_m *is reducible for each* $m \in \boldsymbol{Z}$, *and*

$$\{f_M;\ M = M(m;\, m+n,\, r),\ r \geqq \mathrm{Max}\,\{0,\, -n\}\}$$

is a complete set of the highest weight vectors of the irreducible components of V_m.

3. Remarks

Finally we remark here on a relationship of our representations with the usual Fock representations of the Virasoro algebra. In [10], we proved that $\{\chi_Y(\sqrt{2}x)\}$ exhaust the highest weight vectors in $C[x_1, x_2, \ldots]$, where χ_Y is the character polynomial of the $(r, r+n)$-rectangular Young diagram Y. Under the correspondence between Maya diagrams (or equivalently, Young diagrams) and Schur polynomials, the Virasoro operators turn out to be given by

$$l_j = (1/2) \sum_{n \in \boldsymbol{Z}} (j+2n+1) :\psi_n \psi^*_{j+n}:$$

in the Fermion picture.

Now the change of variables $x \mapsto \sqrt{2}x$ induces a deformation of l_j involving interactions of four Fermions, and our constructions of $\{L_j\}_{j \in \boldsymbol{Z}}$ can also be interpreted along these lines.

References

[1] B. L. Feigin and D. B. Fuks: Invariant skew-symmetric differential operators on the line and Verma modules over the Virasoro algebra, Func. Anal. Appl., **16** (1982), 114–126.

[2] B. L. Feigin and D. B. Fuks: Verma modules over the Virasoro algebra, Func. Anal. Appl., **17** (1983), 241–242.

[3] M. Jimbo and T. Miwa: Solitons and infinite dimensional Lie algebras, Publ. RIMS Kyoto Univ., **19** (1983), 943–1001.

[4] V. G. Kac: Highest weight representations of infinite-dimensional Lie algebras, Proceedings of ICM, Helsinki, 1978, 299–304.

[5] V. G. Kac: Contravariant form for infinite-dimensional Lie algebras and superalgebras, Lect. Notes in Phys. **94**, 1979, 441–445.

[6] V. G. Kac: Some problems on infinite dimensional Lie algebras and their representations, Lect. Notes in Math. **933**, 1982, 117–126.

[7] A. Rocha-Caridi: Vacuum vector representations of the Virasoro algebra, MSRI Publ. **3** (1984), 451–473.

[8] M. Sato: Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold, RIMS-Kokyuroku **439** (1981), 30–46.

[9] A. Tsuchiya and Y. Kanie: Fock space representations of the Virasoro algebra —Intertwining operators—, preprint.

[10] M. Wakimoto and H. Yamada: Irreducible decompositions of Fock representations of the Virasoro algebra, Lett. Math. Phys., **7** (1983), 513–516.

[11] N. R. Wallach: Classical invariant theory and the Virasoro algebra, MSRI Publ. **3** (1984), 475–482.

Department of Mathematics,
Faculty of Science,
Hiroshima University