HIROSHIMA MATH. J.
21 (1991), 557–585

Infinitely many radially symmetric solutions of certain semilinear elliptic equations

Ryuji KAJIKIYA

(Received August 6, 1990)

1. Introduction

In this paper we consider radially symmetric solutions to the semilinear elliptic problem

$$\Delta u + f(u) = 0, \qquad x \in \boldsymbol{R}^n, \tag{1.1}$$

$$\lim_{|x| \to \infty} u(x) = 0, \tag{1.2}$$

where $n \geq 2$ and $f(u)$ is locally Lipschitz continuous. The problem of finding radially symmetric solutions $u = u(t)$, $t = |x|$, of equation (1.1) subject to condition (1.2) is converted to the singular boundary value problem for the ordinary differential equation

$$u'' + \frac{n-1}{t} u' + f(u) = 0, \quad t > 0, \tag{1.3}$$

$$u'(0) = 0, \tag{1.4}$$

$$\lim_{t \to \infty} u(t) = 0. \tag{1.5}$$

Under the condition that

$$sf(s) < 0 \quad \text{for} \quad |s| > 0 \quad \text{sufficiently small,} \tag{1.6}$$

the existence of infinitely many solutions to the problem (1.3)–(1.5) has been obtained by several authors. Assumption (1.6) arises from the study of standing wave solutions of the nonlinear Klein-Gordon or Schrödinger equations (see the references [1], [2], [10]). Berestycki and Lions [1], Berger [2] and Strauss [10] obtained the existence results of infinitely many solutions by means of variational methods. They treated this problem in the case where the function $f(s)$ is odd, $f'(0) < 0$ and satisfies some growth conditions. On the other hand, using a dynamical system approach, Jones and Küpper [5] have proved that for any integer $k \geq 0$ there exists a solution of (1.3)–(1.5) having exactly k zeros in the interval $[0, \infty)$. Under the assumption (1.6) which is weaker than the condition $f'(0) < 0$, McLeod, Troy and Weissler [7] have obtained the same result by applying a shooting method. All of these authors have treated the problem (1.3)–(1.5) under condition (1.6) or else $f'(0) < 0$.

In the present paper we consider the case where assumption (1.6) does not hold. In fact, we treat the function $f(s)$ satisfying

(f1) $$sf(s) > 0 \quad \text{for all} \quad s \neq 0.$$

We furthermore impose assumptions which are concerned with decay of $f(s)$ as s tends to zero.

(f2) There exist constants $q \in ((n+2)/(n-2), +\infty)$, $C > 0$ and $r > 0$ such that

$$sf(s) \leq C|s|^{q+1} \qquad \text{for} \quad |s| \leq r.$$

(f3) $$\liminf_{s \to 0} \frac{sf(s)}{F(s)} > \frac{2n}{n-2}, \qquad \text{where} \quad F(s) \equiv \int_0^s f(\tau)\, d\tau.$$

On the other hand, the next assumptions imply that the function $f(s)$ has a superlinear and subcritical growth order in a neighborhood of $s = \pm\infty$.

(f4) There exist constants $p \in (1, (n+2)/(n-2))$, $c_i > 0$ $(i = 1, 2)$ and $R > 0$ such that

$$C_1|s|^{p+1} \leq sf(s) \leq C_2|s|^{p+1} \qquad \text{for} \quad |s| \geq R.$$

(f5) $$\limsup_{s \to \pm\infty} \frac{sf(s)}{F(s)} < \frac{2n}{n-2}.$$

Our main result is stated as follows.

MAIN THEOREM (I). *Let* $n \geq 3$. *Suppose that* $f(s)$ *is locally Lipschitz continuous and satisfies assumptions* (f1)-(f5). *Then for any integer* $k \geq 0$ *there exists a solution of* (1.3)–(1.5) *which has exactly* k *zeros in the interval* $[0, \infty)$. (II). *Let* $n = 2$. *Suppose that* $sf(s) \geq 0$ *for* $|s|$ *sufficiently small. Then any solution* $u(t)$ *of* (1.3)–(1.5) *must possess infinitely many zeros in* $[0, \infty)$.

Our theorem gives a sufficient condition for the existence of solutions with prescribed numbers of zeros in the case where condition (1.6) is not satisfied. Main Theorem is established by taking three important factors into account, namely, global asymptotic stability of the zero solution $u \equiv 0$, nonoscillation of all solutions to equation (1.3) and the existence of solutions with arbitrarily many zeros in a bounded interval. To investigate these problems, this paper is organized into four sections as below.

In Section 2, we give the proof of Main Theorem (II) and study global asymptotic stability of the zero solution $u \equiv 0$ to equation (1.3). Indeed, in Theorem 1, we prove that for $n \geq 3$ the zero solution $u \equiv 0$ is globally asymptotically stable if and only if condition (f1) holds. Also, in the case of $n = 2$, we give a necessary and sufficient condition for the zero solution to be globally asymptotically stable. This will be given in Theorem 2.

In Section 3, we discuss the nonoscillation of all solutions. A solution $u(t) \not\equiv 0$ on some interval $[t_0, \infty)$ of (1.3) is said to be nonoscillatory if it has no zeros in $[T, \infty)$ for some $T \geq t_0$. We prove that under conditions (f1), (f2) and (f3) all solutions $u \not\equiv 0$ of (1.3) are nonoscillatory when $n \geq 3$.

Finally, in Section 4, we prove Main Theorem (I). To this end, we need a key lemma, Lemma 4.2. This is obtained by applying Sturm's comparison theorem together with a Lyapunov-like function. Moreover, we present our earlier result discussed in [6], which guarantees the existence of solutions with prescribed numbers of zeros in the unit interval [0, 1] under conditions (f4) and (f5). Using this assertion together with the results obtained in Sections 2, 3 and 4, we prove Main Theorem (I).

2. Global asymptotic stability of zero solution

In this section we prove Main Theorem (II) and give a necessary and sufficient condition for global asymptotic stability of the zero solution $u \equiv 0$ to equation (1.3). Throughout this paper, by a solution of (1.3) is meant a function $u(t)$ of C^2 class defined on some interval $[t_0, t_1)$ $(0 \leq t_0 < t_1 \leq +\infty)$ which satisfies equation (1.3). When $u_0 \equiv 0$ is a solution of (1.3), that is, $f(0) = 0$, the zero solution is said to be globally asymptotically stable if it is stable and if all solutions $u(t)$ of (1.3) can be extended to $t = +\infty$ and satisfy

$$\lim_{t \to \infty} u(t) = \lim_{t \to \infty} u'(t) = \lim_{t \to \infty} u''(t) = 0. \tag{2.1}$$

We always suppose that $f(s)$ is locally Lipschitz continuous. In the following, we state the main results in this section.

THEOREM 1. *Let $n \geq 3$. Then $u \equiv 0$ is a solution of* (1.3) *and is globally asymptotically stable if and only if condition* (f1) *holds.*

In the case of $n = 2$, we obtain the next result.

THEOREM 2. *Let $n = 2$. Then the following three statements are equivalent*:

(i) *The function $f(s)$ satisfies condition* (f1) *and*

$$\int_0^\infty e^{\lambda s} f(s)\, ds = +\infty \quad \textit{and} \quad \int_{-\infty}^0 e^{\lambda |s|} f(s)\, ds = -\infty \tag{2.2}$$

for any $\lambda > 0$.

(ii) *The zero solution* $u \equiv 0$ *is globally asymptotically stable.*

(iii) *All nontrivial solutions of* (1.3) *can be extended to* $t = +\infty$ *and are oscillatory.*

By a nontrivial solution we mean a solution $u(t)$ such that $u(t) \not\equiv 0$. For a nontrivial solution $u(t)$ defined on $[T, \infty)$ for some $T > 0$, we say that $u(t)$ is oscillatory if it has an unbounded sequence of zeros in $[T, \infty)$. Otherwise it is said to be nonoscillatory. We give the proof of Main Theorem (II) in the following.

PROOF OF MAIN THEOREM (II). By our assumption on $f(s)$ there is $\delta > 0$ such that

$$sf(s) \geq 0 \quad \text{for} \quad |s| \leq \delta. \tag{2.3}$$

Let $u(t)$ be a solution defined on $[0, \infty)$ of (1.3)–(1.5). We employ the following transformation,

$$s = \log t \quad \text{and} \quad v(s) = u(t).$$

Then equation (1.3) is converted to the equation

$$v'' + e^{2s} f(v) = 0.$$

Since $u(t)$ satisfies condition (1.5), we see that $\lim_{s\to\infty} v(s) = 0$. We suppose that $u(t)$ has at most a finite number of zeros. Then so is $v(s)$. Hence there exists a $T > 0$ such that

$$\delta > v(s) > 0 \quad \text{for} \quad s \geq T \tag{A}$$

or

$$-\delta < v(s) < 0 \quad \text{for} \quad s \geq T. \tag{B}$$

Consider the case (A). Since $v(T) > 0$ and $\lim_{s\to\infty} v(s) = 0$, we apply the mean-value theorem to find a T_1 $(> T)$ such that $v'(T_1) < 0$. It follows from (2.3) that $v'' = -e^{2s} f(v) \leq 0$ for $s \geq T$, which implies that $v'(s) \leq v'(T_1) < 0$ for $s \geq T_1$. Consequently, we have $\lim_{s\to\infty} v(s) = -\infty$. This contradicts the fact that $\lim_{s\to\infty} v(s) = 0$. In the same way as above, we see that the case (B) is also impossible. Hence $u(t)$ has infinitely many zeros, and the proof is complete.

