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The explicit representation of the determinant of Harish-Chandra's C-function in $SL(3, R)$ and $SL(4, R)$ cases

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§ 1. Introduction

Let G be a semisimple Lie group with finite center, K a maximal compact subgroup of G. Let θ be the Cartan involution of G fixing K. Let P be a cuspidal parabolic subgroup and $P=MAN$ its Langlands decomposition.

Let $\pi_{P,\sigma,\nu} = \mathrm{ind}_{MAN}^{G}\sigma \otimes \nu \otimes 1$ (σ in $\hat{M}$, ν a character of A) be the representation of the generalized principal series induced from P to G and $H^{P,\sigma,\nu}$ be its representation space. Then the operator $A(\bar{P}: P: \sigma: \nu)$ defined by the integral

$$(A(\bar{P}: P: \sigma: \nu)f)(x) = \int_{\bar{N}} f(x\bar{n})d\bar{n}\,, \qquad (f \in H^{P,\sigma,\nu})$$

is an intertwining operator between $\pi_{P,\sigma,\nu}(g)$ and $\pi_{\bar{P},\sigma,\nu}(g)$ $(g \in G)$, where $\bar{P} = \theta P$.

In the following we assume that P is a minimal parabolic subgroup of G. For γ in $\hat{K}$ we denote by $H_{\gamma}^{P,\sigma,\nu}$ the γ-isotypic component of $H^{P,\sigma,\nu}$. Let V^{γ} and H^{σ} be the representation spaces of γ and σ respectively. Following Wallach [11], we consider the bijective map $v \otimes A \to L_P(A, v, \nu)$ $(v \in V^{\gamma}, A \in \mathrm{Hom}_M(V^{\gamma}, H^{\sigma}))$ from $V^{\gamma} \otimes \mathrm{Hom}_M(V^{\gamma}, H^{\sigma})$ to $H_{\gamma}^{P,\sigma,\nu}$, where $L_P(A, v, \nu)$ is defined by

$$L_P(A, v, \nu)(kan) = e^{-(\nu+\rho)(\log a)}A(\pi_{\gamma}(k^{-1})v)\,, \qquad (k \in K, a \in A, n \in N)\,,$$

and the operator defined by the integral

$$B_{\gamma}(\bar{P}: P: \nu) = \int_{\bar{N}} \pi_{\gamma}(\kappa(\bar{n}))^{-1}e^{-(\nu+\rho)(H(\bar{n}))}d\bar{n}\,.$$

Then the operator $B_{\gamma}(\bar{P}: P: \nu)$ satisfies

$$A_{\gamma}(\bar{P}: P: \sigma: \nu)L_P(A, v, \nu) = L_{\bar{P}}(A \circ B_{\gamma}(\bar{P}: P: \nu), v, \nu)\,.$$

Moreover, $B_{\gamma}(\bar{P}: P: \nu)$ commutes with $\pi_{\gamma}(m)$ $(m \in M)$ and we can restrict B_{γ} to V_{σ}^{γ}, V_{σ}^{γ} denoting the σ-isotypic component of V^{γ}. We denote by B_{γ}^{σ} the restriction of B_{γ} to V_{σ}^{γ}. Wallach [11] has shown that $B_{\gamma}(\bar{P}: P: \nu)$ is holomorphic in a certain half-space of $\mathfrak{a}_C^*$ and meromorphic in $\mathfrak{a}_C^*$. In the relation with the intertwining operator $A_\gamma(\bar{P}:P:\sigma:\nu)$, it is important to study the nature of the B_γ^σ-function as a meromorphic function, such as its zeroes, poles and their order.

Concerning this problem, Cohn [1] has proved that the determinant of the C-function is a product of some quotient of Γ-factors and gives a conjecture on the rational numbers which appear in these factors.

Our main theorems give the determinant of the B_γ^σ-function explicitly in $SL(3, \boldsymbol{R})$ and $SL(4, \boldsymbol{R})$ cases. In another paper, we shall give an application of the results to the analytical argument of the reducibility of the generalized principal series representation (cf. Speh-Vogan [9]).

In making the conjecture of our results we have used the software "REDUCE" for computers.

§ 2. Notation and preliminaries

Let G be a semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition and θ the Cartan involution defining the decomposition. We introduce an inner product B_θ on $\mathfrak{g}$ in the standard way such that $B_\theta(X, Y)=-B(X, \theta Y)$, where B is the Killing form on $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We fix an order in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and put $\mathfrak{n}=\sum_{\alpha>0}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ denoting the root space for the $\mathfrak{a}$-root α. Then we have an Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ of $\mathfrak{g}$. Let $\mathfrak{v}=\theta\mathfrak{n}$ and $\mathfrak{m}=Z_{\mathfrak{k}}(\mathfrak{a})$, the cetralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

We now let $K=N_G(\mathfrak{k})$ be the normalizer of $\mathfrak{k}$ in G, $M=Z_K(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in K and $M'=N_K(\mathfrak{a})$ the normalizer of $\mathfrak{a}$ in K. Let A, N_0, and V_0 be the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{v}$ respectively. Let $P=MAN$. The conjugates of P are called minimal parabolic subgroups of G. Let $\mathscr{P}(A)$ be the set of all parabolic subgroups P of G such that A is the split component of P. The elements in $\mathscr{P}(A)$ are in obvious one-to-one correspondence with Weyl chambers in $\mathfrak{a}$ and the Weyl group $W=M'/M$ permutes the Weyl chambers transitively. For each w in W, λ a character of A and ξ a representation of M, put

$$w\lambda(a)=\lambda(w^{-1}aw), \qquad w\xi(m)=\xi(w^{-1}mw).$$

Then W acts on characters of A and classes of representation of M.

Let $\hat{K}$ and $\hat{M}$ be the set of all equivalence classes of the irreducible unitary representations of K and M respectively. For each $\sigma\in\hat{M}$ we fix a representation $(\tilde{\sigma}, H^{\tilde{\sigma}})$ in σ and, abusing notation, we use also σ for $\tilde{\sigma}$. For each γ in $\hat{K}$ we fix an element $(\pi_\gamma H^\gamma)$ in γ. Put $\rho = \rho_{P_0} = \frac{1}{2}\sum_{\alpha>0}(\dim \mathfrak{g}_\alpha)\alpha$.

We recall the generalized principal series representations. Let σ be in $\hat{M}$ and ν in $\mathfrak{a}_c^*$ (the complexfication of $\mathfrak{a}^*$). Let $C_{\sigma,\nu}(G)$ be the space of all continuous functions f from G to H^σ such that

$$f(xman) = e^{-(\nu+\rho)(\log a)}\sigma(m)^{-1}f(x) \qquad (x \in G).$$

Let $H^{P_0,\sigma,\nu}$ be the completion of $C_{\sigma,\nu}(G)$ by the norm

$$\|f\|^2 = \int_K \|f(k)\|_\sigma^2 dk, \qquad (f \in C_{\sigma,\nu}(G)).$$

The representation $\pi_{P_0,\sigma,\nu}$ is given by

$$\pi_{P_0,\sigma,\nu}(g)f(x) = f(g^{-1}x), \qquad (g \in G).$$

What we have just described is the "induced picture" for $\pi_{P_0,\sigma,\nu}$.

