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Tests for random-effects covariance structures in the growth curve model with covariates

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1. Introduction

Suppose that we obtain serial measurements for each of N individuals on each of p occasions, yielding an $N \times p$ data matrix of observations X. The growth curve model for the observation matrix X of Potthoff and Roy [8] can be written as

$$X = A\Xi B + U, \tag{1.1}$$

where A is an $N \times k$ design matrix across individuals, Ξ is a $k \times q$ matrix of unknown parameters, B is a $q \times p$ design matrix within individuals, and U is an $N \times p$ unobservable matrix of random errors. It is assumed that A and B have ranks k and q, respectively, and the rows of U are independently and identically distributed as $N_p(\mathbf{0}, \Sigma)$, where Σ is an unknown $p \times p$ positive definite matrix. For an extensive survey of the literature on the model (1.1), see, e.g., Timm [11], Geisser [4] and Woolson [12]. In the model (1.1), suppose that we can use the observations of r covariates for the N individuals. Let Z be the $N \times r$ observation matrix of r covariates. Then the model (1.1) can be extended as

$$X = A\Xi B + Z\Theta + U, \tag{1.2}$$

where Θ is an $r \times p$ matrix of unknown parameters. It is assumed that Z is fixed and $\operatorname{rank}[A, Z] = k + r \leq N - p$. This type of models has been considered in Chinchilli and Elswick [3].

When there is no theoretical or empirical basis for assuming special covariance structures, we need to assume that Σ is an arbitrary positive definite covariance matrix. However, when p is large relative to N, more parsimonious covariance structures are required. Rao [9], [10] introduced a natural candidate for such parsimonious covariance structures, based on random-effects models. As a generalization of his idea we consider a family of covariance structures (see Lange and Laird [7])

$$\Sigma = B_c' \Delta_c B_c + \sigma_c^2 I_p, \qquad 0 \leq c \leq q, \tag{1.3}$$

where Δ_c is an arbitrary positive semi-definite matrix, $\sigma_c^2 > 0$, B_c is the matrix which is composed of the first c rows of B, and I_p is the identity matrix of order p. Without loss of generality, we assume that $BB' = I_q$.

In this paper we consider to test the hypothesis

$$H_{0c}\colon \Sigma = B_c'\Delta_c B_c + \sigma_c^2 I_p \tag{1.4}$$

against alternatives $H_{1c} \neq H_{0c}$ under the model (1.2). In Section 2 we obtain a canonical reduction. It is shown that the problem of obtaining the likelihood ratio (=LR) test under (1.2) can be reduced to the one of obtaining the LR test under (1.1). In Section 3 we obtain the LR test for H_{00} and its asymptotic expansion. The LR test for H_{0c} $(c \geq 1)$ is examined in Section 4. However, since the exact LR test is very complicated, it is suggested to use the LR test for a modified hypothesis.

2. A canonical reduction

Let $B = [B_c', B_{\bar{c}}']'$ and $\bar{B}$ be a $(p-q) \times p$ matrix such that $\bar{B}\bar{B}' = I_{p-q}$ and $B\bar{B}' = O$, i.e.

$$Q = \begin{bmatrix} B_c \\ B_{\bar{c}} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \tag{2.1}$$

is an orthogonal matrix of order p. Further, let $H = [H_1, H_2]$ be an orthogonal matrix such that H_1 is an orthonormal basis matrix on the space spanned by the column vectors of Z. Consider the transformation from X to

$$\begin{pmatrix} \tilde{Y} \\ Y \end{pmatrix} = \begin{pmatrix} H_1' \\ H_2' \end{pmatrix} XQ'. \tag{2.2}$$

Then, the rows of $\tilde{Y}$ and Y are independently distributed, each with a p-variate normal having covariance matrix

$$\Psi = Q\Sigma Q' = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} \end{bmatrix} \tag{2.3}$$

and means

$$E\begin{pmatrix} \tilde{Y} \\ Y \end{pmatrix} = \begin{pmatrix} \mu & \\ \tilde{A}\Xi & O \end{pmatrix}, \tag{2.4}$$

where $\mu = H_1'A[\Xi, O] + H_1'Z\Theta Q'$ and $\tilde{A} = H_2'A$. We can express the hypothesis H_{0c} as

$$H_{0c}\colon \Psi_{11} = \Delta_c + \sigma_c^2 I_c\,, \qquad \Psi_{1(23)} = O\,, \qquad \text{and} \qquad \Psi_{(23)(23)} = \sigma_c^2 I_{p-c}\,, \tag{2.5}$$

where

$$\Psi_{1(23)} = [\Psi_{12}, \Psi_{13}], \qquad \Psi_{(23)(23)} = \begin{pmatrix} \Psi_{22} & \Psi_{23} \\ \Psi_{32} & \Psi_{33} \end{pmatrix}. \tag{2.6}$$