To prove Theorems 1 and 2, we prepare several lemmas.

LEMMA 2.1. *In case of* $n \geq 3$ *assume that* (f1) *holds. In case of* $n = 2$, *suppose that condition* (i) *of Theorem 2 is valid. Let* $u(t)$ *be a nontrivial solution of* (1.3) *defined on some interval* $[t_0, t_1)$. *Then it can be extended to* $t = +\infty$

and $u(t)$, $u'(t)$ *and* $u''(t)$ *are bounded on* $[t_0, \infty)$.

PROOF. For the solution u we define

$$E(t) \equiv \frac{1}{2} u'(t)^2 + F(u(t)),$$

(2.4)

$$F(s) \equiv \int_0^s f(\tau)\, d\tau.$$

Multiplying (1.3) by $u'(t)$, we obtain

$$E'(t) = -\frac{n-1}{t} u'(t)^2 \leq 0. \tag{2.5}$$

Since the critical points of the solution $u(t)$ are isolated, we see that $E(t)$ is strictly decreasing. Noting that $F(s) > 0$ for $s \neq 0$ by (f1), we have

$$\frac{1}{2} u'(t)^2 \leq E(t) \leq E(t_0) \qquad \text{for} \quad t \geq t_0,$$

and so

$$|u'(t)| \leq (2E(t_0))^{1/2}. \tag{2.6}$$

Integrating both sides of this inequality over $[t_0, t]$, we get

$$|u(t)| \leq (2E(t_0))^{1/2}(t - t_0) + |u(t_0)|. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\sup_{t_0 \leq t < T} (|u(t)| + |u'(t)|) < \infty \qquad \text{for any} \quad T \in (t_0, +\infty),$$

which asserts that $u(t)$ can be extended to $[t_0, \infty)$.

We next prove the boundedness of $u(t)$. For this purpose we suppose that $\limsup_{t\to\infty} u(t) = +\infty$. Then there are only two cases to be checked.

(A) There exists a point t_1 such that $u(t_1) > 0$ and $u'(t) > 0$ for all $t \geq t_1$.
(B) There exists a t_1 such that $u(t_1) > 0$ and $u'(t_1) = 0$.

If (B) holds, then

$$F(u(t)) \leq E(t) \leq E(t_1) = F(u(t_1)) \qquad \text{for} \quad t \geq t_1.$$

Since $F(s)$ is strictly increasing in $[0, \infty)$ by (f1), we obtain the inequality $u(t) \leq u(t_1)$ for $t \geq t_1$. This contradicts the assumption $\limsup_{t\to\infty} u(t) = +\infty$. Therefore the case (B) does not occur.

We then consider the case (A). First we treat the case where $n \geq 3$. Since

$u(t) \geq u(t_1) > 0$ for $t \geq t_1$, we have

$$(t^{n-1}u')' = -t^{n-1}f(u) < 0 \qquad \text{for} \quad t \geq t_1, \tag{2.8}$$

and therefore

$$t^{n-1}u'(t) \leq t_1^{n-1}u'(t_1) \qquad \text{for} \quad t \geq t_1. \tag{2.9}$$

Dividing both sides by t^{n-1} and integrating both sides of the resultant inequality over $[t_1, t]$, we obtain

$$u(t) \leq u(t_1) + \frac{t_1}{n-2}u'(t_1) \qquad \text{for} \quad t \geq t_1.$$

This contradicts our assumption that $\limsup_{t\to\infty} u(t) = +\infty$.

Secondly, we deal with the case of $n = 2$. It follows from (2.9) with $n = 2$ that

$$u(t) \leq a\ \log(t/t_1) + b \qquad \text{for} \quad t \geq t_1, \tag{2.10}$$

where $a = t_1 u'(t_1) > 0$ and $b = u(t_1) > 0$. Integrating (2.8) over $[t_1, s]$ with respect to t, we obtain

$$su'(s) = a - \int_{t_1}^{s} tf(u(t))\,dt. \tag{2.11}$$

Since $u'(t) > 0$ on $[t_1, \infty)$ and $u(\cdot)$ maps $[t_1, \infty)$ onto $[b, \infty)$, the inverse function $t = u^{-1}(\tau) = w(\tau)$ is defined on $[b, \infty)$. It follows from (2.10) that

$$w(\tau) \geq t_1 \exp((\tau - b)/a). \tag{2.12}$$

By (2.9) with $n = 2$ and (2.12), we obtain

$$w'(\tau) = \frac{1}{u'(t)} \geq \frac{w(\tau)}{a} \geq \frac{1}{a}t_1 \exp((\tau - b)/a). \tag{2.13}$$

By (2.12) and (2.13) we have

$$\begin{aligned} \int_{t_1}^{s} tf(u(t))\,dt &= \int_{b}^{u(s)} w(\tau)f(\tau)w'(\tau)\,d\tau \\ &\geq \frac{1}{a}t_1^2 \int_{b}^{u(s)} \exp(2(\tau - b)/a)f(\tau)\,d\tau. \end{aligned}$$

This and (2.11) together imply

$$su'(s) \leq a - c\int_{b}^{u(s)} e^{\lambda\tau}f(\tau)\,d\tau,$$

where $c = a^{-1} t_1^2 \exp(-2b/a)$ and $\lambda = 2/a$. Since $\lim_{s\to\infty} u(s) = \infty$ from our assumption on $u(t)$, we see that

$$\lim_{s\to\infty} \int_b^{u(s)} e^{\lambda\tau} f(\tau)\, d\tau = +\infty$$

by (2.2). Therefore we have $\lim_{s\to\infty} su'(s) = -\infty$. This contradicts (A). In either case (A) or case (B), we have a contradiction. Hence it follows that $\limsup_{t\to\infty} u(t) < \infty$. By the same argument, we also obtain $\liminf_{t\to\infty} u(t) > -\infty$ and so $u(t)$ is bounded on $[t_0, \infty)$. The boundedness of $u'(t)$ on $[t_0, \infty)$ follows from the inequality (2.6). Thus we see that the right-hand side of

$$u''(t) = -\frac{1}{t} u'(t) - f(u)$$

is bounded on $[t_0, \infty)$. This means that $u''(t)$ is also bounded. The proof is now complete.

Using the results obtained in Lemma 2.1, we can prove the next lemma.

LEMMA 2.2. *Assume that all of the hypotheses of Lemma* 2.1 *are fulfilled. Let $u(t)$ be a nontrivial solution of* (1.3) *defined on* $[t_0, \infty)$. *Then there exists a sequence* $\{\xi_k\}_{k=1}^{\infty}$ *such that* $\lim_{k\to\infty} \xi_k = +\infty$ *and* $\lim_{k\to\infty} u(\xi_k) = 0$.

PROOF. Suppose that the assertion is false. Then we have positive constants a and T such that

$$\text{(A)} \qquad u(t) \geq a \qquad \text{for} \quad t \geq T$$

or

$$\text{(B)} \qquad u(t) \leq -a \qquad \text{for} \quad t \geq T.$$

On the other hand, by Lemma 2.1 there is a $C > 0$ such that $|u(t)| \leq C$ and $|u'(t)| \leq C$ for $t \geq T$. Therefore in the first case (A) we obtain

$$u''(t) = -\frac{n-1}{t} u' - f(u) \leq \frac{n-1}{t} C - m \qquad \text{for} \quad t \geq T,$$

where $m = \min_{a\leq s\leq C} f(s) > 0$. This implies that $u''(t) \leq -m/2$ for t sufficiently large. From this it follows that $\lim_{t\to\infty} u(t) = -\infty$, which contradicts the boundedness of $u(t)$. In the same way as in the case (A), one can show that the second case (B) is not valid. Hence the proof is complete.

In what follows, we suppose without further mention that the hypotheses of Lemma 2.2 are valid, and that $u(t)$ is a nontrivial solution on $[t_0, \infty)$ of (1.3). For the solution $u(t)$, we define the function $E(t)$ by (2.4). As proved in Lemma 2.1, $E(t)$ is positive and strictly decreasing. Therefore there exists a limit $E_\infty \equiv \lim_{t\to\infty} E(t) \ge 0$. To prove Theorems 1 and 2, we show that $E_\infty = 0$. To the contrary, we assume that $E_\infty > 0$. Then we can choose positive constants α and β so small that

$$\frac{1}{2} E_\infty \ge F(\alpha) = F(-\beta) > 0. \tag{2.14}$$

For a sequence $\{\xi_k\}_{k=1}^\infty$ obtained in Lemma 2.2, we may assume without loss of generality that

$$-\beta < u(\xi_k) < \alpha \qquad \text{for} \quad k \in \boldsymbol{N}. \tag{2.15}$$

LEMMA 2.3. *Assume that the hypotheses of Lemma* 2.2 *are fulfilled, and that* E_∞ *is positive. Suppose that* α *and* β *satisfy* (2.14). *Then the inequality* $|u'(t)| \ge E_\infty^{1/2}$ *holds for all* t *satisfying* $-\beta \le u(t) \le \alpha$.

PROOF. Let t satisfy that $-\beta \le u(t) \le \alpha$. Since $F(s)$ is increasing in $[0, \infty)$ and decreasing in $(-\infty, 0]$ by condition (f1), we have

$$F(u(t)) \le \max(F(\alpha), F(-\beta)) \le \frac{1}{2} E_\infty,$$

and therefore we obtain

$$\frac{1}{2} u'(t)^2 = E(t) - F(u(t)) \ge E_\infty - \frac{1}{2} E_\infty = \frac{1}{2} E_\infty.$$

This is the desired inequality and the proof is complete.