The "compact picture" is the restriction of the induced picture to K. Here the corresponding dense subspace $C_\sigma(K)$ is

$$\{f: K \to H^\sigma | f \text{ is continuous and } f(km) = \sigma(m)^{-1}f(k)\}$$

and is independent of ν. According to the Iwasawa decomposition $G = KAN_0$, each $g \in G$ is written as

$$g = \kappa(g)(\exp H(g))n_0(g), \qquad (\kappa(g) \in K, H(g) \in \mathfrak{a}, n_0(g) \in N_0).$$

Then the representation is given by

$$\pi_{P_0,\sigma,\nu}(g)f(k) = e^{-(\nu+\rho)(H(g^{-1}k))}f(\kappa(g^{-1}k)).$$

If γ is in $\hat{K}$, the projection operator E_γ is defined by

$$E_\gamma f = d(\gamma)\bar{\chi}_\gamma * f, \qquad (f \in C_\sigma(K)),$$

where $d(\gamma)$ and χ_γ denote the dimension and the character of γ respectively. For $\gamma \in \hat{K}$, we put

$$H_\gamma^{P_0,\sigma,\nu} = \{f \in H^{P_0,\sigma,\nu} | E_\gamma f = f\}.$$

§ 3. C-functions and intertwining operators

In this section we recall the Harish-Chandra C-functions, intertwining operators and the relation between them. Let P_0 be as above. Let γ be in $\hat{K}$, σ in $\hat{M}$ and A in $\mathrm{Hom}_M(V^\gamma, H^\sigma)$, where V^γ denotes the representation space of π_γ. For ν in $\mathfrak{a}_{\mathbf{C}}^*$, v in V^γ, let

$$L_{P_0}(A, v, \nu)(kan) = e^{-(\rho+\nu)(\log a)}A(\pi_\gamma(k^{-1})v), \qquad (k \in K, a \in A, n \in N_0).$$

Then an easy computation shows that $L_{P_0}(A, v, \nu)$ is in $H_\gamma^{P_0,\sigma,\nu}$. Furthermore the map $V^\gamma \otimes \mathrm{Hom}_M(V^\gamma, H^\sigma) \to H_\gamma^{P_0,\sigma,\nu}$ given by $v \otimes A \to L_{P_0}(A, v, \nu)$ is a bijective K-intertwining operator.

We introduce formal expressions, often divergent, for operators that implement equivalences among some of these representations. For now, we work in the induced picture. Let $P_1 = MAN_1$, $P_2 = MAN_2$ be in $\mathscr{P}(A)$. For f in $H^{P_1,\sigma,\nu}$, set

$$A(P_2 : P_1 : \sigma : \nu)f(x) = \int_{V_1 \cap N_2} f(xv)dv ,$$

where $V_1 = \theta N_1$ and dv is the normalized Haar measure on $V_1 \cap N_2$ by

$$\int_{V_1 \cap N_2} e^{-\rho_1(H(v))}dv = 1 ,$$

where $\rho_1 = \rho_{P_1}$.

The following result is well known (see e.g. [6]).

PROPOSITION 3.1. *When the indicated integrals are convergent,*

$$A(P_2 : P_1 : \sigma : \nu)\pi_{P_1,\sigma,\nu}(g) = \pi_{P_2,\sigma,\nu}(g)A(P_2 : P_1 : \sigma : \nu)$$

for all g in G.

For w in M', let $R(w)f(x) = f(xw)$. Then it follows from Proposition 3.1 that

$$A_{P_1}(w, \sigma, \nu) = R(w)A(w^{-1}P_1w : P_1 : \sigma : \nu)$$

satisfies

$$\pi_{P_1,w\sigma,w\nu}(\cdot)A_{P_1}(w, \sigma, \nu) = A_{P_1}(w, \sigma, \nu)\pi_{P_1,\sigma,\nu}(\cdot) ,$$

whenever the indicated integrals are convergent.

We denote by $A_\gamma(P_2 : P_1 : \sigma : \nu)$ the restriction of the map $A(P_2 : P_1 : \sigma : \nu)$ to the space $H_\gamma^{P_1,\sigma,\nu}$. Then we have that $A_\gamma(P_2 : P_1 : \sigma : \nu)$ is in $\mathrm{Hom}_K(H_\gamma^{P_1,\sigma,\nu}, H_\gamma^{P_2,\sigma,\nu})$. The inner product B_θ on $\mathfrak{g}$ induces an inner product on $\mathfrak{a}^*$, which we denote by $\langle \cdot, \cdot \rangle$.

PROPOSITION 3.2. *If ν is in $\mathfrak{a}_C^*$, $\langle \mathrm{Re}\, \nu, \alpha \rangle > 0$ for all $\alpha > 0$ then*

$$A_\gamma(P_1 : P_0 : \sigma : \nu)L_{P_0}(A, v, \nu) = L_{P_1}(A \circ B_\gamma(P_1 : P_0 : \nu), v, \nu) ,$$

where

$$B_\gamma(P_1 : P_0 : \nu) = \int_{V_0 \cap N_1} \pi_\gamma(\kappa(v))^{-1}e^{-(\nu+\rho)(H(v))}dv .$$

Furthermore, $B_\gamma(P_1 : P_0 : \nu)$ *satisfies the following conditions,*
(1) $B_\gamma(P_1 : P_0 : \nu)$ *is absolutely convergent.*
(2) $B_\gamma(P_1 : P_0 : \nu)$ *is in* $\mathrm{End}(V^\gamma)$ *and satisfies*

$$B_\gamma(P_1 : P_0 : \nu)\pi_\gamma(m) = \pi_\gamma(m)B_\gamma(P_1 : P_0 : \nu) \qquad (m \in M)\,. \tag{3.1}$$

PROOF. These assertions but (2) are proved by an analogous argument to the proof of 8.11.5 in [11]. We shall prove (3.1). We have

$$\begin{aligned}\pi_\gamma(m)B_\gamma(P_1 : P_0 : \nu) &= \pi_\gamma(m)\int_{V_0 \cap N_1} \pi_\gamma(\kappa(v))^{-1}e^{-(\nu+\rho)(H(v))}dv \\ &= \int_{V_0 \cap N_1} \pi_\gamma(\kappa(v)m^{-1})^{-1}e^{-(\nu+\rho)(H(v))}dv\,.\end{aligned}$$

Since $H(vm^{-1}) = H(v)$, $\kappa(vm^{-1}) = \kappa(v)m^{-1}$ and the measure dv is invariant under $v \to mvm^{-1}$ (note that M is compact), the last expression is

$$\begin{aligned}&= \int_{V_0 \cap N_1} \pi_\gamma(\kappa(vm^{-1}))^{-1}e^{-(\nu+\rho)(H(vm^{-1}))}dv \\ &= \int_{V_0 \cap N_1} \pi_\gamma(\kappa(mv))^{-1}e^{-(\nu+\rho)(H(mv))}dv \\ &= \int_{V_0 \cap N_1} \pi_\gamma(m\kappa(v))^{-1}e^{-(\nu+\rho)(H(v))}dv \\ &= B_\gamma(P_1 : P_0 : \nu)\pi_\gamma(m)\,.\end{aligned}$$

This proves (3.1).

If σ is in $\hat{M}$, we denote the σ-component of V^γ by V^γ_σ. Let

$$B^\sigma_\gamma(P_1 : P_0 : \nu) = B_\gamma(P_1 : P_0 : \nu)|_{V^\gamma_\sigma}\,.$$

Then $B^\sigma_\gamma(P_1 : P_0 : \nu)$ is in $\mathrm{End}(V^\gamma_\sigma)$. Setting $\bar{P} = \theta P$ $(=\theta P_0)$, $\nu \to B^\sigma_\gamma(\bar{P} : P : \nu)$ is called Harish-Chandra's C-function, as is well known, which is continued to $\mathfrak{a}^*_C$ meromorphically.

