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On oscillation of linear neutral differential equations of higher order

Dedicated to Professor Marko Švec on the occasion of his seventieth birthday

Jaroslav JAROŠ and Takaŝi KUSANO

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1. Introduction

We consider linear neutral functional differential equations of the form

$$\text{(A)} \qquad \frac{d^n}{dt^n}[x(t) - h(t)x(\tau(t))] + \sigma p(t)x(g(t)) = 0\,, \qquad t \geqq t_0\,,$$

where $n \geqq 2$, $\sigma = +1$ or -1 and the following conditions are assumed to hold without further mention:

(B) (a) $h : [t_0, \infty) \to (0, \infty)$ is continuous;
(b) $\tau : [t_0, \infty) \to \boldsymbol{R}$ is continuous and strictly increasing, and $\lim_{t\to\infty} \tau(t) = \infty$;
(c) $p : [t_0, \infty) \to (0, \infty)$ is continuous;
(d) $g : [t_0, \infty) \to \boldsymbol{R}$ is continuous and $\lim_{t\to\infty} g(t) = \infty$.

Our aim is to establish new oscillation criteria for equation (A), i.e., sufficient conditions under which all proper solutions of (A) are oscillatory. By a proper solution of (A) we here mean a continuous function $x : [t_x, \infty) \to \boldsymbol{R}$ such that $x(t) - h(t)x(\tau(t))$ is n-times continuously differentiable, $x(t)$ satisfies (A) for all sufficiently large $t \geqq t_x$ and $\sup\{|x(t)| : t \geqq T\} > 0$ for any $T \geqq t_x$. Such a solution is called oscillatory if it has arbitrarily large zeros in $[t_x, \infty)$ and it is called nonoscillatory otherwise.

The problem of oscillation of neutral functional differential equations has received considerable attention in the last few years (see, for example, the papers [1–14, 17–20] and the references cited therein). However, most of the works on the subject has been focused on first and second order equations with constant parameters and very little has been published on higher order neutral equations. For some particular results we refer to [7], [13–14] and [18].

The present paper is an attempt to make a systematic study of oscillatory properties of higher order equations of the form (A) with general arguments $h(t)$ and $\tau(t)$. Our technique here is based on deriving two infinite sequences $\{(\mathrm{I}_k^+, \sigma)\}_{k=0}^{\infty}$ and $\{(\mathrm{I}_m^-, \sigma)\}_{m=1}^{\infty}$ of "non-neutral" functional differential inequalities

with the properties that each inequality (I_k^+, σ), $k = 0, 1, \ldots$, possesses a nonoscillatory solution if equation (A) has a nonoscillatory solution $x(t)$ such that

$$x(t)[x(t) - h(t)x(\tau(t))] > 0 \tag{1.1}$$

for all large t, and each (I_m^-, σ), $m = 1, 2, \ldots$, possesses a nonoscillatory solution if (A) has a nonoscillatory solution $x(t)$ such that

$$x(t)[x(t) - h(t)x(\tau(t))] < 0 \tag{1.2}$$

for all large t. Since (1.1) and (1.2) are the only possibilities for a nonoscillatory solution $x(t)$ of (A), the problem is thus reduced to finding conditions guaranteeing the nonexistence of nonoscillatory solutions for some (I_k^+, σ) and (I_m^-, σ), and such desired conditions will be obtained as a suitable combination of known results on "non-neutral" functional differential inequalities.

It should be emphasized that in contrast to the results contained in the above-mentioned papers, no "a priori" restriction upon h and τ (such as $h(t) < 1$ or $h(t) > 1$ and/or $\tau(t) < t$ or $\tau(t) > t$) is made. Thus the results presented here extend and unify the previous ones. Furthermore, even when specialized to the situation where one of the possibilities $\{h(t) < 1, \tau(t) < t\}$, $\{h(t) < 1, \tau(t) > t\}$, $\{h(t) > 1, \tau(t) < t\}$ and $\{h(t) > 1, \tau(t) > t\}$ occurs, our results are new or improve the known oscillation criteria available for such equations in the literature.