LEMMA 2.4. *Under the hypotheses of Lemma* 2.3, *we have two sequences* $\{\mu_k\}$ *and* $\{\nu_k\}$ *such that* $u(\mu_k) > \alpha$, $u(\nu_k) < -\beta$, $\lim_{k\to\infty} \mu_k = +\infty$ *and* $\lim_{k\to\infty} \nu_k = +\infty$.

PROOF. To prove this lemma, it is sufficient to show the next statement. For each $k \in \boldsymbol{N}$ there exist numbers μ and ν such that

$$\mu, \nu > \xi_k, \quad u(\mu) > \alpha \quad \text{and} \quad u(\nu) < -\beta. \tag{2.16}$$

Fix any $k \in \boldsymbol{N}$. In view of Lemma 2.3 and (2.15) it is sufficient to consider the following two cases:

$$\text{(A)} \quad u'(\xi_k) \ge E_\infty^{1/2} \quad \text{and} \quad \text{(B)} \quad u'(\xi_k) \le -E_\infty^{1/2}.$$

First we treat case (A). If $u(t) \le \alpha$ for all $t \ge \xi_k$, then it follows from Lemma 2.3 that $u'(t) \ge E_\infty^{1/2}$ for $t \ge \xi_k$. This implies that $\lim_{t\to\infty} u(t) = +\infty$, which contradicts the boundedness of $u(t)$. Thus one finds a number $\mu > \xi_k$ satisfying $u(\mu) > \alpha$. Next, we choose an integer m such that $\mu < \xi_m$. Then it follows from (2.15) that $u(\mu) > \alpha > u(\xi_m)$. Hence there is a $t_1 \in (\mu, \xi_m)$ such that

$$u'(t_1) < 0 \quad \text{and} \quad -\beta < u(t_1) < \alpha.$$

Lemma 2.3 asserts that $u'(t_1) \leq -E_\infty^{1/2}$. Using the same argument as in the previous discussion, one finds $\nu \in (t_1, \infty)$ such that $u(\nu) < -\beta$. It turns out that in the first case (A) there exist numbers μ and ν satisfying (2.16). We can deal with the second case (B) in the same way to get the same conclusion. This completes the proof.

Lemma 2.5. *Assume all of the hypotheses of Lemma* 2.4. *Then there exist four sequences* $\{p_k\}$, $\{q_k\}$, $\{r_k\}$, $\{s_k\}$ *and positive constants* m_1, m_2 *and* C *with the following properties*:

(i) $$p_k < q_k < r_k < s_k < p_{k+1} \quad \text{for} \quad k \in N,$$

(ii) $$\lim_{k\to\infty} p_k = +\infty,$$

(iii) $$u(p_k) = u(s_k) = -\beta, \quad u(q_k) = u(r_k) = \alpha,$$

(iv) $$-\beta < u(t) < \alpha, \quad E_\infty^{1/2} \leq u'(t) \leq C \qquad \textit{for} \quad t \in (p_k, q_k),$$

(v) $$-\beta < u(t) < \alpha, \quad -C \leq u'(t) \leq -E_\infty^{1/2} \qquad \textit{for} \quad t \in (r_k, s_k),$$

(vi) $$\alpha < u(t) \leq C, \quad -C \leq u''(t) \leq -m_1 \qquad \textit{for} \quad t \in (q_k, r_k),$$

(vii) $$-C \leq u(t) < -\beta, \quad m_2 \leq u''(t) \leq C \qquad \textit{for} \quad t \in (s_k, p_{k+1}).$$

Proof. By Lemma 2.1 there exists $C > 0$ such that

$$|u(t)|,\ |u'(t)|,\ |u''(t)| \leq C \qquad \text{for} \quad t \geq t_0. \tag{2.17}$$

We set $a_1 = \min_{\alpha \leq s \leq C} f(s)\ (>0)$ and $a_2 = -\max_{-C \leq s \leq -\beta} f(s)\ (>0)$. Choose a number $T\ (\geq t_0)$ so large that

$$C(n-1) < \frac{T}{2}\min(a_1, a_2).$$

For any $t \in [T, \infty)$ with $u(t) \geq \alpha$ it follows that

$$u''(t) = -\frac{n-1}{t}u' - f(u) \leq \frac{C(n-1)}{T} - a_1 \leq -\frac{a_1}{2}. \tag{2.18}$$

For any $t \in [T, \infty)$ with $u(t) \leq -\beta$ we have

$$u''(t) = -\frac{n-1}{t}u' - f(u) \geq -\frac{C(n-1)}{T} + a_2 \geq \frac{a_2}{2}. \tag{2.19}$$

On the other hand, by Lemma 2.4, there is a sequence $\{t_k\}_{k=1}^\infty$ such that

$\lim_{k\to\infty} t_k = +\infty$, $u(t_k) = -\beta$ and $u'(t_k) \geq 0$ for $k \in N$. Then Lemma 2.3 implies that $u'(t_k) > 0$. Therefore the set A defined by

$$A \equiv \{t \in [T, \infty): u(t) = -\beta \text{ and } u'(t) > 0\}$$

is nonempty and is unbounded. If A has an accumulation point τ, then it follows that $u(\tau) = -\beta$ and $u'(\tau) = 0$. This contradicts the assertion of Lemma 2.3. Hence the set A has no accumulation points and so it is a countably infinite set, that is, we can rewrite it as

$$A = \{p_k : k = 1, 2, \cdots\}, \quad \text{where} \quad T \leq p_1 < p_2 < \cdots \uparrow +\infty.$$

In the same way, we define the sequences $\{q_k\}$, $\{r_k\}$ and $\{s_k\}$ by

$$\{q_k : k \in N\} = \{t \in [p_1, \infty): u(t) = \alpha,\ u'(t) > 0\},$$

$$\{r_k : k \in N\} = \{t \in [p_1, \infty): u(t) = \alpha,\ u'(t) < 0\}$$

and

$$\{s_k : k \in N\} = \{t \in [p_1, \infty): u(t) = -\beta,\ u'(t) < 0\},$$

respectively. Then the assertions (i), (ii) and (iii) of Lemma 2.5 follow readily from the definitions of $\{p_k\}$, $\{q_k\}$, $\{r_k\}$ and $\{s_k\}$. By Lemma 2.3 and (2.17) we obtain the assertions (iv) and (v). Combining (2.17), (2.18) and (2.19), we have (vi) and (vii). The proof is complete.

LEMMA 2.6. *Under the assumptions of Lemma 2.5 there exist two positive constants m and M which satisfy the following inequalities for all $k \in N$,*

(i) $$m \leq q_k - p_k \leq M, \qquad m \leq r_k - q_k \leq M,$$

(ii) $$m \leq s_k - r_k \leq M \quad \text{and} \quad m \leq p_{k+1} - s_k \leq M.$$

PROOF. By the application of the mean-value theorem there exists $\lambda_k \in (p_k, q_k)$ such that

$$u'(\lambda_k) = \frac{u(q_k) - u(p_k)}{q_k - p_k} = \frac{\alpha + \beta}{q_k - p_k}.$$

It follows from Lemma 2.5 (iv) that

$$E_\infty^{1/2} \leq \frac{\alpha + \beta}{q_k - p_k} \leq C.$$

This implies the first inequality of the assertion (i).

Let us prove the second inequality. By Lemma 2.5 (iv) and (v), we have

$$-2C \leq u'(r_k) - u'(q_k) \leq -2E_\infty^{1/2}. \tag{2.20}$$

On the other hand, by the mean-value theorem, there is $\tau_k \in (q_k, r_k)$ such that

$$u''(\tau_k) = \frac{u'(r_k) - u'(q_k)}{r_k - q_k}. \tag{2.21}$$

Combining Lemma 2.5 (vi), (2.20) and (2.21), we obtain the inequalities,

$$-C \le \frac{-2E_\infty^{1/2}}{r_k - q_k} \quad \text{and} \quad \frac{-2C}{r_k - q_k} \le -m_1.$$

This means that the second inequality of the assertion (i) holds. The assertion (ii) can be proved in the same argument as in the proof of (i). The proof is thereby complete.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $n \ge 3$ and condition (f1) hold. Then we see that $u \equiv 0$ is a solution of (1.3) since $f(0) = 0$ by (f1). Using a Lyapunov function $E(t)$ defined by (2.4), we readily conclude that the solution $u \equiv 0$ is stable. We then show that any nontrivial solution $u(t)$ can be extended to $t = +\infty$ and satisfies (2.1). Let $u(t)$ be any nontrivial solution defined on some interval $[t_0, t_1)$. Then, by Lemma 2.1, $u(t)$ can be extended to $t = +\infty$. We now suppose that $E_\infty = \lim_{t\to\infty} E(t) > 0$. Let $\{p_k\}$ and $\{q_k\}$ be the sequences defined in Lemma 2.5. Integrating (2.5) over $[p_k, q_k]$, we have

$$E(q_k) - E(p_k) = -(n-1)\int_{p_k}^{q_k} \frac{1}{t} u'(t)^2\, dt. \tag{2.22}$$