COROLLARY 3.3. *If* w *is in* M', ν *in* $\mathfrak{a}^*_C$, $\langle \mathrm{Re}\,\nu, \alpha\rangle > 0$ *for all* $\alpha > 0$ *then*

$$A_{P_0}(w, \sigma, \nu)L(A, v, \nu) = L(A \circ B_\gamma(w, \nu)\pi_\gamma(w)^{-1}, v, \nu)\,,$$

where

$$B_\gamma(w, \nu) = B_\gamma(w^{-1}P_0 w : P_0 : \nu)\,.$$

§ 4. The C-function for the $SL(3, \boldsymbol{R})$ case

In this section we shall specialize to $SL(3, \boldsymbol{R})$ the notation described in the previous sections. Our notation is as follows. Let G be $SL(3, \boldsymbol{R})$, the group of 3-by-3 real matrices of determinant one. Let

$$\theta = -\text{transpose}$$

$$K = SO(3)$$

$$\mathfrak{a} = \{\mathrm{diag}(x_1, x_2, x_3) | x_i \in \boldsymbol{R},\ x_1 + x_2 + x_3 = 0\}$$

$$M = \left\{ \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right\}$$

$$A = \exp \mathfrak{a}$$

$$N = \left\{ g \in G \,\middle|\, g = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} \right\}$$

$$P = MAN$$

and define linear functions e_i $(1 \leq i \leq 3)$ on $\mathfrak{a}_C$ by

$$e_i(\mathrm{diag}(x_1, x_2, x_3)) = x_i .$$

Then each ν in $\mathfrak{a}_C^*$ can be written in the form

$$\nu = \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3 \qquad (\nu_i \in \boldsymbol{C},\ 1 \leq i \leq 3),$$

and we sometimes write (ν_1, ν_2, ν_3) for ν. The $\mathfrak{a}$-roots of $\mathfrak{g}$ are $\pm(e_1 - e_2)$, $\pm(e_2 - e_3)$, $\pm(e_1 - e_3)$, and the simple $\mathfrak{a}$-roots are $e_1 - e_2$, $e_2 - e_3$. Let

$$w_1 = \begin{bmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & -1 & 0 \end{bmatrix}.$$

Their adjoint actions on $\mathfrak{a}$ are corresponding to the simple reflections. We set $w_0 = w_1 w_2 w_1$. Then we have

$$A(\bar{P} : P : \sigma : \nu) = R(w_0) A_P(w_0, \sigma, \nu) .$$

By the relation

$$A_P(w_0, \sigma, \nu) = A_P(w_1, w_2 w_1 \sigma, w_2 w_1 \nu) A_P(w_2, w_1 \sigma, w_1 \nu) A_P(w_1, \sigma, \nu)$$

and Corollary 3.3, we have for γ in $\hat{K}$

$$B_\gamma(\bar{P}:P:\nu) = B_\gamma(w_1, \nu)\pi_\gamma(w_1)B_\gamma(w_2, w_1\nu)\pi_\gamma(w_2)B_\gamma(w_1, w_2w_1\nu)\pi_\gamma(w_1)\pi_\gamma(w_0).$$

LEMMA 4.1. *If ν is in $\mathfrak{a}_C^*$, $\langle \mathrm{Re}\,\nu, \alpha\rangle > 0$ for all $\alpha > 0$, we have*

$$B_\gamma(w_1, \nu) = \mathrm{Const}\cdot\int_{-\infty}^{\infty} f(x)^{-(\nu_1-\nu_2)-1}\pi_\gamma\left[\frac{1}{f(x)}\begin{bmatrix}1 & -x & \\ x & 1 & \\ & & f(x)\end{bmatrix}\right]^{-1}dx,$$

$$B_\gamma(w_2, \nu) = \mathrm{Const}\cdot\int_{-\infty}^{\infty} f(x)^{-(\nu_2-\nu_3)-1}\pi_\gamma\left[\frac{1}{f(x)}\begin{bmatrix}f(x) & & \\ & 1 & -x \\ & x & 1\end{bmatrix}\right]^{-1}dx,$$

where $f(x) = (1 + x^2)^{1/2}$.

Since the results are obtained by an easy computation, we leave the proof to the reader.

We shall recall irreducible unitary representations of K.

LEMMA 4.2. *Set*

$$X_1 = \frac{\sqrt{-1}}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad X_3 = \frac{\sqrt{-1}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then $\{X_i\}_{1\le i\le 3}$ is a basis of $\mathfrak{su}(2)$ and satisfies the following relations

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2.$$

By the basis of $\mathfrak{su}(2)$ given in Lemma 4.2, $(SU(2), \mathrm{Ad})$ can be considered to be the universal covering group of K. If n is a nonnegative even integer, we set

$$V^n = \{p \in C[z_1, z_2] \mid p \text{ is a homogeneous polynomial of degree } n\}.$$

Then V^n is a Hilbert space of dimension $n+1$ equipped with the inner product (p_1, p_2) defined by

$$\left(\textstyle\sum_{k=0}^n a_k z_1^k z_2^{n-k}, \sum_{k=0}^n b_k z_1^k z_2^{n-k}\right) = \sum_{k=0}^n k!(n-k)!a_k\bar{b}_k.$$

For each $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SU(2)$, we assign

$$(\tilde{\pi}_n(g)p)(z_1, z_2) = p(az_1 + cz_2, bz_1 + dz_2) \qquad (f \in V^n).$$

Then it is known that $(\tilde{\pi}_n, V^n)$ $(n \ge 0)$ are irreducible representations of $SU(2)$ and exaust $SU(2)^\wedge$. Moreover, as is well known (see e.g. [10]), for each γ in $\hat{K}$ there exists a unique nonnegative even integer n satisfying

$$\tilde{\pi}_n \simeq \pi_\gamma \circ \mathrm{Ad}\,, \qquad \text{(unitarily equivalent)}. \tag{4.1}$$

LEMMA 4.3. *Suppose γ is in $\hat{K}$, n is the nonnegative even integer satisfying* (4.1) *and V^n is defined as above. Let*

$$v_i = z_1^{n-i} z_2^i \qquad (0 \le i \le n)\,,$$

$$C = 2^{-1/2}\sqrt{-1}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in SU(2)\,.$$

Then we have

(1) $$\pi_\gamma\left[f(x)^{-1}\begin{bmatrix} 1 & -x & \\ x & 1 & \\ & & f(x)\end{bmatrix}\right]^{-1} v_i = \left(\frac{1-\sqrt{-1}x}{f(x)}\right)^{n/2-i} v_i\,,$$

(2) $$\pi_\gamma(\mathrm{Ad}(C))\pi_\gamma\left[f(x)^{-1}\begin{bmatrix} f(x) & & \\ & 1 & -x \\ & x & 1\end{bmatrix}\right]^{-1}\pi_\gamma(\mathrm{AD}(C))$$
$$= \pi_\gamma\left[f(x)^{-1}\begin{bmatrix} 1 & -x & \\ x & 1 & \\ & & f(x)\end{bmatrix}\right]^{-1}.$$

PROOF. We first prove formula (1). From Proposition 4.2, we have

$$\mathrm{Ad}(\exp tX_3) = \begin{bmatrix} \cos t & -\sin t & \\ \sin t & \cos t & \\ & & 1\end{bmatrix}, \qquad (t \in \boldsymbol{R})$$

and by an easy computation, we obtain

$$\tilde{\pi}_n(\exp tX_3) = e^{(n/2-i)\sqrt{-1}\,t} v_i, \qquad (0 \le i \le n)\,.$$

Therefore we have

$$\pi_\gamma\left[\begin{bmatrix} \cos t & -\sin t & \\ \sin t & \cos t & \\ & & 1\end{bmatrix}\right] v_i = e^{(n/2-i)\sqrt{-1}\,t} v_i, \qquad (0 \le i \le n) \tag{3.2}$$

If we put $\cos t = f(x)^{-1}$, $\sin t = x/f(x)$, (3.2) is equal to

$$\pi_\gamma\left[f(x)^{-1}\begin{bmatrix} 1 & -x & \\ x & 1 & \\ & & f(x)\end{bmatrix}\right] v_i = \left(\frac{1+\sqrt{-1}\,x}{f(x)}\right)^{n/2-i} v_i \quad (0 \le i \le n)\,.$$

Therefore we have

$$\pi_\gamma\left[f(x)^{-1}\begin{bmatrix}1 & -x & \\ x & 1 & \\ & & f(x)\end{bmatrix}\right]^{-1} v_i = \left(\frac{1-\sqrt{-1}x}{f(x)}\right)^{n/2-i} v_i \qquad (0 \le i \le n).$$

We next prove (2). We note that

$$C^{-1}X_1C = X_3,$$

$$\mathrm{Ad}(\exp tX_1) = \begin{bmatrix}1 & & \\ & \cos t & -\sin t \\ & \sin t & \cos t\end{bmatrix}.$$