Since the elements of μ are free parameters, it can be easily seen that the LR statistic for H_{0c} is equal to the LR statistic formed by considering only the density of

$$Y = H_2' X Q' = [Y_1, Y_2, Y_3] = [Y_{(12)}, Y_3].$$

The model for $Y\colon n \times p$ is

$$Y \sim N_{n\times p}([\tilde{A}\Xi, O], \Psi \otimes I_n), \tag{2.7}$$

where $n = N - r$. Let $L(\Xi, \Psi)$ be the likelihood function of Y. The maximum of $L(\Xi, \Psi)$ when Ξ and Ψ are unrestricted was first obtained by Khatri [5] and can be written as

$$\max L(\Xi, \Psi) = (2\pi)^{-pn/2} \left| \frac{1}{n} S_{(12)(12)\cdot 3} \right|^{-n/2} \times \left| \frac{1}{n} Y_3' Y_3 \right|^{-n/2} \exp\left(-\frac{1}{2} np \right), \tag{2.8}$$

where $S_{(12)(12)\cdot 3} = S_{(12)(12)} - S_{(12)3} S_{33}^{-1} S_{3(12)}$ and

$$S = Y'(I_n - \tilde{A}(\tilde{A}'\tilde{A})^{-1}\tilde{A}')Y = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}. \tag{2.9}$$

The result (2.8) is also obtained by considering the conditional density of $Y_{(12)}$ given Y_3. In order to express S_{ij} in terms of the original notations, let

$$V = [X, Z, A]'[X, Z, A] = \begin{bmatrix} V_{xx} & V_{xz} & V_{xa} \\ V_{zx} & V_{zz} & V_{za} \\ V_{ax} & V_{az} & V_{aa} \end{bmatrix}. \tag{2.10}$$

Noting that $H_2 H_2' = I_N - Z(Z'Z)^{-1}Z'$, it can be shown that

$$Y_i' Y_j = Q_i V_{xx\cdot z} Q_j'\,, \qquad S_{ij} = Q_i V_{xx\cdot za} Q_j'\,, \tag{2.11}$$

where $V_{xx\cdot z} = V_{xx} - V_{xz} V_{zz}^{-1} V_{zx}$, and

$$V_{xx\cdot za} = V_{xx} - [V_{xz}, V_{xa}] \begin{bmatrix} V_{zz} & V_{za} \\ V_{az} & V_{aa} \end{bmatrix}^{-1} \begin{bmatrix} V_{zx} \\ V_{ax} \end{bmatrix},$$

which is equal to $V_{xx\cdot z} - V_{xa\cdot z} V_{aa\cdot z}^{-1} V_{ax\cdot z}$.

3. Test for H_{00}

Khatri [6] obtained the LR test for H_{00} in the model (1.1). Therefore, using the canonical reduction in Section 2, we can obtain the LR test for H_{00} in the model (1.2). On the other hand, it is easily seen that

$$\max_{H_{00}} L(\Xi, \sigma_0^2 I_p) = (2\pi)^{-pn/2} \exp\left(-\frac{1}{2}np\right) \times \left\{\frac{1}{np}(\mathrm{tr}\, S_{(12)(12)} + \mathrm{tr}\, Y_3' Y_3)\right\}^{-np/2}. \quad (3.1)$$

Therefore, from (2.8) we can write the LR statistic for H_{00} as

$$\lambda_0 = \frac{|S_{(12)(12)\cdot 3}||Y_3' Y_3|}{\left\{\frac{1}{p}(\mathrm{tr}\, S_{(12)(12)} + \mathrm{tr}\, Y_3' Y_3)\right\}^p}. \quad (3.2)$$