The integral on the right-hand side is estimated as

$$\int_{p_k}^{q_k} \frac{1}{t} u'(t)^2\, dt \ge E_\infty(q_k - p_k)/q_k \ge mE_\infty/q_k, \tag{2.23}$$

where we have used Lemma 2.5 (iv) and Lemma 2.6 (i). On the other hand, there is a positive constant a such that

$$q_k \le ak \qquad \text{for all} \quad k \in N. \tag{2.24}$$

We here prove this fact. Summing up all of the inequalities stated in Lemma 2.6 (i) and (ii), we obtain $p_{k+1} - p_k \le 4M$ for $k \in N$. Therefore it follows that $p_k \le 4(k-1)M + p_1$. This implies that $q_k \le p_k + M \le ak$ for $k \in N$, where $a = 4M + p_1$. Thus we get the inequality (2.24). Combining (2.22), (2.23) and (2.24), we have

$$E(q_k) - E(p_k) \le -\frac{C}{k}, \tag{2.25}$$

where C is a positive constant independent of $k \in N$. Summing up (2.25) from $k=1$ to $k=j$, we get

$$\sum_{k=1}^{j}(E(q_k)-E(p_k)) \leq -C\sum_{k=1}^{j}\frac{1}{k}. \tag{2.26}$$

Since $E(t)$ is strictly decreasing, it follows that $E(q_{k-1}) > E(p_k)$ for all $k \geq 2$. Using this inequality, we estimate the left-hand side of (2.26) as

$$\begin{aligned}&\sum_{k=1}^{j}(E(q_k)-E(p_k)) \\ &\quad = E(q_j) + \sum_{k=2}^{j}(E(q_{k-1})-E(p_k)) - E(p_1) \\ &\quad \geq E(q_j) - E(p_1).\end{aligned} \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$E(q_j) - E(p_1) \leq -C\sum_{k=1}^{j}\frac{1}{k} \qquad \text{for} \quad j \in N.$$

This inequality implies that $\lim_{j\to\infty} E(q_j) = -\infty$. This contradicts the fact that $E(t) > 0$ for all t. Consequently, we see that $E_\infty = \lim_{t\to\infty} E(t) = 0$. Noting that $F(s) > 0$ for $s \neq 0$ and $F(0) = 0$, we conclude that $\lim_{t\to\infty} u(t) = 0$ and $\lim_{t\to\infty} u'(t) = 0$. Therefore it follows that

$$u'' = -\frac{n-1}{t}u' - f(u) \longrightarrow 0 \qquad \text{as} \quad t \to \infty.$$

Thus we obtain (2.1).

Conversely, we suppose that $u \equiv 0$ is a solution of (1.3) and is globally asymptotically stable. We aim to prove that condition (f1) holds. If $f(s_0) = 0$ for some $s_0 \neq 0$, then $u(t) \equiv s_0$ is a solution of (1.3) but it does not satisfy (2.1), which contradicts the global asymptotic stability of the zero solution. This means that $f(s) \neq 0$ for $s \neq 0$. Therefore we have to check the following four cases:

(A) $\qquad f(s) > 0 \quad \text{for} \quad s > 0,$

(B) $\qquad f(s) < 0 \quad \text{for} \quad s > 0,$

(C) $\qquad f(s) > 0 \quad \text{for} \quad s < 0$

and

(D) $\qquad f(s) < 0 \quad \text{for} \quad s < 0.$

We show that the cases (B) and (C) do not occur. Suppose that the case (B) holds. We consider the solution $u(t)$ of (1.3) satisfying the initial condition

$u'(0) = 0$ and $u(0) = 1$. Then in the case (B) we have that $(t^{n-1}u')' = -t^{n-1}f(u) > 0$, and so it follows that $t^{n-1}u'(t) > 0$ and $u'(t) > 0$. We conclude that $u(t) \geq u(0) = 1$ for all $t \geq 0$. This contradicts (2.1). Consequently, the case (B) is impossible. In the same way, we see that the case (C) does not occur. Now the cases (A) and (D) both assert that (f1) holds. The proof of Theorem 1 is now complete.

To prove Theorem 2, we need the next lemma.

LEMMA 2.7. *Let $n = 2$ and* (f1) *hold. Suppose that there exists a $\lambda > 0$ such that*

$$\text{(i)} \quad \int_0^\infty e^{\lambda s} f(s)\, ds < +\infty \qquad \textit{or} \quad \text{(ii)} \quad \int_{-\infty}^0 e^{\lambda|s|}|f(s)|\, ds < +\infty.$$

Then there exist $T > 0$ and a solution $u(t)$ of (1.3) *such that $u(t)$ is defined on* $[T, \infty)$ *and* $\lim_{t\to\infty}|u(t)| = +\infty$.

PROOF. We first consider the case (i). We set $a = 4/\lambda$ to obtain

$$\int_0^\infty \exp(4s/a) f(s)\, ds < +\infty.$$

Next, we choose a number $T\,(>0)$ so large that

$$e^{-2T}\int_{aT}^\infty \exp(4s/a) f(s)\, ds < \frac{a^2}{4}. \tag{2.28}$$

For these numbers a and T, we denote by $y(s)$ the solution of the initial value problem

$$y'' + e^{2s} f(as + y) = 0, \qquad s > T, \tag{2.29}$$

$$y'(T) = 0, \qquad y(T) = 0. \tag{2.30}$$

Then it is seen that the solution $y(s)$ is defined on $[T, \infty)$ and satisfies that $-a/2 < y'(s) < 0$ for all $s > T$. We here prove this assertion. To this end, it is sufficient to show that

$$-\frac{a}{2} < y'(s) < 0 \qquad \text{for} \quad s \in (T, T^*), \tag{2.31}$$

where T^* is the right end point of the maximal interval of existence for the solution $y(s)$. Indeed, once the inequality (2.31) is proved, then we obtain from (2.30) and (2.31) that

$$-\frac{a}{2}(s - T) < y(s) < 0 \qquad \text{for} \quad s \in (T, T^*). \tag{2.32}$$

It follows from (2.31) and (2.32) that $T^* = +\infty$, and so $y(s)$ is defined on $[T, \infty)$ and satisfies (2.31) and (2.32) for all $s \in (T, \infty)$. We then show the inequality (2.31). Since $y''(T) = -e^{2T} f(aT) < 0$, it follows from (2.30) that (2.31) holds for $s \in (T, T+\varepsilon)$ with $\varepsilon > 0$ sufficiently small. We now suppose that (2.31) does not hold for some $s \in (T, T^*)$. Then we define

$$T_1 = \sup\left\{t \colon -\frac{a}{2} < y'(s) < 0 \quad \text{for} \quad s \in (T, t)\right\}.$$

This definition implies that

$$-\frac{a}{2} < y'(s) < 0 \quad \text{for} \quad s \in (T, T_1), \tag{2.33}$$

and that one of the following two cases,

$$\text{(A)} \quad y'(T_1) = 0 \quad \text{or} \quad \text{(B)} \quad y'(T_1) = -\frac{a}{2}$$

must be satisfied. Integrating (2.33) over $[T, s]$, we obtain

$$-\frac{a}{2}(s - T) < y(s) < 0 \quad \text{for} \quad s \in (T, T_1). \tag{2.34}$$

It follows from equation (2.29) that

$$y'(T_1) = -\int_T^{T_1} e^{2s} f(as + y(s))\, ds. \tag{2.35}$$

This implies that $y'(T_1) < 0$ since $as + y(s) > 0$ by (2.34). Thus the case (A) is impossible. Next, we set $v(s) = as + y(s)$. It follows from (2.33) and (2.34) that

$$v(s) \geq \frac{a}{2}(s + T) \tag{2.36}$$

and

$$v'(s) \geq \frac{a}{2} \quad \text{for} \quad s \in [T, T_1]. \tag{2.37}$$

Since $v'(s) > 0$ on $[T, T_1]$ and $v(\cdot)$ maps the interval $[T, T_1]$ onto $[aT, v(T_1)]$, the inverse functions $s = v^{-1}(\tau) = w(\tau)$ is well defined on the interval $[aT, v(T_1)]$. Combining (2.36) and (2.37), we obtain

$$w(\tau) \leq \frac{2}{a}\tau - T \tag{2.38}$$

and

$$(2.39) \qquad w'(\tau) = \frac{1}{v'(s)} \le \frac{2}{a} \qquad \text{for} \quad \tau \in [aT, v(T_1)].$$

Using the change of variables $\tau = v(s) = as + y(s)$, $s = w(\tau)$, and applying (2.38) and (2.39), we have

$$(2.40) \qquad \int_T^{T_1} e^{2s} f(as + y(s))\, ds = \int_{aT}^{v(T_1)} \exp(2w(\tau)) f(\tau) w'(\tau)\, d\tau$$

$$\le \frac{2}{a} e^{-2T} \int_{aT}^{\infty} \exp(4\tau/a) f(\tau)\, d\tau < \frac{a}{2},$$

where we have used (2.28) in the last inequality. The identity (2.35) together with (2.40) implies that $y'(T_1) > -a/2$. Therefore the case (B) does not occur, too. Hence we conclude that $y(s)$ is defined on $[T, \infty)$ and the inequalities (2.31) and (2.32) hold for all $s > T$. We set $s = \log t$ and $u(t) = as + y(s)$. Then the function $u(t)$ is a solution on $[e^T, \infty)$ of equation (1.3) with $n = 2$. Furthermore it follows from (2.32) that

$$u(t) \ge \frac{a}{2} \log t + \frac{a}{2} T \qquad \text{for} \quad t > e^T,$$

which implies that $\lim_{t\to\infty} u(t) = +\infty$. This completes the proof under condition (i). In the case where (ii) holds, one finds a solution $u(t)$ satisfying $\lim_{t\to\infty} u(t) = -\infty$ in the same way. The proof is thereby complete.

We conclude this section by proving Theorem 2.