We thus have

$$\begin{aligned}\pi_\gamma\left[\begin{bmatrix}1 & & \\ & \cos t & -\sin t \\ & \sin t & \cos t\end{bmatrix}\right] &= \pi_\gamma(\mathrm{Ad}(\exp tX_1)) \\ &= \pi_\gamma(\mathrm{Ad}(C^{-1}(\exp tX_3)C)).\end{aligned}$$

Since $\mathrm{Ad}(C^2)$ is equal to the identity, $\mathrm{Ad}(C)^{-1} = \mathrm{Ad}(C)$ and the last expression equals

$$\pi_\gamma(\mathrm{Ad}(C))\pi_\gamma\left[\begin{bmatrix}\cos t & -\sin t & \\ \sin t & \cos t & \\ & & 1\end{bmatrix}\right]\pi_\gamma(\mathrm{Ad}(C)).$$

Therefore we have

$$\begin{aligned}&\pi_\gamma\left[\begin{bmatrix}1 & & \\ & \cos t & -\sin t \\ & \sin t & \cos t\end{bmatrix}\right]^{-1} \\ &\qquad = \pi_\gamma(\mathrm{Ad}(C))\pi_\gamma\left[\begin{bmatrix}\cos t & -\sin t & \\ \sin t & \cos t & \\ & & 1\end{bmatrix}\right]^{-1}\pi_\gamma(\mathrm{Ad}(C)).\end{aligned}$$

Putting $\cos t = f(x)^{-1}$, $\sin t = x/f(x)$, we obtain (2).

As is easily seen, the element $\mathrm{Ad}(C)$ in K normalizes $\mathfrak{a}_{\mathbf{C}}^*$. Thus $\mathrm{Ad}(C)$ is in M'. Then we have the following

COROLLARY 4.4. *Suppose ν is an element in $\mathfrak{a}_{\mathbf{C}}^*$ such that $\langle \mathrm{Re}\,\nu, \alpha\rangle > 0$ for all $\alpha > 0$ and let γ, n be as in Lemma 4.3. Then*

(1) $$B_\gamma(w_1, \nu) = \alpha(\nu_1 - \nu_2, n),$$

where $\alpha(s, n) = \mathrm{diag}(\alpha_0(s, n), \dots, \alpha_n(s, n))$ $(s \in \boldsymbol{C})$ *with respect to the basis* $\{v_i\}_{0 \le i \le n}$ *of* V^n *and for* $0 \le i \le n$,

$$\alpha_i(s, n) = \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-(n/2-i)}{2}\right)\Gamma\left(\frac{s+1+(n/2-i)}{2}\right)}.$$

We have also

$$\pi_\gamma(\mathrm{Ad}(C))B_\gamma(w_2, \nu)\pi_\gamma(\mathrm{Ad}(C)) = B_\gamma(w_1, -(\mathrm{Ad}(C)\cdot\nu)), \tag{2}$$

where $\mathrm{Ad}(C)\cdot\nu$ *is the element in* $\mathfrak{a}_{\boldsymbol{C}}^*$ *defined by*

$$(\mathrm{Ad}(C)\cdot\nu)(H) = \nu(\mathrm{Ad}(C)^{-1}H), \qquad (H \in \mathfrak{a}_{\boldsymbol{C}}).$$

PROOF. (2) is a direct consequence of Lemma 4.3 (2). We shall prove (1). From Lemma 4.1, we have

$$B_\gamma(w_1, \nu)v_i = \int_{-\infty}^{\infty} f(x)^{-(\nu_1-\nu_2)-1}\pi_\gamma\left[\frac{1}{f(x)}\begin{bmatrix} 1 & -x & \\ x & 1 & \\ & & 1\end{bmatrix}\right]v_i dx$$

by Lemma 4.3

$$= \int_{-\infty}^{\infty} f(x)^{-(\nu_1-\nu_2)-1}\left(\frac{1-\sqrt{-1}\,x}{f(x)}\right)^{n/2-i} dx v_i$$

by A.3

$$= \alpha_i(\nu_1 - \nu_2, n)v_i .$$

This proves (1).

§5. The M-isotypic components of γ

In this section we shall describe the M-isotypic components of γ in $\hat{K}$. Let n be a nonnegative even integer satisfying (4.1) and V^n be the space defined in §4. For any integer ξ, we write $\xi \equiv 0$ (resp. $\xi \equiv 1$) if ξ is even (resp. odd). Let

$$V^n_{(0,+)} = \sum_{\substack{k \equiv 0\,(2) \\ 0 \le k \le n}} \boldsymbol{C}(z_1^{n-k}z_2^k + z_1^k z_2^{n-k}),$$

$$V^n_{(0,-)} = \sum_{\substack{k \equiv 0\,(2) \\ 0 \le k \le n}} \boldsymbol{C}(z_1^{n-k}z_2^k - z_1^k z_2^{n-k}),$$

$$V^n_{(1,+)} = \sum_{\substack{k\equiv 1\,(2)\\ 0\le k\le n}} C(z_1^{n-k}z_2^k + z_1^k z_2^{n-k}),$$

$$V^n_{(1,-)} = \sum_{\substack{k\equiv 1\,(2)\\ 0\le k\le n}} C(z_1^{n-k}z_2^k - z_1^k z_2^{n-k}).$$

Then we have

$$V^n = V^n_{(0,+)} + V^n_{(0,-)} + V^n_{(1,+)} + V^n_{(1,-)}, \tag{5.1}$$

(orthogonal direct).

LEMMA 5.1. *Let C be the matrix defined in* §4. *Then we have*

(1) $$V^n_{(0,+)} \xrightarrow{\tilde{\pi}_n(C)} V^n_{(0,+)}$$

(2) $$V^n_{(0,-)} \longrightarrow V^n_{(1,+)}$$

(3) $$V^n_{(1,+)} \longrightarrow V^n_{(0,-)}$$

(4) $$V^n_{(1,-)} \longrightarrow V^n_{(1,-)}$$

PROOF. For any integer $r \ge 0$, we observe that

$$(z_1+z_2)^r + (z_1-z_2)^r = 2\sum_{\substack{0\le p\le r\\ p\equiv 0\,(2)}} \binom{r}{p} z_1^{r-p}z_2^p, \tag{5.2}$$

$$(z_1+z_2)^r - (z_1-z_2)^r = 2\sum_{\substack{0\le p\le r\\ p\equiv 1\,(2)}} \binom{r}{p} z_1^{r-p}z_2^p. \tag{5.3}$$

Suppose $n-2k \ge 0$. Then we have

$$\begin{aligned}
&\tilde{\pi}_n(C)(z_1^{n-k}z_2^k + z_1^k z_2^{n-k})\\
&\quad = \mathrm{Const}\cdot((z_1+z_2)^{n-k}(z_1-z_2)^k + (z_1+z_2)^k(z_1-z_2)^{n-k})\\
&\quad = \mathrm{Const}\cdot(z_1^2-z_2^2)^k((z_1+z_2)^{n-2k} + (z_1-z_2)^{n-2k}).
\end{aligned}$$

By (5.2) we have

$$(z_1+z_2)^{n-2k} + (z_1-z_2)^{n-2k} \in V^{n-2k}_{(0,+)}.$$

Therefore, for the proof of (1) and (3), it is enough to show the following relations

$$(z_1^2-z_2^2)^k\cdot V^{n-2k}_{(0,+)} \subset V^n_{(0,+)} \qquad \text{for } k\equiv 0\,(2) \tag{5.4}$$

and

$$(z_1^2-z_2^2)^k\cdot V^{n-2k}_{(0,+)} \subset V^n_{(0,-)} \qquad \text{for } k\equiv 1\,(2). \tag{5.5}$$

Now for each interger $s \geq 0$ such that $s \equiv 0$ (2), we have

$$(z_1^2 - z_2^2)(z_1^{r-s}z_2^s + z_1^s z_2^{r-s}) = (z_1^{(r+2)-s}z_2^s - z_1^{s+2}z_2^{(r+2)-s}) \\ + (z_1^{s+2}z_2^{(r+2)-(s+2)} - z_1^{(r+2)-(s+2)}z_2^{s+2}).$$

Therefore for an even integer $r \geq 0$, we have

$$(z_1^2 - z_2^2) \cdot V_{(0,+)}^r \subset V_{(0,-)}^{r+2}.$$

In the same way as above, for an even integer $r \geq 0$, we have

$$(z_1^2 - z_2^2) \cdot V_{(0,-)}^r \subset V_{(0,+)}^{r+2}.$$

This proves (5.4), (5.5). Namely (1) and (3) are proved. Similarly, we can prove (2) and (4).