The statistic λ_0 can be written as

$$\lambda_0 = \frac{|W_1||W_2|}{\left\{\frac{1}{p}(\mathrm{tr}\, W_1 + \mathrm{tr}\, W_2 + \mathrm{tr}\, W_3)\right\}^p}, \quad (3.3)$$

where $W_1 = S_{(12)(12)\cdot 3}$, $W_2 = Y_3' Y_3$ and $W_3 = S_{(12)3} S_{33}^{-1} S_{3(12)}$. It is easy to verify that under H_{00}, W_1, W_2 and W_3 are independent, $W_1 \sim W_q(n - k - (p - q), \sigma_0^2 I_q)$, $W_2 \sim W_{p-q}(n, \sigma_0^2 I_{p-q})$ and $W_3 \sim W_q(p - q, \sigma_0^2 I_q)$. Khatri [6] has given the hth moment of this statistic. However, his result should be corrected as follows:

$$E(\lambda_0^h) = p^{ph} \frac{\Gamma_q(\frac{1}{2}(m - p + q) + h)\Gamma_{p-q}(\frac{1}{2}(m + k) + h)\Gamma(\frac{1}{2}\{mp + k(p - q)\})}{\Gamma_q(\frac{1}{2}(m - p + q))\Gamma_{p-q}(\frac{1}{2}(m + k))\Gamma(\frac{1}{2}\{mp + k(p - q)\} + ph)}, \quad (3.4)$$

where $m = n - k$ and $\Gamma_p(n/2) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma((n - j + 1)/2)$. From this, we can obtain an asymptotic expansion of the null distribution of $-(m + k)\rho \log \lambda_0$ by expanding its characteristic function. For the method, see, e.g., Anderson [1].

THEOREM 3.1. *When the hypothesis H_{00}: $\Sigma = \sigma_0^2 I_p$ is true, the distribution function of $-(m + k)\rho \log \lambda_0$ can be expanded for large $M = \rho(m + k)$ as*

$$P(-(m + k)\rho \log \lambda_0 \le x) = P(\chi_f^2 \le x) + O(M^{-2}),$$

where $f = \frac{1}{2}(p - 1)(p + 2)$, $m = N - r - k$ and ρ is defined by

$$f(m + k)(1 - \rho) = \frac{1}{12p}(p - 1)(p + 2)(2p^2 + p + 2)$$

$$+ \frac{1}{2p} q\{2p^2 + p - qp - 2 + (p - q)k\}k.$$

In a special case $q = p$,

$$\rho = 1 - \frac{1}{6(m+k)p}\{2p^2 + (6k+1)p + 2\}.$$

4. Test for H_{0c}

For testing the hypothesis H_{0c} in (1.4), we may start from the model (2.7) for Y, in which the hypothesis is equivalent to (2.5). Under H_{0c},

$$\begin{aligned} Y_1 &\sim N_{n\times c}(\tilde{A}\Xi_1, \Psi_{11}\otimes I_n), \\ Y_{(23)} &\sim N_{n\times(p-c)}([\tilde{A}\Xi_2, O], \sigma_c^2 I_{p-c}\otimes I_n), \end{aligned} \tag{4.1}$$

where $\Psi_{11} = \Delta_c + \sigma_c^2 I_c$ and $\Xi = [\Xi_1, \Xi_2]$, $\Xi_1 : k\times c$. The log-likelihood after maximizing with respect to Ξ can be written as

$$\begin{aligned} l^*(\Psi_{11}, \sigma_c^2) = -\frac{n}{2}\Bigg[& p\log(2\pi) + \log|\Psi_{11}| + \operatorname{tr}\Psi_{11}^{-1}\frac{1}{n}S_{11} \\ & + (p-c)\left\{\log\sigma_c^2 + \frac{1}{\sigma_c^2}\cdot\frac{1}{n(p-c)}(\operatorname{tr}S_{22} + \operatorname{tr}Y_3'Y_3)\right\}\Bigg]. \end{aligned} \tag{4.2}$$

As is seen later on, the maximization of (4.2) in the space

$$\omega = \{(\Psi_{11}, \sigma_c^2);\ \Psi_{11} - \sigma_c^2 I_c \geq O, \sigma_c^2 > 0\} \tag{4.3}$$

is complicated. For simplicity, we consider the maximization of (4.2) in the space $\tilde{\omega} = \{(\Psi_{11}, \sigma_c^2);\ \Psi_{11} > O, \sigma_c^2 > 0\}$. This is equivalent to considering the LR test for a modified hypothesis

$$\tilde{H}_{0c}:\ \Psi_{11} > O,\ \Psi_{1(23)} = O \quad \text{and} \quad \Psi_{(23)(23)} = \sigma_c^2 I_{p-c}. \tag{4.4}$$