Proof of Theorem 2. (i) $\Rightarrow$ (ii). We first note that all of Lemmas 2.1 through 2.6 are valid. Then under condition (i), the assertion (ii) follows from the same argument as in the proof of Theorem 1.

(ii) $\Rightarrow$ (i). Assume condition (ii). In the proof of Theorem 1 we have already seen that (f1) holds. Next, using Lemma 2.7, we obtain (2.2). Hence (i) is valid.

(i), (ii) $\Rightarrow$ (iii). Suppose that (i) and (ii) hold. Then by Main Theorem (II) we get assertion (iii).

(iii) $\Rightarrow$ (i). We assume condition (iii). If $f(s_0) = 0$ for some $s_0 \ne 0$, then $u(t) \equiv s_0$ is a solution of (1.3) which is nonoscillatory. Therefore we see that $f(s) \ne 0$ for $s \ne 0$ and we have to check the four cases (A) through (D) stated in the proof of Theoem 1. As shown in the proof of Theorem 1, we obtain a solution $u(t)$ satisfying $u(t) \ge 1$ for all $t \ge 0$ under condition (B). This is nonoscillatory. Thus the case (B) is impossible. The same argument implies that the case (C) does not occur, too. The two cases (A) and (D) are now remained, and we see that (f1) holds. We then prove (2.2). To the contrary, suppose it is false. Then by Lemma 2.7 there exists a solution $u(t)$ such that $\lim_{t\to\infty}|u(t)| = +\infty$, and so $u(t)$ is nonoscillatory. This contradicts the assumption (iii). It turns out that (2.2) is valid and we obtain the desired assertion (i). This completes the proof of Theorem 2.

3. Nonoscillation of solutions

The purpose of this section is to investigate nonoscillation of solutions. We give a sufficient condition for nonoscillation of all solution to equation (1.3).

THEOREM 3. *Let $n \geq 3$. Suppose that conditions* (f1), (f2) *and* (f3) *hold. Then all nontrivial solutions of* (1.3) *can be extended to $t = +\infty$ and are nonoscillatory.*

REMARK 3.1. We state some remarks and references related to nonoscillation of solutions of equation (1.3). For $n \geq 3$ we employ the following standard Liouville transformation (see [11]):

$$y(s) = su(t), \qquad s = t^{n-2},$$

which reduces (1.3) to the equation

$$y'' + as^{-\sigma} f(s^{-1}y) = 0, \quad s > 0, \tag{3.1}$$

where $a = 1/(n-2)^2$ and $\sigma = (n-4)/(n-2)$. If $f(s)$ is odd (hence $f(0) = 0$) and is differentiable at $s = 0$, then we set

$$g(r, s) = as^{-\sigma} f(s^{-1}r^{1/2})/r^{1/2}, \qquad r \geq 0,$$

which is continuous on $(r, s) \in [0, \infty) \times (0, \infty)$. Therefore equation (3.1) is rewritten as the second order equation

$$y'' + yg(y^2, s) = 0, \qquad s > 0. \tag{3.2}$$

Under the monotonicity assumptions on the functions $g(r, s)$ or $rg(r^2, s)$ with respect to r, Nehari [8], Coffman and Wong [3] and many other authors gave sufficient conditions for nonoscillation of solutions of equation (3.2). For the details, we refer the readers to [3], [8] and [11]. These results are not applicable to equation (1.3) unless $f(s)$ is odd. But our result, Theorem 3, is valid without the hypotheses that $f(s)$ is odd and monotone. Theorem 3 is applicable to the following type of $f(s)$, for example

(i) $f(s) \equiv |s|^{q-1}s$,

(ii) $f(s) = |s|^{p-1}s \quad (s \geq 0), \qquad = |s|^{q-1}s \quad (s < 0)$

and

(iii) $f(s) = se^{-1/|s|} \quad (s \geq 0), \qquad = |s|^{q-1}s \quad (s < 0)$,

where p, $q > (n+2)/(n-2)$.

PROOF OF THEOREM 3. We first choose numbers α and r_0 ($< r$) such that

$$\frac{n-2}{2} > \alpha > \frac{2}{q-1} \quad \text{and} \quad \frac{sf(s)}{F(s)} \geq \frac{2\alpha+2}{\alpha} > \frac{2n}{n-2} \tag{3.3}$$

for $0 < |s| \leq r_0$. Let $u(t)$ be any nontrivial solution of (1.3). Then it follows from Theorem 1 that $u(t)$ can be extended to $t = +\infty$ and (2.1) holds. Let $[T_0, \infty)$ be the interval of existence for $u(t)$. We now apply the change of variables

$$u(t) = e^{-\alpha s}w(s) \quad \text{and} \quad t = e^s. \tag{3.4}$$

Equation (1.3) is then reduced to the following second order equation,

$$w'' + aw' - bw + e^{(\alpha+2)s}f(e^{-\alpha s}w) = 0, \quad s > T_1, \tag{3.5}$$

where $a = n - 2 - 2\alpha$ (> 0), $b = \alpha(n-2-\alpha)$ (> 0) and $T_1 = \log T_0$. Now, we suppose that $u(t)$ is oscillatory; hence so is $w(s)$. In this case we obtain the next lemma.

LEMMA 3.1. *If $w(s)$ is oscillatory, then*

$$\lim_{s\to\infty} w(s) = 0.$$

Lemma 3.1 will be proved later on. This lemma and condition (f2) together imply that

$$0 \leq e^{(\alpha+2)s}f(e^{-\alpha s}w(s))/w(s) \leq Ce^{-\beta s}|w(s)|^{q-1} \longrightarrow 0$$

as $s \to \infty$, where C is a positive constnat and $\beta = \alpha(q-1) - 2$ (> 0). Hence we can choose a constant T^* ($> T_1$) such that

$$-b + e^{(\alpha+2)s}f(e^{-\alpha s}w(s))/w(s) \leq -\frac{b}{2} \quad \text{for} \quad s \geq T^*. \tag{3.6}$$

We rewrite equation (3.5) as the Sturm-Liouville equation

$$(p(s)w')' + \{-b + e^{(\alpha+2)s}f(e^{-\alpha s}w)/w\}\ p(s)w = 0, \tag{3.7}$$

where $p(s) = e^{as}$. Noting (3.6) and applying Sturm's comparison theorem (see

[4, p335]) to equation (3.7), we deduce that all solutions of the equation

$$(p(s)y')' - \frac{b}{2}p(s)y = 0, \quad s \geq T^*, \tag{3.8}$$

must be oscillatory since $w(s)$ is oscillatory. Equation (3.8) can be reduced to

$$y'' + ay' - \frac{b}{2}y = 0. \tag{3.9}$$

This equation has a general solution of the form,

$$y(t) = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t), \quad C_1, C_2 \in \boldsymbol{R}, \tag{3.10}$$

where $\lambda = \lambda_1, \lambda_2$ are the roots of the characteristic equation

$$\lambda^2 + a\lambda - \frac{b}{2} = 0.$$

Note that λ_1 and λ_2 are real roots. Therefore $y(t)$ represented by (3.10) is nonoscillatory. This is a contradiction. Consequently, we obtain that $w(s)$ is nonoscillatory, and so is $u(t)$. This proves Theorem 3.

To complete the proof of Theorem 3, it suffices to prove Lemma 3.1.

PROOF OF LEMMA 3.1. Using (2.1) and (3.4), we can choose a number T($> T_1$) so large that

$$|e^{-\alpha s}w(s)| \leq r_0 \quad \text{for} \quad s \geq T. \tag{3.11}$$

Now, we define a Lyapunov-like function V by

$$V(s) \equiv \frac{1}{2}w'(s)^2 - \frac{b}{2}w(s)^2 + e^{(2\alpha+2)s}F(e^{-\alpha s}w(s)) \tag{3.12}$$

for $s \geq T$. We wish to show that $V(s)$ is decreasing. To this end, we multiply (3.5) by $w'(s)$. Then we obtain

$$\frac{d}{ds}\left\{\frac{1}{2}w'(s)^2 - \frac{b}{2}w(s)^2\right\} + aw'(s)^2 + e^{(\alpha+2)s}f(e^{-\alpha s}w)w' = 0.$$

The last term on the left-hand side is written as

$$e^{(\alpha+2)s}f(e^{-\alpha s}w)w' = \frac{d}{ds}\{e^{(2\alpha+2)s}F(e^{-\alpha s}w(s))\} + \zeta(s),$$

where

$$\zeta(s) \equiv \alpha e^{(\alpha+2)s}f(e^{-\alpha s}w)w - (2\alpha+2)e^{(2\alpha+2)s}F(e^{-\alpha s}w).$$

Therefore we have

$$V'(s) + aw'(s)^2 + \zeta(s) = 0.$$

Using the inequalities (3.3) and (3.11), we see that

$$\zeta(s) = \{\alpha uf(u) - (2\alpha + 2)F(u)\}e^{(2\alpha+2)s} \geq 0 \qquad \text{for} \quad s \geq T,$$

where $u = e^{-\alpha s}w(s)$. Hence it follows that

(3.13) $$V'(s) + aw'(s)^2 \leq 0 \qquad \text{for} \quad s \geq T.$$

This means that $V(s)$ is decreasing. Integrating both sides of (3.13) over $[T, s]$, we obtain

(3.14) $$V(s) + a\int_T^s w'(\tau)^2 d\tau \leq V(T) \qquad \text{for} \quad s \geq T.$$