2. Classification of nonoscillatory solutions

Let $x(t)$ be a nonoscillatory solution of the equation (A). From (A) and the hypothesis (B-c) it follows that the function

$$y(t) = x(t) - h(t)x(\tau(t)) \tag{2.1}$$

has to be eventually of constant sign, so that either

$$x(t)y(t) > 0 \tag{2.2}$$

or

$$x(t)y(t) < 0 \tag{2.3}$$

for all sufficiently large t. Assume first that (2.2) holds. Then the function $y(t)$ satisfies $\sigma y(t)y^{(n)}(t) < 0$ eventually and from the well-known Kiguradze's lemma (see, for example, [15]) it follows that there is an integer $l \in \{0, 1, \ldots, n\}$ and a $t_1 \geqq t_0$ such that $(-1)^{n-l-1}\sigma = 1$ and

$$(2.4)_l \qquad \begin{aligned} &y(t)y^{(i)}(t) > 0 \quad \text{for} \quad 0 \leqq i \leqq l \quad \text{and} \quad t \geqq t_1, \\ &(-1)^{i-l}y(t)y^{(i)}(t) > 0 \quad \text{for} \quad l \leqq i \leqq n \quad \text{and} \quad t \geqq t_1. \end{aligned}$$

A function $y(t)$ satisfying $(2.4)_l$ is said to be a nonoscillatory function of degree l. In what follows, the set of all solutions $x(t)$ of (A) satisfying (2.2) and $(2.4)_l$ will be denoted by $\mathcal{N}_l^+$.

Now assume that (2.3) holds. Then $y(t)$ satisfies $(-\sigma)y(t)y^{(n)}(t) < 0$ for all large t and so it is a function of degree l for some $l \in \{0, 1, \dots, n\}$ with $(-1)^{n-l}\sigma = 1$. The totality of nonoscillatory solutions of (A) which satisfy (2.3) and $(2.4)_l$ will be denoted by $\mathcal{N}_l^-$.

Consequently, if we denote by $\mathcal{N}$ the set of all possible nonoscillatory solutions of (A), then

$$
\begin{aligned}
&\mathcal{N} = \mathcal{N}_1^+ \cup \mathcal{N}_3^+ \cup \cdots \cup \mathcal{N}_{n-1}^+ \cup \mathcal{N}_0^- \cup \mathcal{N}_2^- \cup \cdots \cup \mathcal{N}_n^- \\
&\qquad\qquad \text{for } \sigma = 1 \text{ and } n \text{ even}, \\
&\mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \cdots \cup \mathcal{N}_{n-1}^+ \cup \mathcal{N}_1^- \cup \mathcal{N}_3^- \cup \cdots \cup \mathcal{N}_n^- \\
&\qquad\qquad \text{for } \sigma = 1 \text{ and } n \text{ odd}, \\
&\mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \cdots \cup \mathcal{N}_n^+ \cup \mathcal{N}_1^- \cup \mathcal{N}_3^- \cup \cdots \cup \mathcal{N}_{n-1}^- \\
&\qquad\qquad \text{for } \sigma = -1 \text{ and } n \text{ even}, \\
&\mathcal{N} = \mathcal{N}_1^+ \cup \mathcal{N}_3^+ \cup \cdots \cup \mathcal{N}_n^+ \cup \mathcal{N}_0^- \cup \mathcal{N}_2^- \cup \cdots \cup \mathcal{N}_{n-1}^- \\
&\qquad\qquad \text{for } \sigma = -1 \text{ and } n \text{ odd}.
\end{aligned}
\tag{2.5}
$$

It is now clear that the oscillation of all proper solutions of (A) is equivalent to the situation in which all the nonoscillatory solution classes appearing in the above classification scheme are empty. Sufficient conditions for this situation to hold will be given in the next section.

3. Oscillation of all proper solutions

We begin with an elementary but useful lemma which will be needed in deriving our main results.

The following notation is employed:

$$\tau^0(t) = t\,, \quad \tau^i(t) = \tau(\tau^{i-1}(t))\,, \quad \tau^{-i}(t) = \tau^{-1}(\tau^{-(i-1)}(t))\,, \qquad i = 1, 2, \dots,$$

where $\tau^{-1}(t)$ denotes the inverse function of $\tau(t)$,

$$H_0(t) \equiv 1\,, \qquad H_i(t) = \prod_{j=1}^{i-1} h(\tau^j(t))\,, \qquad i = 1, 2, \dots.$$

LEMMA 3.1. (i) *Let $x(t)$ be a nonoscillatory solution of* (A) *satisfying* (2.2) *for all large t. Then for any integer $k \geqq 0$ there is a $T_k \geqq t_0$ such that*

$$|x(t)| \geqq \sum_{i=0}^{k} H_i(t)|y(\tau^i(t))| \tag{3.1}$$

for $t \geqq T_k$, *where* $y(t)$ *is defined by* (2.1).