The maximum is achieved at

$$\hat{\Psi}_{11} = \frac{1}{n}S_{11}, \qquad \hat{\sigma}_c^2 = \frac{1}{n(p-c)}(\operatorname{tr}S_{22} + \operatorname{tr}Y_3'Y_3). \tag{4.5}$$

Therefore, we can suggest a test statistic

$$\tilde{\lambda}_c = \frac{|S_{(12)(12)\cdot 3}|\,|Y_3'Y_3|}{|S_{11}|\left\{\dfrac{1}{p-c}(\operatorname{tr}S_{22} + \operatorname{tr}Y_3'Y_3)\right\}^{p-c}} \tag{4.6}$$

for testing H_{0c} against alternatives $H_{1c} \neq H_{0c}$. Rao [9] proposed this statistic in a special case $k = 1$, $r = 0$ and $p = q$. We can decompose $\tilde{\lambda}_c$ as

$$\tilde{\lambda}_c = \tilde{\lambda}_c^{(1)}\tilde{\lambda}_c^{(2)}, \tag{4.7}$$

where

$$\tilde{\lambda}_c^{(1)} = \frac{|S_{11\cdot(23)}|}{|S_{11}|} = \frac{|S_{11\cdot(23)}|}{|S_{11\cdot(23)} + S_{1(23)}S_{(23)(23)}^{-1}S_{(23)1}|}$$

and

$$\tilde{\lambda}_c^{(2)} = \frac{|S_{22\cdot 3}||Y_3'Y_3|}{\left\{\frac{1}{p-c}(\mathrm{tr}\, S_{22\cdot 3} + \mathrm{tr}\, Y_3'Y_3 + \mathrm{tr}\, S_{23}S_{33}^{-1}S_{32})\right\}^{p-c}}.$$

The statistics $\tilde{\lambda}_c^{(1)}$ and $\tilde{\lambda}_c^{(2)}$ are the LR statistics for $\Psi_{1(23)} = O$ and $\Psi_{(23)(23)} = \sigma_c^2 I_{p-c}$, respectively.

LEMMA 4.1. *When the hypothesis H_{0c} is true, it holds that*

(i) *$\tilde{\lambda}_c^{(1)}$ and $\tilde{\lambda}_c^{(2)}$ are independent,*

(ii) $$E(\{\tilde{\lambda}_c^{(1)}\}^h) = \frac{\Gamma_c(\frac{1}{2}(m-p+c)+h)\Gamma_c(\frac{1}{2}m)}{\Gamma_c(\frac{1}{2}(m-p+c))\Gamma_c(\frac{1}{2}m+h)},$$

(iii) $$E(\{\tilde{\lambda}_c^{(2)}\}^h) = (p-c)^{(p-c)h}\frac{\Gamma_{q-c}(\frac{1}{2}(m-p+q)+h)}{\Gamma_{q-c}(\frac{1}{2}(m-p+q))} \times \frac{\Gamma_{p-q}(\frac{1}{2}(m+k)+h)\Gamma(\frac{1}{2}\{m(p-c)+k(p-q)\})}{\Gamma_{p-q}(\frac{1}{2}(m+k))\Gamma(\frac{1}{2}\{m(p-c)+k(p-q)\}+(p-c)h)},$$

where $m = n - k$.

PROOF. It is easy to verify that under H_{0c}, $S_{11\cdot(23)} \sim W_c(n-k-(p-c), \Psi_{11})$, $S_{1(23)}S_{(23)(23)}^{-1}S_{(23)1} \sim W_c(p-c, \Psi_{11})$, $S_{22\cdot 3} \sim W_{q-c}(n-k-(p-q), \sigma_c^2 I_{q-c})$, $Y_3'Y_3 \sim W_{p-q}(n, \sigma_c^2 I_{p-q})$ and $S_{23}S_{33}^{-1}S_{32} \sim W_{q-c}(p-q, \sigma_c^2 I_{q-c})$. Further, these five statistics are independent. Therefore, $\tilde{\lambda}_c^{(1)}$ and $\tilde{\lambda}_c^{(2)}$ are independent. The hth moment of $\tilde{\lambda}_c^{(1)}$ is obtained from that $\tilde{\lambda}_c^{(1)}$ is distributed as a lambda distribution $\Lambda_{c,p-c,n-k-(p-c)}$. The hth moment of $\tilde{\lambda}_c^{(2)}$ is obtained from the one of λ_0 by changing p and q as $p-c$ and $q-c$, respectively.