Let $\{s_k\}_{k=1}^{\infty}$ $(T < s_1 < s_2 < \cdots \uparrow +\infty)$ denote the sequence of the zeros of $w(s)$. From the uniqueness of the zero solution to equation (3.5) it follows that $\{s_k\}_{k=1}^{\infty}$ are simple zeros, that is, $w'(s_k) \neq 0$ and $w(s_k) = 0$. This implies that $V(s_k) = (1/2)w'(s_k)^2 > 0$ for $k \in \boldsymbol{N}$. Since V is decreasing, we get

(3.15) $$V(s) > 0 \qquad \text{for all} \quad s \geq T,$$

It follows from (3.14) and (3.15) that

(3.16) $$\int_T^{\infty} w'(\tau)^2 d\tau \leq \frac{1}{a}V(T) < +\infty.$$

We wish to prove that $\lim_{s\to\infty} w(s) = 0$. To the contrary, we suppose that

(3.17) $$\limsup_{s\to\infty} w(s) > \varepsilon \qquad \text{for some} \quad \varepsilon > 0.$$

Since $w(s)$ is oscillatory, we can find two sequences $\{p_k\}_{k=1}^{\infty}$ and $\{q_k\}_{k=1}^{\infty}$ which have the following properties,

(3.18) $$T < p_1 < q_1 < p_2 < q_2 < \cdots,$$

(3.19) $$\lim_{k\to\infty} p_k = \lim_{k\to\infty} q_k = +\infty,$$

(3.20) $$w(p_k) = \frac{\varepsilon}{2}, \quad w(q_k) = \varepsilon \qquad \text{for} \quad k \in \boldsymbol{N},$$

(3.21) $$\frac{\varepsilon}{2} < w(s) < \varepsilon \qquad \text{for} \quad s \in (p_k, q_k) \text{ and } k \in \boldsymbol{N}.$$

Moreover there exist positive constants C_1 and C_2 depending upon ε such that

(3.22) $$C_1 \leq w'(s) \leq C_2 \qquad \text{for} \quad s \in (p_k, q_k) \text{ and } k \in \boldsymbol{N}.$$

We here prove (3.22). Since $F(s) \geq 0$ for $s \in \boldsymbol{R}$, we have

$$V(T) \geq V(s) \geq \frac{1}{2} w'(s)^2 - \frac{b}{2} w(s)^2 \qquad \text{for} \quad s \geq T,$$

and therefore we get

$$w'(s)^2 \leq 2V(T) + b\varepsilon^2 \qquad \text{for} \quad s \in (p_k, q_k), \tag{3.23}$$

where we have used (3.21). On the other hand, it follows from (f2) that there exists $C > 0$ such that $F(s) \leq C|s|^{q+1}$ for $|s| \leq r_0$. Using this inequality, we obtain

$$\begin{aligned} 0 < V(s) &\leq \frac{1}{2} w'(s)^2 - \frac{b}{2} w(s)^2 + Ce^{-\beta s}|w|^{q+1} \\ &\leq \frac{1}{2} w'(s)^2 - \left(\frac{b}{2} - C\varepsilon^{q-1}\right) w(s)^2 \\ &\leq \frac{1}{2} w'(s)^2 - \left(\frac{b}{2} - C\varepsilon^{q-1}\right) \frac{\varepsilon^2}{4}, \end{aligned} \tag{3.24}$$

for $s \in (p_k, q_k)$, where $\beta = \alpha(q-1) - 2 > 0$. We may take a smaller positive number ε if necessary. Therefore we may assume that ε satisfies the inequality

$$\frac{b}{2} - C\varepsilon^{q-1} \geq \frac{b}{4}. \tag{3.25}$$

From the inequalities (3.24) and (3.25) we deduce

$$\frac{b}{16} \varepsilon^2 \leq \frac{1}{2} w'(s)^2 \qquad \text{for} \quad s \in (p_k, q_k). \tag{3.26}$$

By (3.23) and (3.26) there are positive constants A and B which are independent of k and such that $A \leq w'(s)^2 \leq B$ for $s \in (p_k, q_k)$ and $k \in \boldsymbol{N}$. This together with the property (3.20) implies that $w'(s) > 0$ for $s \in (p_k, q_k)$, and so (3.22) follows. We now apply the mean-value theorem to find a number $\xi_k \in (p_k, q_k)$ which satisfies

$$w'(\xi_k) = \frac{w(q_k) - w(p_k)}{q_k - p_k} = \frac{\varepsilon}{2(q_k - p_k)}.$$

Combining this identity and (3.22), we get positive constants C_3 and C_4 such that

$$C_3 \leq q_k - p_k \leq C_4 \qquad \text{for} \quad k \in \boldsymbol{N}. \tag{3.27}$$

By virtue of (3.22) and (3.27) we obtain

$$\int_T^\infty w'(s)^2\,ds \ge \sum_{k=1}^j \int_{p_k}^{q_k} w'(s)^2\,ds \ge C_1^2 C_3 j \qquad \text{for} \quad j \in \boldsymbol{N}.$$

Passing to the limit as $j \to \infty$, we have

$$\int_T^\infty w'(s)^2\,ds = +\infty,$$

which contradicts (3.16). This means that (3.17) is false, and so $\limsup_{s\to\infty} w(s) \le 0$. In the same way, it follows that $\liminf_{s\to\infty} w(s) \ge 0$. Thus we conclude that $\lim_{s\to\infty} w(s) = 0$. This completes the proof of Lemma 3.1.

4. Proof of Main Theorem

In this section we prove Main Theorem (I) stated in Section 1. Our approach is based on the so-called shooting method. To proceed this argument, we consider the initial value problem

$$u'' + \frac{n-1}{t}u' + f(u) = 0, \qquad t \in (0, \infty), \tag{4.1}$$

$$u'(0) = 0, \qquad u(0) = \gamma. \tag{4.2}$$

This problem has a local and unique solution since $f(s)$ is locally Lipschitz continuous. Theorem 1 asserts that the solution can be extended to $t = +\infty$ and satisfies (2.1) if $n \ge 3$ and if (f1) is satisfied. Throughout this section we suppose that $n \ge 3$ and (f1) holds. We begin by introducing some notation which will be used in the subsequent discussions.

DEFINITION 1. For each $\gamma \in \boldsymbol{R}$ we denote by $u(t, \gamma)$ the solution of the initial value problem (4.1)–(4.2).

DEFINITION 2. For a continuous function $u(t)$ on $[0, \infty)$ and each $R > 0$, $N_R[u]$ and $N[u]$ denote the numbers of zeros of $u(t)$ in the intervals $[0, R]$ and $[0, \infty)$, respectively, that is,

$$N_R[u] = \#\{t \in [0, R] : u(t) = 0\} \quad (\le +\infty)$$

and

$$N[u] = \#\{t \in [0, \infty) : u(t) = 0\} \quad (\le +\infty).$$

In Theorem 3 we have already proved that $N[u(\cdot, \gamma)] < +\infty$ for all $\gamma \ne 0$ provided that $n \ge 3$ and (f1), (f2) and (f3) are satisfied. The next lemma plays an important role in the proof of the key lemma, i.e., Lemma 4.2 below.

LEMMA 4.1. *Let* $n \geq 3$ *and* (f1) *hold. Suppose that* $\{\gamma_j\}_{j=1}^{\infty}$ *converges to* γ_0. *Then for any* $\varepsilon > 0$ *there exists a* $T^* > 0$ *such that*

$$|u(t, \gamma_j)| < \varepsilon \quad \text{for} \quad t \geq T^* \text{ and } j \geq 0. \tag{4.3}$$

PROOF. For simplicity we write

$$u_j(t) = u(t, \gamma_j) \qquad \text{for} \quad j \geq 0.$$

Let $\varepsilon > 0$. First, we choose a positive number $\delta > 0$ that is smaller than both $F(\varepsilon)$ and $F(-\varepsilon)$. We define

$$E_j(t) \equiv \frac{1}{2} u_j'(t)^2 + F(u_j(t)) \qquad \text{for} \quad j \geq 0.$$

Then it follows from (2.5) that each $E_j(t)$ is decreasing. Since $u_0(t)$ satisfies (2.1), one can find a $T > 0$ such that $E_0(T) < \delta/2$. We also note that solutions to (4.1)–(4.2) depend continuously on the initial value γ. More precisely, the mapping $\gamma \to u(\cdot, \gamma)$ from $\boldsymbol{R}$ into $C_{loc}^2[0, \infty)$ is continuous. This implies that $\{E_j(T)\}_{j=1}^{\infty}$ converges to $E_0(T)$. Therefore there exists a positive integer j_0 such that

$$E_j(T) < E_0(T) + \frac{\delta}{2} < \delta \qquad \text{for} \quad j > j_0.$$

From this we obtain

$$F(u_j(t)) \leq E_j(t) \leq E_j(T) < \delta < \min(F(\varepsilon), F(-\varepsilon))$$

for $t \geq T$ and $j > j_0$. This together with (f1) implies that

$$|u_j(t)| < \varepsilon \qquad \text{for} \quad t \geq T \text{ and } j > j_0.$$

By (2.1) there exist constants T_j $(0 \leq j \leq j_0)$ such that

$$|u_j(t)| < \varepsilon \qquad \text{for} \quad t \in [T_j, \infty) \text{ and } 0 \leq j \leq j_0.$$

Consequently we set

$$T^* = \max(T_0, T_1, \dots, T_{j_0}, T)$$

to obtain the desired inequality (4.3). This completes the proof.

To prove Main Theorem (I) we need the next key lemma, which has been proved in [7] under condition (1.6). We establish the result in the case where condition (1.6) does not hold.