Let

$$m_1 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}.$$

Then M is generated by m_1, m_2 and we have

$$m_1 = \mathrm{Ad}(\exp \pi X_3) = \mathrm{Ad}\left(\begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}\right) \tag{5.6}$$

and

$$m_2 = \mathrm{Ad}(\exp \pi X_1) = \mathrm{Ad}(C)\mathrm{Ad}\left(\begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}\right)\mathrm{Ad}(C). \tag{5.7}$$

Since M is abelian and m_1, m_2 are of order two, for each γ in $\hat{K}$, its representation space V^γ is decomposed as

$$V^\gamma = V_{(+,+)}^\gamma + V_{(+,-)}^\gamma + V_{(-,+)}^\gamma + V_{(-,-)}^\gamma, \tag{5.8}$$

(orthogonal direct),
where

$$\begin{aligned} \pi_\gamma(m_1)|_{V_{(+,+)}^\gamma} &= 1, & \pi_\gamma(m_2)|_{V_{(+,+)}^\gamma} &= 1, \\ \pi_\gamma(m_1)|_{V_{(+,-)}^\gamma} &= 1, & \pi_\gamma(m_2)|_{V_{(+,-)}^\gamma} &= -1, \\ \pi_\gamma(m_1)|_{V_{(-,+)}^\gamma} &= -1, & \pi_\gamma(m_2)|_{V_{(-,+)}^\gamma} &= 1, \\ \pi_\gamma(m_1)|_{V_{(-,-)}^\gamma} &= -1, & \pi_\gamma(m_2)|_{V_{(-,-)}^\gamma} &= -1. \end{aligned}$$

LEMMA 5.2. *There is an M-intertwining correspondence between* (5.1) *and* (5.8), *that is, whenever $n/2$ is odd,*

$$V^n_{(0,+)} \to V^\gamma_{(-,-)},$$

$$V^n_{(0,-)} \to V^\gamma_{(-,+)},$$

$$V^n_{(1,+)} \to V^\gamma_{(+,-)},$$

$$V^n_{(1,-)} \to V^\gamma_{(+,+)},$$

whenever $n/2$ is even,

$$V^n_{(0,+)} \to V^\gamma_{(+,+)},$$

$$V^n_{(0,-)} \to V^\gamma_{(+,-)},$$

$$V^n_{(1,+)} \to V^\gamma_{(-,+)},$$

$$V^n_{(1,-)} \to V^\gamma_{(-,-)}.$$

PROOF. According to (4.1) and (5.6), we have

$$\begin{aligned}\pi_\gamma(m_1)(z_1^{n-k}z_2^k) &= \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}\right)(z_1^{n-k}z_2^k)\\ &= (-1)^{n/2+k}z_1^{n-k}z_2^k.\end{aligned}$$

The signature of $\pi_\gamma(m_1)$ is determined by $n/2+k$. Combining this fact with (5.7) and Lemma 5.1, we have the desired results.

For simplicity, we denote $V^\gamma_{(\cdot,\cdot)}$ by $(\cdot, \cdot)$.

LEMMA 5.3. *Let w_i $(i=1, 2)$ be as in §4. Then $\pi_\gamma(\mathrm{Ad}(C))$ and $\pi_\gamma(w_i)$ satisfy the following diagram.*

$$\begin{aligned}(+,+) &\xrightarrow{\pi_\gamma(\mathrm{Ad}(C))} (+,+)\\ (+,-) &\longrightarrow (-,+)\\ (-,+) &\longrightarrow (+,-)\\ (-,-) &\longrightarrow (-,-)\end{aligned}$$

$$\begin{aligned}(+,+) &\xrightarrow{\pi_\gamma(w_1)} (+,+) & (+,+) &\xrightarrow{\pi_\gamma(w_2)} (+,+)\\ (+,-) &\longrightarrow (+,-) & (+,-) &\longrightarrow (-,-)\\ (-,+) &\longrightarrow (-,-) & (-,+) &\longrightarrow (-,+)\\ (-,-) &\longrightarrow (-,+) & (-,-) &\longrightarrow (+,-)\end{aligned}$$

Since the proof is simple, we leave it to the reader.

§6. The determinant of the C-function

In this section, we shall give an explicit formula of the determinant of $B_\gamma^\sigma(\bar{P}:P:\nu)$. Let

$$\pi_\gamma^\sigma(w) = \pi_\gamma(w)|_{V_\sigma^\gamma}, \qquad (w \in M')$$

and for ν in $\mathfrak{a}_C^*$,

$$\alpha^{+,+}(\nu, \gamma) = \prod_{\substack{n/2-k\equiv 0 \\ 0\le k\le n/2}} \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-(n/2-k)}{2}\right)\Gamma\left(\frac{s+1+(n/2-k)}{2}\right)},$$

$$\alpha^{+,-}(\nu, \gamma) = \prod_{\substack{n/2-k\equiv 0 \\ 0\le k< n/2}} \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-(n/2-k)}{2}\right)\Gamma\left(\frac{s+1+(n/2-k)}{2}\right)},$$

$$\alpha^{-,+}(\nu, \gamma) = \prod_{\substack{n/2-k\equiv 1 \\ 0\le k\le n/2}} \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-(n/2-k)}{2}\right)\Gamma\left(\frac{s+1+(n/2-k)}{2}\right)},$$

$$\alpha^{-,-}(\nu, \gamma) = \prod_{\substack{n/2-k\equiv 1 \\ 0\le k< n/2}} \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-(n/2-k)}{2}\right)\Gamma\left(\frac{s+1+(n/2-k)}{2}\right)},$$

where $s = \nu_1 - \nu_2$, and n is a nonnegative even integer satisfying (4.1).

LEMMA 6.1. *Suppose γ is in $\hat{K}$ and σ is in $\hat{M}$ such that $V_\sigma^\gamma \neq \{0\}$. Then we have*

$$\det(B_\gamma^\sigma(w_1, \nu)) = \alpha^\sigma(\nu, \gamma).$$

The assertion of the lemma is an immediate consequence of Corollary 4.4 (1) and Lemma 5.2. The proof is left to the reader.

THEOREM 6.2. *Suppose γ is in $\hat{K}$ and σ is in $\hat{M}$ such that $V_\sigma^\gamma \neq \{0\}$. Then we have*

(1) *if* $\sigma = (+, +)$,

$$\det(B_\gamma^\sigma(\bar{P}:P:\nu)) = \mathrm{Const}\cdot\alpha^{+,+}(w_2 w_1 \nu, \gamma)\alpha^{+,+}(w_1\nu, \gamma)\alpha^{+,+}(\nu, \gamma),$$

(2) *if* $\sigma = (+, -)$,

$$\det(B_\gamma^\sigma(\bar{P}:P:\nu)) = \mathrm{Const}\cdot\alpha^{-,-}(w_2 w_1 \nu, \gamma)\alpha^{-,+}(w_1\nu, \gamma)\alpha^{+,-}(\nu, \gamma),$$

(3) *if* $\sigma=(-,+)$,

$$\det(B_\gamma^\sigma(\bar{P}:P:\nu))=\mathrm{Const}\cdot\alpha^{+,-}(w_2w_1\nu,\gamma)\alpha^{-,-}(w_1\nu,\gamma)\alpha^{-,+}(\nu,\gamma)\,,$$