(ii) *Let* $x(t)$ *be a nonoscillatory solution of* (A) *satisfying* (2.3) *for all large* t. *Then for any integer* $m \geqq 1$ *there is a* $T_m \geqq t_0$ *such that*

$$|x(t)| \geqq \sum_{j=1}^{m} \frac{y(\tau^{-j}(t))}{H_j(\tau^{-j}(t))} \tag{3.2}$$

for $t \geqq T_m$.

PROOF. Assume that a nonoscillatory solution $x(t)$ of (A) satisfies $x(t)[x(t) - h(t)x(\tau(t))] > 0$ for $t \geqq T \geqq t_0$. From the assumption $\lim_{t\to\infty} \tau(t) = \infty$ it follows that for every integer $k \geqq 0$ there exists a $T_k \geqq T$ such that $\tau^i(t) \geqq T$ for $t \geqq T_k$ and $i = 0, 1, \ldots, k+1$. Now using the relation

$$x(t) = y(t) + h(t)x(\tau(t))$$

repeatedly, we find

$$\begin{aligned} |x(t)| &= \sum_{i=0}^{k} H_i(t)|y(\tau^i(t))| + H_{k+1}(t)|x(\tau^{k+1}(t))| \\ &\geqq \sum_{i=0}^{k} H_i(t)|y(\tau^i(t))| \end{aligned}$$

for $t \geqq T_k$, which proves our claim in the case (i).

Similarly, if $x(t)$ is a nonoscillatory solution of (A) satisfying $x(t)[x(t) - h(t)x(\tau(t))] < 0$ for $t \geqq T$, then, for any integer $m \geqq 1$ there is a $T_m \geqq T$ such that $\tau^{-j}(t) \geqq T$ for $t \geqq T_m$ and $j = 1, \ldots, m$, and the repeated use of

$$x(t) = \frac{x(\tau^{-1}(t)) - y(\tau^{-1}(t))}{h(\tau^{-1}(t))}$$

yields

$$\begin{aligned} |x(t)| &= \frac{|x(\tau^{-m}(t))|}{H_m(\tau^{-m}(t))} + \sum_{j=1}^{m} \frac{|y(\tau^{-j}(t))|}{H_j(\tau^{-j}(t))} \\ &\geqq \sum_{j=1}^{m} \frac{|y(\tau^{-j}(t))|}{H_j(\tau^{-j}(t))} \end{aligned}$$

for $t \geqq T_m$. The proof of the lemma is complete.

Let $x(t)$ be a nonoscillatory solution of (A). Lemma 3.1 then implies that if (2.2) hold, then for any integer $k \geqq 0$ the function $y(t)$ defined by (2.1) satisfies

the functional differential inequality

$$(\mathrm{I}_k^+, \sigma) \qquad \left\{\sigma y^{(n)}(t) + p(t) \sum_{i=0}^{k} H_i(g(t)) y(\tau^i(g(t)))\right\} \operatorname{sgn} y(t) \leqq 0$$

for $t \geqq T_k$, and that if (2.3) holds, then for any integer $m \geqq 1$, the function $y(t)$ is a nonoscillatory solution of

$$(\mathrm{I}_m^-, \sigma) \qquad \left\{\sigma y^{(n)}(t) - p(t) \sum_{j=1}^{m} [H_j(\tau^{-j}(g(t)))]^{-1} y(\tau^{-j}(g(t)))\right\} \operatorname{sgn} y(t) \geqq 0$$

for $t \geqq T_m$.

As an immediate consequence of the above observation we have the following oscillation theorem.

THEOREM 3.1. *Assume that there are integers $k \geqq 0$ and $m \geqq 1$ such that the functional differential inequalities (I_k^+, σ) and (I_m^-, σ) have no nonoscillatory solutions. Then all proper solutions of* (A) *are oscillatory.*

In order to ensure the nonexistence of nonoscillatory solutions of the inequalities (I_k^+, σ) and (I_m^-, σ) for some $k \geqq 0$ and $m \geqq 1$, respectively, we shall use the results (Lemmas 3.2–3.5 below) due to Kitamura [16] specialized to functional differential inequalities of the form

$$(\mathrm{I}, \sigma) \qquad \left\{\sigma u^{(n)}(t) + \sum_{i=1}^{N} p_i(t) u(g_i(t))\right\} \operatorname{sgn} u(t) \leqq 0\,,$$

where $n \geqq 2$, $\sigma = +1$ or -1, $p_i : [t_0, \infty) \to (0, \infty)$, $i = 1, \ldots, N$, are continuous, $g_i : [t_0, \infty) \to \boldsymbol{R}$ are continuous and $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, \ldots, N$. We use the notation:

$$g_i^*(t) = \max\{g_i(t), t\}\,, \qquad \alpha[g_i](t) = \min_{s \geqq t} g_i^*(s)\,,$$

$$g_{i*}(t) = \min\{g_i(t), t\}\,, \qquad \rho[g_i](t) = \max_{t_0 \leqq s \leqq t} g_{i*}(s)\,.$$