Using Lemma 4.1, we can obtain an asymptotic expansion of $-(m+k)\rho_c \log \tilde{\lambda}_c$.

THEOREM 4.1. *When the hypothesis H_{0c} is true, the distribution function of $-(m+k)\rho_c \log \tilde{\lambda}_c$ can be expanded for large $M = (m+k)\rho_c$ as*

$$P(-(m+k)\rho_c \log \tilde{\lambda}_c \leq x) = P(\chi_{f_c}^2 \leq x) + O(M^{-2}),$$

where $f_c = c(p-c) + \frac{1}{2}(p-c-1)(p-c+2)$, $m = n-k$ and ρ_c is given by

$$f_c(m+k)(1-\rho_c)=\frac{1}{12(p-c)}[(p-c-1)(p-c+2)\{2(p-c)^2+p-c+2\}$$
$$+6c(p+1)(p-c)^2]+\frac{1}{2(p-c)}\{2q(p-c)^2$$
$$-(q-c-1)(q-c)(p-c)-2(q-c)+(q-c)(p-q)k\}k\,.$$

Next we obtain the exact LR criterion $\lambda_c^{n/2}$ for H_{0c}, based on the distribution of Y. For the case $\hat{\Psi}_{11}-\hat{\sigma}_c^2 I_c \geq O$, the LR statistic λ_c is equal to $\tilde{\lambda}_c$. However, if it is not the case, we need to obtain the maximum of (4.2) in the space ω. This is equivalent to solving the problem of minimizing

$$g(\Delta_c, \sigma_c^2) = \log|\Delta_c+\sigma_c^2 I_c| + \mathrm{tr}(\Delta_c+\sigma_c^2 I_c)^{-1}\hat{\Psi}_{11} + (p-c)\,(\log\sigma_c^2 + \hat{\sigma}_c^2/\sigma_c^2). \quad (4.8)$$

Let $\delta_1 \geq \cdots \geq \delta_c$ $(\geq \sigma_c^2)$ and $t_1 > \cdots > t_c$ be the characteristic roots of $\Delta_c+\sigma_c^2 I_c$ and $\hat{\Psi}_{11}$, respectively. Then, from Anderson, Anderson and Olkin [2] it is seen that

$$\min_{\Delta_c\geq O,\,\sigma_c^2>0} g(\Delta_c,\sigma_c^2) = \min_{\delta_1\geq\cdots\geq\delta_c\geq\delta^*>0}\left[\sum_{i=1}^{c}\left(\log\delta_i+\frac{t_i}{\delta_i}\right)+(p-c)\left(\log\delta^*+\frac{t^*}{\delta^*}\right)\right], \quad (4.9)$$

where $\delta^*=\sigma_c^2$ and $t^*=\hat{\sigma}_c^2$. If $t_c \geq t^*$, then the minimum is achieved at $\delta_i=t_i$, $i=1,\ldots,c$ and $\delta^*=t^*$, and hence $\lambda_c=\tilde{\lambda}_c$. For the case $t^*>t_c$, such a minimum may be found in a boundary-value situation, but becomes very complicated. As a simple bound for λ_c, consider

$$\bar{\lambda}_c = \begin{cases} \tilde{\lambda}_c & (t_c \geq t^*)\,, \\ \dfrac{|S_{(12)(12)\cdot 3}|\,|Y_3'Y_3|}{|S_{11}|\{nt_c\exp(t^*/t_c-1)\}^{p-c}} & (t^*>t_c)\,. \end{cases} \quad (4.10)$$

We note that $\bar{\lambda}_c$ is obtained by letting $\delta_i=t_i$, $i=1,\ldots,c$, and $\delta^*=t_c$ in (4.9) if $t^*>t_c$. Then, from (4.9) and $\omega\subset\tilde{\omega}$ we have

$$\bar{\lambda}_c \leq \lambda_c \leq \tilde{\lambda}_c\,. \quad (4.11)$$

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