(4) *if* $\sigma=(-,-)$,

$$\det(B_\gamma^\sigma(\bar{P}:P:\nu))=\mathrm{Const}\cdot\alpha^{-,+}(w_2w_1\nu,\gamma)\alpha^{+,-}(w_1\nu,\gamma)\alpha^{-,-}(\nu,\gamma)\,,$$

PROOF. We shall prove (2). By Lemma 5.3 we have

$$\begin{aligned}B_\gamma^{(+,-)}(\bar{P}:P:\nu)&=B_\gamma^{(+,-)}(w_1,\nu)\pi_\gamma^{(+,-)}(w_1)B_\gamma^{(+,-)}(w_2,w_1\nu)\pi_\gamma^{(-,-)}(w_2)\\&\quad\cdot B_\gamma^{(-,-)}(w_1,w_2w_1\nu)\pi_\gamma^{(-,+)}(w_1)\pi_\gamma^{(+,-)}(w_0)\,.\end{aligned}\tag{6.1}$$

By Corollary 4.4 (2) and Lemma 5.3 we obtain

$$B_\gamma^{(+,-)}(w_2,w_1\nu)=\pi_\gamma(\mathrm{Ad}(C))B_\gamma^{(-,+)}(w_1,-(\mathrm{Ad}(C)\cdot w_1\nu))\pi_\gamma(\mathrm{Ad}(C))\,.$$

Therefore (6.1) is equal to

$$\begin{aligned}&B_\gamma^{(+,-)}(w_1,\nu)\pi_\gamma^{(+,-)}(w_1)\pi_\gamma^{(-,+)}(\mathrm{Ad}(C))\\&\cdot B_\gamma^{(-,+)}(w_1,-(\mathrm{Ad}(C)\cdot w_1\nu))\pi_\gamma^{(+,-)}(\mathrm{Ad}(C))\pi_\gamma^{(-,-)}(w_2)\\&\cdot B_\gamma^{(-,-)}(w_1,w_2w_1\nu)\pi_\gamma^{(-,+)}(w_1)\pi_\gamma^{(+,-)}(w_0)\,.\end{aligned}\tag{6.2}$$

Let $i=0$ or 1 and σ' in M'. We extend $B_\gamma^{\sigma'}(w_i,\cdot)$ to an operator $\tilde{B}_\gamma^\sigma(w_i,\cdot)$ of V^γ by

$$\tilde{B}_\gamma^{\sigma'}(w_i,\cdot)=\begin{cases}B_\gamma^{\sigma'}(w_i,\cdot) & \text{on } V_{\sigma'}^\gamma\\ \text{identity} & \text{otherwise}\end{cases}\tag{6.3}$$

and define

$$\begin{aligned}B_\gamma^\sigma(\bar{P}:P:\nu)&=\tilde{B}_\gamma^{(+,-)}(w_1,\nu)\pi_\gamma(w_1)\pi_\gamma(\mathrm{Ad}(C))\\&\quad\cdot\tilde{B}_\gamma^{(-,+)}(w_1,-(\mathrm{Ad}(C)\cdot w_1\nu))\pi_\gamma(\mathrm{Ad}(C))\pi_\gamma(w_2)\\&\quad\cdot\tilde{B}_\gamma^{(-,-)}(w_1,w_2w_1\nu)\pi_\gamma(w_1)\pi_\gamma(w_0)\,.\end{aligned}\tag{6.4}$$

Then from (6.2) we have

$$\tilde{B}_\gamma^\sigma(\bar{P}:P:\nu)|_{V_\sigma^\gamma}=B_\gamma^\sigma(\bar{P}:P:\nu)\tag{6.5}$$

and

$$\det(B_\gamma^\sigma(\bar{P}:P:\nu)=d_1\cdot\det(B_\gamma^\sigma(\bar{P}:P:\nu))\,,\tag{6.6}$$

where d_1 is a nonzero constant which is independent of ν. On the other hand, from (6.3) and (6.4) we have

$$\det(B_\gamma^\sigma(\bar{P}:P:\nu)) = d_2\cdot\det(B_\gamma^{(+,-)}(w_1,\nu))\det(B_\gamma^{(-,+)}(w_1,-(\mathrm{Ad}(C)\cdot w_1\nu))) \cdot\det(B_\gamma^{(-,-)}(w_1,w_2w_1\nu), \tag{6.7}$$

where d_2 is a constant such that $|d_2|=1$. By Lemma 6.1 and (6.6), (6.7), we can prove (2). Similarly, we can prove the others.

§7. The C-function for $SL(4, \boldsymbol{R})$

Let G be $SL(4, \boldsymbol{R})$, the group of 4-by-4 real matrices of determinant one. Let

$$\theta = -\text{transpose},$$

$$K = SO(4),$$

$$\mathfrak{a} = \{\mathrm{diag}(x_1, x_2, x_3, x_4) | x_i \in \boldsymbol{R}, \textstyle\sum_{i=1}^4 x_i = 0\},$$

$$M = Z_K(\mathfrak{a}),$$

$$N = \left\{ g\in G \,\middle|\, g = \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix} \right\},$$

$$P = MAN,$$

and define linear functions e_i $(1\le i\le 4)$ on $\mathfrak{a}_C$ by

$$e_i(\mathrm{diag}(x_1, x_2, x_3, x_4)) = x_i.$$

Then each ν in $\mathfrak{a}_C^*$ can be written in the form

$$\nu = \nu_1e_1 + \nu_2e_2 + \nu_3e_3 + \nu_4e_4 \qquad (\nu_i\in \boldsymbol{C},\ 1\le i\le 4),$$

and we write $(\nu_1, \nu_2, \nu_3, \nu_4)$ for ν. The $\mathfrak{a}$-roots of $\mathfrak{g}$ are $e_i - e_j$ $(1\le i, j\le 4, i\ne j)$ and the simple $\mathfrak{a}$-roots are e_1-e_2, e_2-e_3, e_3-e_4. Let

$$w_1 = \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \qquad w_2 = \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 1 \end{bmatrix},$$

$$w_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{bmatrix}.$$

Their adjoint actions on $\mathfrak{a}$ are corresponding to the simple reflections. We set $w_0 = w_1 w_2 w_1 w_3 w_2 w_1$, then we have

$$A(\bar{P} : P : \sigma : \nu) = R(w_0) A_P(w_0, \sigma, \nu).$$

By the relation

$$\begin{aligned} A_P(w_0, \sigma, \nu) = {} & A_P(w_1, w_2 w_1 w_3 w_2 w_1 \sigma, w_2 w_1 w_3 w_2 w_1 \nu) \\ & \cdot A_P(w_2, w_1 w_3 w_2 w_1 \sigma, w_1 w_3 w_2 w_1 \nu) A_P(w_1, w_3 w_2 w_1 \sigma, w_3 w_2 w_1 \nu) \\ & \cdot A_P(w_3, w_2 w_1 \sigma, w_2 w_1 \nu) A_P(w_2, w_1 \sigma, w_1 \nu) A_P(w_1, \sigma, \nu) \end{aligned}$$

and Corollary 3.3, we have

$$\begin{aligned} B_\gamma(\bar{P} : P : \nu) = {} & B_\gamma(w_1, \nu) \pi_\gamma(w_1) B_\gamma(w_2, w_1 \nu) \pi_\gamma(w_2) B_\gamma(w_3, w_2 w_1 \nu) \pi_\gamma(w_3) \\ & \cdot B_\gamma(w_1, w_3 w_2 w_1 \nu) \pi_\gamma(w_1) B_\gamma(w_2, w_1 w_3 w_2 w_1 \nu) \pi_\gamma(w_2) \\ & \cdot B_\gamma(w_1, w_2 w_1 w_3 w_2 w_1 \nu) \pi_\gamma(w_1) \pi_\gamma(w_0). \end{aligned} \tag{7.1}$$

LEMMA 7.1. *If ν is in $\mathfrak{a}_C^*$, $\langle \mathrm{Re}\, \nu, \alpha \rangle > 0$ for all $\alpha > 0$, we have*

(1) $$B_\gamma(w_1, \nu) = \int_{-\infty}^{\infty} f(x)^{-(\nu_1 - \nu_2) - 1} \pi_\gamma \left[\frac{1}{f(x)} \begin{bmatrix} 1 & -x & & \\ x & 1 & & \\ & & f(x) & \\ & & & f(x) \end{bmatrix} \right]^{-1} dx,$$

(2) $$B_\gamma(w_2, \nu) = \int_{-\infty}^{\infty} f(x)^{-(\nu_2 - \nu_3) - 1} \pi_\gamma \left[\frac{1}{f(x)} \begin{bmatrix} f(x) & & & \\ & 1 & -x & \\ & x & 1 & \\ & & & f(x) \end{bmatrix} \right]^{-1} dx,$$

(3) $$B_\gamma(w_3, \nu) = \int_{-\infty}^{\infty} f(x)^{-(\nu_3 - \nu_4) - 1} \pi_\gamma \left[\frac{1}{f(x)} \begin{bmatrix} f(x) & & & \\ & f(x) & & \\ & & 1 & -x \\ & & x & 1 \end{bmatrix} \right]^{-1} dx,$$

where $f(x) = (1 + x^2)^{1/2}$.