LEMMA 4.2. *Let* $n \geq 3$. *Suppose that conditions* (f1), (f2) *and* (f3) *hold. If* $\gamma_0 \neq 0$ *and* $N[u(\cdot, \gamma_0)] = k \geq 0$, *then there exists an* $\varepsilon > 0$ *such that* $N[u(\cdot, \gamma)]$ *is* *equal to k or* $k+1$ *for each* γ *satisfying* $|\gamma-\gamma_0|<\varepsilon$.

PROOF. Let $\{t_i\}_{i=1}^k$ $(0<t_1<t_2<\cdots<t_k)$ denote the sequence of the zeros of $u(t,\gamma_0)$. We choose a number T so that $T>t_k$. Since $u(t,\gamma)$ converges to $u(\cdot,\gamma_0)$ in $C^2[0,T]$ as γ tends to γ_0, we have

$$N[u(\cdot,\gamma)]\geq N_T[u(\cdot,\gamma)]=N_T[u(\cdot,\gamma_0)]=k$$

provided that γ is close to γ_0. In order to give the proof of Lemma 4.2, it is sufficient to show that $N[u(\cdot,\gamma)]\leq k+1$ when $|\gamma-\gamma_0|$ is sufficiently small. Suppose to the contrary that it does not hold. Then there is a sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\{\gamma_j\}$ converges to γ_0 and $N[u(\cdot,\gamma_j)]\geq k+2$ for all $j\in \boldsymbol{N}$. For simplicity in notation we set

$$u_j(t)=u(t,\gamma_j)\qquad \text{for}\quad j\geq 0.$$

It should be noted that for $j\geq 1$, $u_j(t)$ has at least $k+2$ zeros in $[0,\infty)$ but $u_0(t)$ has exactly k zeros in $[0,\infty)$ and $\{u_j\}$ converges to $u_0(t)$ in $C^2_{loc}[0,\infty)$. First we denote the smallest $k+2$ zeros of $u_j(t)$ for $j\geq 1$ by

$$0<t_{1j}<t_{2j}<\cdots<t_{kj}<a_j<b_j.$$

Fix any $T(>t_k)$. Then the convergence of $\{u_j\}$ to u_0 in $C^2[0,T]$ implies that $\{t_{ij}\}$ converges to t_i as j tends to $+\infty$ for $1\leq i\leq k$. Since $N_T[u_j]=k$ for sufficiently large j, we obtain that $T<a_j$ for j large. This means that $\lim_{j\to\infty}a_j=+\infty$. Next, we define the functions $\{w_j\}_{j=0}^\infty$ by

$$w_j(s)=e^{\alpha s}u_j(t),\ t=e^s,\ \ j=0,1,2,\dots,$$

where α is a postive constant chosen as in (3.3). Since $u_j(t)$ is a solution of (4.1), the function $w_j(s)$ satisfies equation (3.5) for all $s\in(-\infty,+\infty)$. We set

$$p_j=\log a_j\quad\text{and}\quad q_j=\log b_j\qquad\text{for}\quad j\in\boldsymbol{N}.$$

Then p_j and q_j are the $(k+1)^{\text{th}}$ and the $(k+2)^{\text{th}}$ zeros of $w_j(s)$, respectively. Since $\{a_j\}$ tends to $+\infty$ as mentioned above, we obtain

$$\lim_{j\to\infty}p_j=\lim_{j\to\infty}q_j=+\infty. \tag{4.4}$$

Employing the same idea as in (3.12), we define a function $V_j(s)$ for each $j\geq 0$ by

$$V_j(s)\equiv\frac{1}{2}w_j'(s)^2-\frac{b}{2}w_j(s)^2+e^{(2\alpha+2)s}F(e^{-\alpha s}w_j(s)), \tag{4.5}$$

where $b=\alpha(n-2-\alpha)$ (>0). Let r_0 be the positive constant chosen as in (3.3). By Lemma 4.1 there is a $T^*>0$ such that

$$|u_j(t)|<r_0\qquad\text{for}\quad t\geq T^*\ \text{and}\ j\geq 0.$$

We set $T = \log T^*$ to obtain

$$|e^{-\alpha s} w_j(s)| \leq r_0 \quad \text{for} \quad s \geq T \text{ and } j \geq 0. \tag{4.6}$$

Using (4.6) and applying the same method as in the proof of Theorem 3, we see that $V_j(s)$ is decreasing in $[T, \infty)$ for each $j \geq 0$.

We now fix a constant $M > 0$ so large that

$$M^2 > \frac{1}{b}(V_0(T) + 1)\max(1, 2a^{-1}(6b)^{1/2}), \tag{4.7}$$

where $a = n - 2 - 2\alpha$ (> 0) and $b = \alpha(n - 2 - \alpha)$ (> 0). Since $\lim_{j \to \infty} p_j = +\infty$ by (4.4), there exists a positive integer j_0 such that $T < p_j$ for each $j \geq j_0$. We here present a technical lemma, which asserts that a sequence $\{w_j\}$ is uniformly bounded on $[p_j, q_j]$.

LEMMA 4.3. *There exists a positive integer j^* $(\geq j_0)$ such that*

$$\max_{p_j \leq s \leq q_j} |w_j(s)| \leq 2M \quad \text{for all} \quad j \geq j^*. \tag{4.8}$$

This lemma will be proved later on. From (4.8) and (f2) it follows that

$$\begin{aligned} 0 &\leq e^{(\alpha+2)s} f(e^{-\alpha s} w_j(s))/w_j(s) \leq Ce^{-\beta s}|w_j(s)|^{q-1} \\ &\leq C'\exp(-\beta p_j) \longrightarrow 0 \quad \text{for} \quad s \in [p_j, q_j], \end{aligned}$$

as j tends to $+\infty$. Here C and C' are positive constants independent of j, and $\beta = \alpha(q - 1) - 2 > 0$. This implies that there exists a positive integer m_0 such that

$$-b + e^{(\alpha+2)s} f(e^{-\alpha s} w_j(s))/w_j(s) \leq -\frac{b}{2} \tag{4.9}$$

for $s \in [p_j, q_j]$ and $j \geq m_0$. We here set $p(s) = e^{as}$ to reduce equation (3.5) to the following Sturm-Liouville equation

$$(p(s)w_j')' + \{-b + e^{(\alpha+2)s} f(e^{-\alpha s} w_j)/w_j\} p(s) w_j = 0. \tag{4.10}$$

Applying Sturm's comparison theorem to this equation and using (4.9), we see that all solutions of the equation

$$(p(s)y')' - \frac{b}{2} p(s) y = 0, \tag{4.11}$$

must possess at least one zero in the interval $[p_j, q_j]$ with $j \geq m_0$. But (4.11) can be rewritten as equation (3.9) which has a general solution of the form (3.10). In particular, if we choose a solution $y(t) = \exp(\lambda_1 t)$, then it has no zeros in $[p_j, q_j]$. This is a contradiction, which is caused by the assumption that $N[u(\cdot, \gamma_j)] \geq k+2$. Therefore we conclude that $N[u(\cdot, \gamma)] \leq k+1$ for all γ with $|\gamma - \gamma_0|$ sufficiently small. This proves Lemma 4.2.

In order to complete the proof of Lemma 4.2, we have to prove Lemma 4.3.

Proof of Lemma 4.3. Suppose that the assertion of Lemma 4.3 is false. Then there must exist a subsequence (again denoted $\{w_j\}_{j=1}^{\infty}$ for simplicity) of $\{w_j\}$ such that

$$\max_{p_j \leq s \leq q_j} |w_j(s)| > 2M \qquad \text{for all} \quad j \in N. \tag{4.12}$$

We here observe that there are only two cases, namely,

$$w_j(s) > 0 \qquad \text{for all} \quad s \in (p_j, q_j) \text{ and } j \in N \tag{4.13}$$

or else

$$w_j(s) < 0 \qquad \text{for all} \quad s \in (p_j, q_j) \text{ and } j \in N. \tag{4.14}$$

In fact, since p_j is the $(k+1)^{\text{th}}$ zero of w_j, the case (4.13) occurs if either $\gamma_0 > 0$ and k is odd, or $\gamma_0 < 0$ and k is even. Otherwise, we have only the case (4.14). We here deal with the case (4.13) only, since (4.14) can also be treated in the same argument as in (4.13). Combining (4.12) and (4.13), we obtain numbers ξ_j and ζ_j such that

$$p_j < \xi_j < \zeta_j < q_j, \quad w_j(\xi_j) = M, \quad w_j(\zeta_j) = 2M$$

and

$$M < w_j(s) < 2M \qquad \text{for} \quad s \in (\xi_j, \zeta_j).$$

Since $V_j(s)$ is decreasing in $[T, \infty)$ for each j, we have

$$V_j(s) \geq V_j(q_j) = \frac{1}{2} w_j'(q_j)^2 > 0 \qquad \text{for} \quad s \in [T, q_j]. \tag{4.15}$$

It follows from this inequality and condition (f2) that

$$\begin{aligned} 0 < V_j(s) &\leq \frac{1}{2} w_j'(s)^2 - \frac{b}{2} w_j(s)^2 + Ce^{-\beta s} |w_j|^{q+1} \\ &\leq \frac{1}{2} w_j'(s)^2 - \frac{b}{2} M^2 + C' \exp(-\beta p_j) M^{q+1} \end{aligned} \tag{4.16}$$

for $s \in [\xi_j, \zeta_j]$, where $\beta = \alpha(q-1) - 2 > 0$ and C, C' are positive constants independent of j. Since $\lim_{j \to \infty} p_j = \infty$ by (4.4), one finds a positive integer j_1 by (4.16) such that

$$\frac{b}{4}M^2 \leq \frac{1}{2}w_j'(s)^2 \quad \text{for} \quad s \in [\xi_j, \zeta_j] \text{ and } j \geq j_1. \tag{4.17}$$