LEMMA 3.2. *Let $\sigma = 1$ and n be even. Suppose that there exists an integer $i \in \{1, \ldots, N\}$ such that*

$$(3.3) \qquad \int_{t_0}^{\infty} [g_{i*}(t)]^{n-1} [g_i(t)]^{-\varepsilon} p_i(t)\, dt = \infty$$

for some $\varepsilon > 0$. Then all proper solutions of $(\mathrm{I}, +1)$ *are oscillatory.*

LEMMA 3.3. *Let $\sigma = 1$ and n be odd. If there is an integer $i \in \{1, \ldots, N\}$ such that*

$$\int_{t_0}^{\infty} [g_{i*}(t)]^{n-2}[g_i(t)]^{1-\varepsilon}p_i(t)\,dt = \infty \tag{3.4}$$

for some $\varepsilon > 0$, *then all possible nonoscillatory solutions of* (I, $+1$) *are of degree* 0. *Such nonoscillatory solutions are precluded if there is* $j \in \{1, \dots, N\}$ *such that*

$$\limsup_{t\to\infty} \int_{\rho[g_j](t)}^{t} \frac{\{s - \rho[g_j](t)\}^{n-\nu-1}}{(n-\nu-1)!} \frac{\{\rho[g_j](t) - g_j(s)\}^{\nu}}{\nu!} p_j(s)\,ds > 1 \tag{3.5}$$

for some $\nu \in \{0, 1, \dots, n-1\}$.

LEMMA 3.4. *Let* $\sigma = -1$ *and* n *be odd. If there is an integer* $i \in \{1, \dots, N\}$ *such that*

$$\int_{t_0}^{\infty} t[g_{i*}(t)]^{n-2}[g_i(t)]^{-\varepsilon}p_i(t)\,dt = \infty \tag{3.6}$$

for some $\varepsilon > 0$, *then all possible nonoscillatory solutions of* (I, -1) *are of degree* n. *Such nonoscillatory solutions are precluded if there is an integer* $k \in \{1, \dots, N\}$ *such that*

$$\limsup_{t\to\infty} \int_{t}^{\alpha[g_k](t)} \frac{\{g_k(s) - \alpha[g_k](t)\}^{n-\mu-1}}{(n-\mu-1)!} \frac{\{\alpha[g_k](t) - s\}^{\mu}}{\mu!} p_k(s)\,ds > 1 \tag{3.7}$$

for some $\mu \in \{0, 1, \dots, n-1\}$.

LEMMA 3.5. *Let* $\sigma = -1$ *and* n *be even. If there exists an integer* $i \in \{1, \dots, N\}$ *such that*

$$\int_{t_0}^{\infty} t[g_{i*}(t)]^{n-3}[g_i(t)]^{1-\varepsilon}p_i(t)\,dt = \infty \tag{3.8}$$

for some $\varepsilon > 0$, *then every nonoscillatory solution of* (I, -1) *is either of degree* 0 *or of degree* n. *Solutions of degree* 0 [*resp. degree* n] *are precluded provided that* (3.5) *holds for some* $j \in \{1, \dots, N\}$ *and* $\nu \in \{0, 1, \dots, n-1\}$ [*resp.* (3.7) *holds for some* $k \in \{1, \dots, N\}$ *and* $\mu \in \{0, 1, \dots, n-1\}$].

The main results of this paper will now be stated and proved.