Since the results are obtained by an easy computation, we leave the proof to the reader.

We shall recall irreducible unitary representations of K. Let $\boldsymbol{H}$ be the field of quaternion numbers, the algebra over $\boldsymbol{R}$ with basis 1, i, j, k and multiplication law $i^2 = j^2 = k^2 = -1$, $ij = k$, $ki = j$, $jk = i$. If $z = z_1 + iz_2 + jz_3 + kz_4$, we set $\bar{z} = z_1 - iz_2 - jz_3 - kz_4$ and $|z|^2 = z\bar{z}$. Then we have

$$|z|^2 = z_1^2 + z_2^2 + z_3^2 + z_4^2 .$$

Let $Sp(1)$ be the group of all quaternion numbers such that $|z|^2 = 1$.

We identify $Sp(1)$ with $SU(2)$ as follows. Each element $z = z_1 + iz_2 + jz_3 + kz_4$ in $Sp(1)$ can be written as

$$z = x + jy , \qquad (x = z_1 + iz_2, y = z_3 - iz_4) .$$

Using the above notation, we set

$$\varphi(z) = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \in SU(2) .$$

Then φ is an isomorphism from $Sp(1)$ to $SU(2)$.

We identify $\boldsymbol{H}$ with $\boldsymbol{R}^4$ by the map $w \to (w_1, w_2, w_3, w_4)$, $(w = w_1 + iw_2 + jw_3 + kw_4)$. For each (z, z') in $SP(1) \times SP(1)$ and (w_1, w_2, w_3, w_4) in $\boldsymbol{R}^4$, we define a map ψ from $Sp(1) \times Sp(1)$ to K by

$$\psi(z, z')((w_1, w_2, w_3, w_4)) = z \cdot w \cdot z'^{-1} .$$

Then $(Sp(1) \times Sp(1), \psi)$ is the universal covering group of K.

We set

$$\imath = \psi \cdot (\varphi^{-1} \times \varphi^{-1}) .$$

LEMMA 7.2. *$(SU(2) \times SU(2), \imath)$ is the universal covering group of K. Furthermore, for each γ in $\hat{K}$ there exist unique nonnegative integers m, n satisfying the following relations,*

$$m + n \equiv 0 \ (2) \tag{7.2}$$

and

$$\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n \simeq \pi_\gamma \circ \imath , \qquad \text{(unitarily equivalent)} \tag{7.3}$$

where $\tilde{\pi}_n$ $(n \geq 0)$ are defined in §4 and $\hat{\otimes}$ denotes the exterior tensor product.

We identify V^γ with $V^m \otimes V^n$ by Lemma 7.2.

LEMMA 7.3. *Suppose γ is in $\hat{K}$, m, n are nonnegative inegers satisfying (7.2) and (7.3) and V^m, V^n are the spaces defined in §4. Let*

$$u_i = z_1^{m-i} z_2^i \quad (0 \leq i \leq m) , \qquad v_j = z_1^{n-j} z_2^j \quad (0 \leq j \leq n) .$$

Then for x in $\boldsymbol{R}$ we have the following relations,

$$(1)\qquad \pi_\gamma\left[\frac{1}{f(x)}\begin{bmatrix}1 & -x & & \\ x & 1 & & \\ & & f(x) & \\ & & & f(x)\end{bmatrix}\right]^{-1}(u_i\otimes v_j)$$

$$=\left(\frac{1-\sqrt{-1}x}{f(x)}\right)^{(m-n)/2-(i-j)}(u_i\otimes v_j)\,,$$

$$(2)\qquad \pi_\gamma\left[\frac{1}{f(x)}\begin{bmatrix}f(x) & & & \\ & f(x) & & \\ & & 1 & -x \\ & & x & 1\end{bmatrix}\right]^{-1}(u_i\otimes v_j)$$

$$=\left(\frac{1-\sqrt{-1}x}{f(x)}\right)^{(m+n)/2-(i+j)}(u_i\otimes v_j)\,,\qquad (0\le i\le m,\ 0\le j\le n).$$

PROOF. We shall prove (1). Let $\{X_i\}_{1\le i\le 3}$ be the basis of $\mathfrak{su}(2)$ given in Lemma 4.2. We then have for t in $\boldsymbol{R}$

$$\begin{aligned}&\tilde{\pi}_m\mathbin{\hat{\otimes}}\tilde{\pi}_n(\exp tX_3,\exp -tX_3)(u_i\otimes v_j)\\&=\tilde{\pi}_m(\exp tX_3)u_i\otimes\tilde{\pi}_n(\exp -tX_3)v_j\\&=(e^{(m/2-i)\sqrt{-1}t}u_i)\otimes(e^{-(n/2-j)\sqrt{-1}t}v_j)\\&=e^{((m-n)/2-(i-j))\sqrt{-1}t}u_i\otimes v_j\,,\qquad(0\le i\le m,0\le j\le n)\end{aligned}\tag{7.4}$$

and by an easy computation, we obtain

$$\iota(\exp tX_3,\exp -tX_3)=\begin{bmatrix}\cos t & -\sin t & & \\ \sin t & \cos t & & \\ & & 1 & \\ & & & 1\end{bmatrix},\tag{7.5}$$

$$\iota(\exp tX_3,\exp tX_3)=\begin{bmatrix}1 & & & \\ & 1 & & \\ & & \cos t & -\sin t \\ & & \sin t & \cos t\end{bmatrix}.\tag{7.6}$$

From (7.4), (7.5) and Lemma 7.2 we have

$$\pi_\gamma\begin{bmatrix}\cos t & -\sin t & & \\ \sin t & \cos t & & \\ & & 1 & \\ & & & 1\end{bmatrix}u_i\otimes v_j$$

$$=e^{((m-n)/2-(i-j))\sqrt{-1}t}u_i\otimes v_j\,,\qquad(0\le i\le m,0\le j\le n)\,.$$

Putting $\cos t = f(x)^{-1}$, $\sin t = x/f(x)$, we obtain (1). (2) can be proved similarly.