Next, noting that $F(s) \geq 0$ for $s \in \boldsymbol{R}$, we have

$$V_j(T) \geq V_j(s) \geq \frac{1}{2}w_j'(s)^2 - \frac{b}{2}w_j(s)^2 \quad \text{for} \quad s \geq T. \tag{4.18}$$

Since $\{V_j\}$ converges to V_0 in $C^1_{loc}(-\infty, +\infty)$, there exists a positive integer j_2 $(\geq j_1)$ such that

$$V_j(T) \leq V_0(T) + 1 \quad \text{for} \quad j \geq j_2. \tag{4.19}$$

Hence it follows from (4.7), (4.18) and (4.19) that

$$\begin{aligned} \frac{1}{2}w_j'(s)^2 &\leq \frac{b}{2}w_j(s)^2 + V_j(T) \\ &\leq 2bM^2 + V_0(T) + 1 \leq 3bM^2 \end{aligned} \tag{4.20}$$

for $s \in [\xi_j, \zeta_j]$ and $j \geq j_2$. By (4.17) and (4.20) we obtain

$$\left(\frac{b}{2}\right)^{1/2} M \leq |w_j'(s)| \leq (6b)^{1/2} M \tag{4.21}$$

for $s \in [\xi_j, \zeta_j]$ and $j \geq j_2$. Applying the mean-value theorem, we find $\tau_j \in (\xi_j, \zeta_j)$ such that

$$w_j'(\tau_j) = \frac{w_j(\zeta_j) - w_j(\xi_j)}{\zeta_j - \xi_j} = \frac{M}{\zeta_j - \xi_j}.$$

This relation together with (4.21) implies that

$$\zeta_j - \xi_j \geq (6b)^{-1/2} \quad \text{for} \quad j \geq j_2. \tag{4.22}$$

Since $V_j(s)$ satisfies the inequality (3.13), we obtain

$$V_j(\zeta_j) + a\int_{\xi_j}^{\zeta_j} |w_j'|^2\, ds \leq V_j(\xi_j) \leq V_j(T) \leq V_0(T) + 1,$$

for $j \geq j_2$. On the other hand, using (4.15), (4.21) and (4.22), one can estimate the left-hand side as

$$V_j(\zeta_j) + a\int_{\xi_j}^{\zeta_j} |w_j'|^2\, ds \geq \frac{ab}{2}M^2(\zeta_j - \xi_j) \geq \frac{a}{2}M^2\left(\frac{b}{6}\right)^{1/2}.$$

Consequently, we have

$$\frac{a}{2}M^2\left(\frac{b}{6}\right)^{1/2} \leq V_0(T) + 1.$$

But this contradicts the choice of M in (4.7). Thus the inequality (4.8) must hold. The proof is thereby complete.

Finally, to end up the proof of Main Theorem (I), we need one more lemma which means that the solution $u(t, \gamma)$ has sufficiently many zeros for $|\gamma|$ large.

LEMMA 4.4. *Let $n \geq 3$. Suppose that conditions* (f4) *and* (f5) *hold. Then* $\lim_{\gamma \to \pm\infty} N[u(\cdot, \gamma)] = +\infty$.

PROOF. In [6, Proposition 6.1] we have already proved that $\lim_{\gamma \to \pm\infty} N_1[u(\cdot, \gamma)] = +\infty$. We note that $N[u(\cdot, \gamma)] \geq N_1[u(\cdot, \gamma)]$, and this proves the lemma.

We are now in a position to give the proof of Main Theorem (I). Our approach is basic to the method used in [7].

PROOF OF MAIN THEOREM (I). Let r_0 be the positive constant chosen in (3.3). Then it is proved that

$$N[u(\cdot, \gamma)] = 0 \qquad \text{for} \quad \gamma \in (0, r_0).$$

In fact, suppose that for some $\gamma \in (0, r_0)$, $u(t, \gamma)$ has a zero in $[0, \infty)$. In what follows, we write $u(t)$ for $u(t, \gamma)$ for simplicity. Let t_1 be the first zero of $u(t)$. Then we have

$$u(t_1) = 0 \quad \text{and} \quad u(t) > 0 \qquad \text{for} \quad t \in [0, t_1).$$

This implies

$$(t^{n-1}u')' = -t^{n-1}f(u) < 0 \quad \text{and} \quad t^{n-1}u'(t) < 0$$

for $t \in (0, t_1)$. Therefore we obtain

$$0 < u(t, \gamma) < \gamma < r_0 \qquad \text{for} \quad t \in (0, t_1). \tag{4.23}$$

On the other hand, Pohozaev's identity [9] gives

$$\int_0^{t_1} \{2nF(u) - (n-2)f(u)u\}t^{n-1}\,dt = u'(t_1)^2 t_1^n. \tag{4.24}$$

The relation (4.24) is obtained in the following way. First, multiplying (4.1) by $u(t)t^{n-1}$ and integrating both sides of the resultant identity over $[0, t_1]$, we obtain

$$\int_0^{t_1} |u'|^2 t^{n-1}\,dt = \int_0^{t_1} uf(u)t^{n-1}\,dt. \tag{4.25}$$

Secondly, multiplying (4.1) by $u'(t)t^n$ and integrating over $[0, t_1]$, we have

$$\frac{1}{2}u'(t_1)^2 t_1^n + \left(\frac{n}{2} - 1\right)\int_0^{t_1} |u'|^2 t^{n-1}\,dt - n\int_0^{t_1} F(u)t^{n-1}\,dt = 0. \tag{4.26}$$

Now, combining (4.25) and (4.26) yields the identity (4.24). Using (4.23) and (3.3), we see that the left-hand side of (4.24) is negative. This is a contradiction. Hence it follows that $u(t, \gamma)$ has no zeros in $[0, \infty)$ when $0 < \gamma < r_0$.

Now, we define

$$\gamma_0 = \sup\{\gamma > 0 \colon N[u(\cdot, \gamma)] = 0\}.$$

It follows from Lemma 4.4 that $0 < \gamma_0 < +\infty$. Lemma 4.2 then implies that $N[u(\cdot, \gamma_0)] = 0$, and moreover there exists an $\varepsilon_0 > 0$ such that

$$N[u(\cdot, \gamma)] = 1 \qquad \text{for} \quad \gamma \in (\gamma_0, \gamma_0 + \varepsilon_0). \tag{4.27}$$

Next, we set

$$\gamma_1 = \sup\{\gamma > 0 \colon N[u(\cdot, \gamma)] = 1\}.$$

It follows from (4.27) and Lemma 4.4 that $0 < \gamma_0 < \gamma_1 < \infty$. Lemma 4.2 then implies that $N[u(\cdot, \gamma_1)] = 1$, and there is an $\varepsilon_1 > 0$ such that

$$N[u(\cdot, \gamma)] = 2 \qquad \text{for} \quad \gamma \in (\gamma_1, \gamma_1 + \varepsilon_1).$$

Repeating this argument, we obtain the sequences $\{\gamma_k\}_{k=0}^{\infty}$ and $\{\varepsilon_k\}_{k=0}^{\infty}$ such that

$$0 < \gamma_0 < \gamma_1 < \gamma_2 < \cdots \uparrow +\infty,$$

$$\varepsilon_k > 0, \qquad N[u(\cdot, \gamma_k)] = k$$

and

$$N[u(\cdot, \gamma)] = k + 1 \qquad \text{for} \quad \gamma \in (\gamma_k, \gamma_k + \varepsilon_k).$$

Since $u(t, \gamma_k)$ satisfies (2.1) by Theorem 1, this is the desired solution. The proof is now complete.

Acknowledgment

The author would like to thank Professor S. Oharu for valuable comments.

References

[1] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations, II. Existence of infinitely many solutions, Arch. Rational Mech. Anal. **82** (1983), 347–375.

[2] M. S. Berger, On the existence and structure of stationary states for a nonlinear Klein-Gordon equation, J. Funct. Anal. **9** (1972), 249–261.

[3] C. V. Coffman and J. S. W. Wong, Oscillation and nonoscillation of solutions of generalized Emden-Fowler equations, Trans. Amer. Math. Soc. **167** (1972), 399–434.

[4] P. Hartman, Ordinary Differential Equations, second edition, Birkhäuser, Boston, 1982.

[5] C. Jones and T. Küpper, On the infinitely many solutions of a semilinear elliptic equation, SIAM J. Math. Anal. **17** (1986), 803–835.

[6] R. Kajikiya, Radially symmetric solutions of semilinear elliptic equations, existence and Sobolev estimates, Hiroshima Math. J. **21** (1991), 111–161.

[7] K. McLeod, W. C. Troy and F. B. Weissler, Radial solutions of $\Delta u + f(u) = 0$ with prescribed number of zeros, J. Differential Equations **83** (1990), 368–378.

[8] Z. Nehari, A nonlinear oscillation problem, J. Differential Equations **5** (1969), 452–460.

[9] S. I. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **5** (1965), 1408–1411.

[10] W. A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. **55** (1977), 149–162.

[11] J. S. W. Wong, On the generalized Emden-Fowler equations, SIAM Rev. **17** (1975), 339–360.

Department of Mathematics,
Faculty of Engineering,
Niigata University