THEOREM 3.2. *Let* $\sigma = 1$ *and* n *be even. Then all proper solutions of* (A) *are oscillatory if there exist a nonnegative integer* i *and positive integers* j, k *and* m *such that*

$$\int_{t_0}^{\infty} [(\tau^i \circ g)_*(t)]^{n-1}[(\tau^i \circ g)(t)]^{-\varepsilon}p(t)H_i(g(t))\,dt = \infty \tag{3.9}$$

for some $\varepsilon > 0$,

$$(3.10)\qquad \int_{t_0}^{\infty} t[(\tau^{-j}\circ g)_*(t)]^{n-3}[(\tau^{-j}\circ g)(t)]^{1-\delta}\frac{p(t)}{H_j(\tau^{-j}\circ g(t))}\,dt=\infty$$

for some $\delta>0$,

$$(3.11)\qquad \limsup_{t\to\infty}\int_{\rho[\tau^{-k}\circ g](t)}^{t}\frac{\{s-\rho[\tau^{-k}\circ g](t)\}^{n-\nu-1}}{(n-\nu-1)!}\,\frac{\{\rho[\tau^{-k}\circ g](t)-(\tau^{-k}\circ g)(s)\}^{\nu}}{\nu!}\cdot$$

$$\cdot\frac{p(s)}{H_k(\tau^{-k}\circ g(s))}\,ds>1$$

for some $\nu\in\{0,1,\dots,n-1\}$, *and*

$$(3.12)\qquad \limsup_{t\to\infty}\int_{t}^{\alpha[\tau^{-m}\circ g](t)}\frac{\{(\tau^{-m}\circ g)(s)-\alpha[\tau^{-m}\circ g](t)\}^{n-\mu-1}}{(n-\mu-1)!}\,\frac{\{\alpha[\tau^{-m}\circ g](t)-s\}^{\mu}}{\mu!}$$

$$\cdot\frac{p(s)}{H_m(\tau^{-m}\circ g(s))}\,ds>1$$

for some $\mu\in\{0,1,\dots,n-1\}$.

THEOREM 3.3. *Let* $\sigma=1$ *and* n *be odd. Then all proper solutions of* (A) *are oscillatory if there exist nonnegative integers* i *and* k *and positive integers* j *and* m *such that*

$$(3.13)\qquad \int_{t_0}^{\infty}[(\tau^{i}\circ g)_*(t)]^{n-2}[(\tau^{i}\circ g)(t)]^{1-\varepsilon}p(t)H_i(g(t))\,dt=\infty$$

for some $\varepsilon>0$,

$$(3.14)\qquad \int_{t_0}^{\infty} t[(\tau^{-j}\circ g)_*(t)]^{n-2}[(\tau^{-j}\circ g)(t)]^{-\delta}\frac{p(t)}{H_j(\tau^{-j}\circ g(t))}\,dt=\infty$$

for some $\delta>0$,

$$(3.15)\qquad \limsup_{t\to\infty}\int_{\rho[\tau^{k}\circ g](t)}^{t}\frac{\{s-\rho[\tau^{k}\circ g](t)\}^{n-\nu-1}}{(n-\nu-1)!}\,\frac{\{\rho[\tau^{k}\circ g](t)-(\tau^{k}\circ g)(s)\}^{\nu}}{\nu!}$$

$$\cdot p(s)H_k(g(s))\,ds>1$$

for some $\nu\in\{0,1,\dots,n-1\}$, *and* (3.12) *holds for some* $\mu\in\{0,1,\dots,n-1\}$.

THEOREM 3.4. *Let* $\sigma=-1$ *and* n *be even. Then all proper solutions of* (A) *are oscillatory if there exist nonnegative integers* i, k *and* m *and a positive integer* j *such that*

$$(3.16)\qquad \int_{t_0}^{\infty} t[(\tau^{i}\circ g)_*(t)]^{n-3}[(\tau^{i}\circ g)(t)]^{1-\varepsilon}p(t)H_i(g(t))\,dt=\infty$$

for some $\varepsilon > 0$,

$$\int_{t_0}^{\infty} [(\tau^{-j} \circ g)_*(t)]^{n-1} [(\tau^{-j} \circ g)(t)]^{-\delta} \frac{p(t)}{H_j(\tau^{-j} \circ g(t))}\, dt = \infty \tag{3.17}$$

for some $\delta > 0$, (3.15) *holds for some* $v \in \{0, 1, \ldots, n-1\}$, *and*

$$\limsup_{t \to \infty} \int_t^{\alpha[\tau^m \circ g](t)} \frac{\{(\tau^m \circ g)(s) - \alpha[\tau^m \circ g](t)\}^{n-\mu-1}}{(n-\mu-1)!} \frac{\{\alpha[\tau^m \circ g](t) - s\}^{\mu}}{\mu!} \cdot p(s) H_m(g(s))\, ds > 1 \tag{3.18}$$

for some $\mu \in \{0, 1, \ldots, n-1\}$.

THEOREM 3.5. *Let* $\sigma = -1$ *and* n *be odd. Then all proper solutions of* (A) *are oscillatory if there exist nonnegative integers* i *and* m *and positive integers* j *and* k *such that*

$$\int_{t_0}^{\infty} t[(\tau^i \circ g)_*(t)]^{n-2} [(\tau^i \circ g)(t)]^{-\varepsilon} p(t) H_i(g(t))\, dt = \infty \tag{3.19}$$

for some $\varepsilon > 0$,

$$\int_{t_0}^{\infty} [(\tau^{-j} \circ g)_*(t)]^{n-2} [(\tau^{-j} \circ g)(t)]^{1-\delta} \frac{p(t)}{H_j(\tau^{-j} \circ g(t))}\, dt = \infty \tag{3.20}$$

for some $\delta > 0$, *and* (3.11) *and* (3.18) *are satisfied for some* $v \in \{0, 1, \ldots, n-1\}$ *and* $\mu \in \{0, 1, \ldots, n-1\}$, *respectively.*

PROOF OF THEOREM 3.2. According to (2.5), $\mathcal{N}_1^+$, $\mathcal{N}_3^+$, $\ldots$, $\mathcal{N}_{n-1}^+$ and $\mathcal{N}_0^-$, $\mathcal{N}_2^-$, $\ldots$, $\mathcal{N}_n^-$ are the possible nonoscillatory solution classes for equation (A) with $\sigma = 1$ and n even. We shall show that under the conditions of the theorem none of these solution classes has a member.

Suppose first that $\mathcal{N}_l^+ \neq \emptyset$ for some $l \in \{1, 3, \ldots, n-1\}$. Then each of the inequalities $(\mathrm{I}_N^+, +1)$, $N = 0, 1, \ldots$, has a nonoscillatory solution of degree l. However, this is impossible, because from Lemma 3.2 applied to $(\mathrm{I}_N^+, +1)$ with $N \geqq i$ it follows that (3.9) prevents $(\mathrm{I}_N^+, 1)$ from having a nonoscillatory solution of any kind. Thus we must have $N_l^+ = \emptyset$ for all $l \in \{1, 3, \ldots, n-1\}$.

If $\mathcal{N}_l^- \neq \emptyset$ for some $l \in \{2, 4, \ldots, n-2\}$, then all inequalities $(\mathrm{I}_M^-, +1)$, $M = 1, 2, \ldots$, must possess nonoscillatory solutions of degree l. On the other hand, from (3.10) and the first part of Lemma 3.5 it follows that $(\mathrm{I}_M^-, +1)$ with $M \geqq j$ cannot have a nonoscillatory solution of degree $l \in \{2, 4, \ldots, n-2\}$. This contradiction shows that $\mathcal{N}_2^- = \mathcal{N}_4^- = \cdots = \mathcal{N}_{n-2}^- = \emptyset$.

If $\mathcal{N}_0^- \neq \emptyset$, then the inequalities $(\mathrm{I}_M^-, +1)$, $M = 1, 2, \ldots$, have nonoscillatory solutions of degree 0. However, from (3.11) and the second statement of

Lemma 3.5 we see that $(\mathrm{I}_M^-, +1)$ with $M \geqq k$ cannot have a solution of degree 0. Thus, $\mathcal{N}_0^- = \varnothing$ as well.

Finally, suppose that $\mathcal{N}_n^- \neq \varnothing$. Then the inequalities $(\mathrm{I}_M^-, +1)$, $M = 1, 2, \ldots$, have nonoscillatory solutions of degree n. But this contradicts the fact that $(\mathrm{I}_M^-, +1)$ with $M \geqq m$ do not admit solutions of degree n because of (3.12) and the second statement of Lemma 3.5. Thus the class $\mathcal{N}_n^-$ must be empty. This completes the proof of Theorem 3.2.

PROOF OF THEOREM 3.3. *Let* $\mathcal{N}_l^+ \neq \varnothing$ for some $l \in \{0, 2, \ldots, n-1\}$. Then each of the inequalities $(\mathrm{I}_N^+, +1)$, $N = 0, 1, \ldots$, must possess a nonoscillatory solution of degree l. However, this is impossible, because the first part of Lemma 3.3 implies that under (3.13), $(\mathrm{I}_N^+, +1)$ with $N \geqq i$ cannot possess a nonoscillatory solution of degree $l \in \{2, 4, \ldots, n-1\}$, while the second part of Lemma 3.3 implies that, under (3.15), none of $(\mathrm{I}_N^+, +1)$ with $N \geqq k$ admits a solution of degree 0.

Next, by Lemma 3.4, (3.14) guarantees the nonexistence of solutions of degree $l \in \{1, 3, \ldots, n-2\}$ for each of the inequalities $(\mathrm{I}_M^-, +1)$ with $M \geqq j$ and (3.12) implies that none of $(\mathrm{I}_M^-, +1)$ with $M \geqq m$ may have nonoscillatory solutions of degree n. Consequently, $\mathcal{N}_1^- = \mathcal{N}_3^- = \cdots = \mathcal{N}_n^- = \varnothing$. The conclusion of the theorem now follows from (2.5).

PROOF OF THEOREM 3.4. From (3.16) and the first part of Lemma 3.5 we see that none of the inequalities $(\mathrm{I}_N^+, -1)$ with $N \geqq i$ has nonoscillatory solutions of degree $l \in \{2, 4, \ldots, n-2\}$, which implies that $\mathcal{N}_l^+ = \varnothing$, $l \in \{2, 4, \ldots, n-2\}$, for (A) with $\sigma = -1$ and n even. Similarly, the second part of Lemma 3.5 applied to $(\mathrm{I}_N^+, -1)$ with $N \geqq \max\{k, m\}$ shows that both $\mathcal{N}_0^+$ and $\mathcal{N}_n^+$ have to be empty provided that (3.15) and (3.18) hold.

Furthermore, we claim that the condition (3.17) ensures that $\mathcal{N}_l^- = \varnothing$ for all $l = 1, 3, \ldots, n-1$. In fact, if $\mathcal{N}_l^- \neq \varnothing$ for some $l \in \{1, 3, \ldots, n-1\}$, then for any integer $M \geqq 1$ the inequality $(\mathrm{I}_M^-, -1)$ has a solution of degree l. However, because of (3.17), this contradicts the assertion of Lemma 3.2 applied to $(\mathrm{I}_M^-, -1)$ with $M \geqq j$.

Since $\mathcal{N}_0^+, \ldots, \mathcal{N}_n^+$ and $\mathcal{N}_1^-, \ldots, \mathcal{N}_{n-1}^-$ are the only possible nonoscillatory solution classes for (A) with $\sigma = -1$ and n even, we conclude that (A) may have only oscillatory proper solutions in this case.

PROOF OF THEOREM 3.5. In view of (2.5), it suffices to prove that

$$\mathcal{N}_1^+ = \mathcal{N}_3^+ = \cdots = \mathcal{N}_n^+ = \mathcal{N}_0^- = \mathcal{N}_2^- = \cdots = \mathcal{N}_{n-1}^- = \varnothing .$$

If $\mathcal{N}_l^+ \neq \varnothing$ for some $l \in \{1, 3, \ldots, n\}$, then all the inequalities $(\mathrm{I}_N^+, -1)$, $N = 0, 1, \ldots$, have nonoscillatory solutions of degree l. This, however, is

impossible since Lemma 3.4 shows that all proper solutions of $(\mathrm{I}_N^+, -1)$ with $N \geqq \max\{i, m\}$ must be oscillatory provided that (3.18) and (3.19) are satisfied.

Similarly, if $\mathcal{N}_l^- \neq \emptyset$ for some $l \in \{0, 2, \ldots, n-1\}$, then all $(\mathrm{I}_M^-, -1)$, $M = 1, 2, \ldots$, must possess a nonoscillatory solution of degree l. This is also a contradiction, since, according to Lemma 3.3, (3.11) and (3.20) guarantee the oscillation of all proper solutions of $(\mathrm{I}_M^-, -1)$ with $M \geqq \max\{j, k\}$. This completes the proof.

REMARK. The oscillation criteria given in the above theorems become somewhat simpler if the functions $h(t)$ and $\tau(t)$ in (A) satisfy the additional conditions

$$0 < h(t) \leqq h^* < 1 \quad \text{and} \quad \tau(t) < t \quad \text{for } t \geqq t_0 \tag{3.21}$$

or

$$h^* \geqq h(t) \geqq h_* > 1 \quad \text{and} \quad \tau(t) > t \quad \text{for } t \geqq t_0\,, \tag{3.22}$$

where h^* and h_* are constants. In fact, if (3.21) holds, then, as is easily verified, a nonoscillatory solution $x(t)$ of (A) such that $x(t)[x(t) - h(t)x(\tau(t))] < 0$ for all large t satisfies

$$\lim_{t\to\infty} x(t) = 0\,, \qquad \lim_{t\to\infty} [x(t) - h(t)x(\tau(t))] = 0\,, \tag{3.23}$$

which implies that $\mathcal{N}_k^- = \emptyset$ for all $k \geqq 1$. Thus, in this case, the classes $\mathcal{N}_k^-$, $k \geqq 1$, are automatically excluded from the classification list (2.5), and so the conditions (3.10), (3.12), (3.14), (3.17) and (3.20) in Theorems 3.2–3.5 and superfluous and need not be considered.

Similarly, if (3.22) holds, then a nonoscillatory solution $x(t)$ of (A) such that $x(t)[x(t) - h(t)x(\tau(t))] > 0$ for all large t satisfies (3.23), so that the classes $\mathcal{N}_k^+$, $k \geqq 1$, are absent in (2.5). This shows in this case that the conditions (3.9), (3.13), (3.16), (3.18) and (3.19) can be deleted from Theorems 3.2–3.5.

EXAMPLES. Consider the equation

$$\frac{d^n}{dt^n}[x(t) - h(t)x(\log t)] + \sigma p(t)x(\log(\log(et))) = 0 \tag{3.24}$$

which is a special case of (A) in which

$$\tau(t) = \log t\,, \qquad g(t) = \log(\log(et))\,.$$

Suppose that there are positive constants h_*, h^* and p_* such that

$$h_* \leqq h(t) \leqq h^*\,, \qquad p(t) \geqq p_*\,, \qquad t \geqq 1\,. \tag{3.25}$$

We claim that if $\sigma = +1$, then all proper solutions of (3.24) are oscillatory. In fact, since

$$\tau^{-1} \circ g(t) = \log(et), \qquad \rho[\tau^{-1} \circ g](t) = \log(et),$$
$$\tau^{-2} \circ g(t) = et, \qquad \alpha[\tau^{-2} \circ g](t) = et,$$

for t sufficiently large, and since simple calculations show that

$$\int_1^\infty [\log(\log(et))]^{n-1-\varepsilon} p(t)\, dt = \infty,$$

$$\int_1^\infty t[\log(et)]^{n-2-\delta} p(t)\, dt = \infty,$$

$$\limsup_{t\to\infty} \int_{\log(et)}^t \frac{[s - \log(et)]^{n-1}}{(n-1)!} p(s)\, ds = \infty,$$

and

$$\limsup_{t\to\infty} \int_t^{et} \frac{[e(s-t)]^{n-1}}{(n-1)!} p(s)\, ds = \infty,$$

the conditions (3.9)($i = 0$), (3.10)($j = 1$), (3.11)($k = 1, \nu = 0$) and (3.12)($m = 2$, $\mu = 0$) are satisfied. The desired conclusion for the case of even n then follows from Theorem 3.2. The oscillation of all solutions of (3.24) in the case of odd n is guaranteed by Theorem 3.3, since the conditions (3.13)($i = 0$), (3.14)($j = 1$) and (3.15)($k = 0, \nu = 0$) are easily verified.

The situation changes when $\sigma = -1$; neither Theorem 3.4 nor Theorem 3.5 is applicable to this case. It suffices to observe that the condition (3.18) is not satisfied, because $\alpha[\tau^m \circ g](t) = t$ for $m = 1, 2, \ldots$, so that the class $\mathcal{N}_n^+$ can by no means be eliminated. As a matter of fact, (3.25) is not sufficient to eliminate the class $\mathcal{N}_n^+$ for (3.24) with $\sigma = -1$; one such example is given by the equation

$$\frac{d^n}{dt^n}[x(t) - h_0 x(\log t)] - \frac{e^t}{\log(et)} x(\log(\log(et))) = 0,$$

where $h_0 > 0$ is a constant. This equation has a nonoscillatory solution $x(t) = e^t$ which clearly belongs to $\mathcal{N}_n^+$.

The following is an example of equations to which Theorems 3.4 and 3.5 apply:

$$\frac{d^n}{dt^n}[x(t) - h(t)x(t^\alpha)] + \sigma p(t) x(t^\beta) = 0,$$

where $h(t)$ and $p(t)$ satisfy (3.25), and α and β are positive constants such that $\alpha > 1$, $\beta < 1$ and $\alpha\beta > 1$.

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Department of Mathematical Analysis,
Faculty of Mathematics and Physics,
Komensky University
(*Bratislava, Czechoslovakia*)
and
Department of Mathematics,
Faculty of Science,
Hiroshima University