We put

$$C = 2^{-1/2}\sqrt{-1}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in SU(2).$$

Then we have the following

LEMMA 7.4. *In the setting of the last lemma we have for x in* $\boldsymbol{R}$

$$\pi_\gamma\left[\frac{1}{f(x)}\begin{bmatrix} f(x) & & & \\ & 1 & -x & \\ & x & 1 & \\ & & & f(x) \end{bmatrix}\right]^{-1}$$

$$= \pi_\gamma(\imath(C, C))\pi_\gamma\left[\frac{1}{f(x)}\begin{bmatrix} 1 & -x & & \\ x & 1 & & \\ & & f(x) & \\ & & & f(x) \end{bmatrix}\right]^{-1}\pi_\gamma(\imath(C, C))^{-1}.$$

PROOF. We note that

$$C^{-1}X_1C = X_3, \tag{7.7}$$

and we have for t in $\boldsymbol{R}$

$$\begin{aligned} &\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(\exp -tX_1, \exp -tX_1) \\ &\quad = \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C(\exp -tX_3)C^{-1}, C(\exp -tX_3)C^{-1}) \\ &\quad = \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C, C)\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(\exp -tX_3, \exp -tX_3)\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C, C)^{-1}. \end{aligned} \tag{7.8}$$

By an easy computation we obtain for t in $\mathbb{R}$

$$\imath(\exp -tX_1, \exp -tX_1) = \begin{bmatrix} 1 & & & \\ & \cos t & -\sin t & \\ & \sin t & \cos t & \\ & & & 1 \end{bmatrix}. \tag{7.9}$$

Therefore, from (7.6), (7.8) and (7.9) we have

$$\pi_\gamma\left[\begin{bmatrix} 1 & & & \\ & \cos t & -\sin t & \\ & \sin t & \cos t & \\ & & & 1 \end{bmatrix}\right] = \pi_\gamma(\imath(C, C))\pi_\gamma\left[\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cos t & \sin t \\ & & -\sin t & \cos t \end{bmatrix}\right]\pi_\gamma(\imath(C, C))^{-1}.$$

Puting $\cos t = f(x)^{-1}$, $\sin t = x/f(x)$, we obtain the desired relation, and this proves the assertion of the lemma.

Let m, n be nonnegative integers and s in $\boldsymbol{C}$. For each integer i, j such that $0 \leq i \leq m$, $0 \leq j \leq n$, we set

$$\alpha_{i,j}(s, (m, n)) = \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-((m-n)/2-(i-j))}{2}\right)\Gamma\left(\frac{s+1+((m-n)/2-(i-j))}{2}\right)},$$

$$\beta_{i,j}(s, (m, n)) = \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-((m+n)/2-(i+j))}{2}\right)\Gamma\left(\frac{s+1+((m+n)/2-(i+j))}{2}\right)}.$$

As is easily seen, the element $\imath(C, C)$ in K normalizes $\mathfrak{a}_{\boldsymbol{C}}^*$. Thus $\imath(C, C)$ is in M'. We then have the following

LEMMA 7.5. *Suppose ν is in $\mathfrak{a}_{\boldsymbol{C}}^*$ such that $\langle \operatorname{Re}\nu, \alpha\rangle > 0$ for all $\alpha > 0$ and let γ, m and n be as in Lemma 7.3. Then*

(1)
$$B_\gamma(w_1, \nu)(u_i \otimes v_j) = \alpha_{i,j}(\nu_1 - \nu_2, (m, n))(u_i \otimes v_j), \tag{7.10}$$
$$B_\gamma(w_3, \nu)(u_i \otimes v_j) = \beta_{i,j}(\nu_3 - \nu_4, (m, n))(u_i \otimes v_j), \tag{7.11}$$

$(0 \leq i \leq m, 0 \leq j \leq n)$.

(2)
$$\pi_\gamma(\imath(C, C))B_\gamma(w_2, \nu)\pi_\gamma(\imath(C, C))^{-1} = B_\gamma(w_3, -(\imath(C, C)\cdot\nu)), \tag{7.12}$$

where $\imath(C, C)\cdot\nu$ is the element in $\mathfrak{a}_{\boldsymbol{C}}^*$ defined by

$$\imath(C, C)\cdot\nu(H) = \nu(\imath(C, C)^{-1}\cdot H\cdot\imath(C, C)), \qquad (H \in \mathfrak{a}_{\boldsymbol{C}}).$$

PROOF. We shall prove (1). We have

$$B_\gamma(w_1, \nu)(u_i \otimes v_j) = \int_{-\infty}^{\infty} f(x)^{-(\nu_1-\nu_2)-1}\pi_\gamma\left(\frac{1}{f(x)}\begin{bmatrix} 1 & -x & & \\ x & 1 & & \\ & & f(x) & \\ & & & f(x)\end{bmatrix}\right)^{-1}(u_i \otimes v_j)dx,$$

and by Lemma 7.3 (1)

$$= \int_{-\infty}^{\infty} f(x)^{-(\nu_1-\nu_2)-1}\left(\frac{1-\sqrt{-1}x}{f(x)}\right)^{(m-n)/2-(i-j)}(u_i \otimes v_j)dx\,,$$

and by A.3

$$= \alpha_{i,j}(\nu_1 - \nu_2, (m, n))(u_i \otimes v_j)\,.$$

This proves (7.10). In the same way, we can prove (7.11). This proves (1). Furthermore, (2) is a direct consequence of Lemma 7.4.

§8. The M-isotypic components of γ

In this section we shall describe the M-isotypic components of γ in $\hat{K}$. Let m, n be the nonnegative integers given in Lemma 7.2 and $k=0$ or 1. If i, j ($i, j \in N$) satisfy $(m-n)/2-(i-j) \equiv k$, we write $(i, j) \equiv k$ for the above relation.

$$V^{(m,n)}_{(k,+,+)} = \sum_{\substack{(i,j)\equiv k \\ 0\le i\le m,\, 0\le j\le n}} C(z_1^{m-i}z_2^i + z_1^i z_2^{m-i}) \otimes (z_1^{n-j}z_2^j + z_1^j z_2^{n-j})\,,$$

$$V^{(m,n)}_{(k,+,-)} = \sum_{\substack{(i,j)\equiv k \\ 0\le i\le m,\, 0\le j\le n}} C(z_1^{m-i}z_2^i + z_1^i z_2^{m-i}) \otimes (z_1^{n-j}z_2^j - z_1^j z_2^{n-j})\,,$$

$$V^{(m,n)}_{(k,-,+)} = \sum_{\substack{(i,j)\equiv k \\ 0\le i\le m,\, 0\le j\le n}} C(z_1^{m-i}z_2^i - z_1^i z_2^{m-i}) \otimes (z_1^{n-j}z_2^j + z_1^j z_2^{n-j})\,,$$

$$V^{(m,n)}_{(k,-,-)} = \sum_{\substack{(i,j)\equiv k \\ 0\le i\le m,\, 0\le j\le n}} C(z_1^{m-i}z_2^i - z_1^i z_2^{m-i}) \otimes (z_1^{n-j}z_2^j - z_1^j z_2^{n-j})\,.$$

Then we have

$$V^m \otimes V^n = \textstyle\sum_{k=0}^1 V^{(m,n)}_{(k,+,+)} + V^{(m,n)}_{(k,+,-)} + V^{(m,n)}_{(k,-,+)} + V^{(m,n)}_{(k,-,-)}\,, \tag{8.1}$$

(orthogonal direct).

Let

$$m_1 = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix},$$

$$m_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}.$$

Then M is generated by m_1, m_2, m_3 and, from (7.5), (7.6) and (7.7), we have

$$m_1 = \iota(\exp \pi X_3, \exp -\pi X_3)$$
$$= \iota\left(\begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}^{-1}\right), \tag{8.2}$$

$$m_2 = \iota(\exp -\pi X_1, \exp -\pi X_1)$$
$$= \iota(C, C)\iota\left(\begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}^{-1}, \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}^{-1}\right)\iota(C, C)^{-1}, \tag{8.3}$$

and

$$m_3 = \iota\left(\begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}\right). \tag{8.4}$$

Since M is abelian and m_1, m_2, m_3 are of order two, for each γ in $\hat{K}$ its representation space V^γ is decomposed as

$$\begin{aligned} V^\gamma = {} & V^\gamma_{(+,+,+)} + V^\gamma_{(+,+,-)} + V^\gamma_{(+,-,+)} + V^\gamma_{(+,-,-)} \\ & + V^\gamma_{(-,+,+)} + V^\gamma_{(-,+,-)} + V^\gamma_{(-,-,+)} + V^\gamma_{(-,-,-)}, \end{aligned} \tag{8.5}$$

(orthogonal direct),
where

$$\begin{aligned} \pi_\gamma(m_1)|_{V^\gamma_{(+,*,*)}} &= 1, & \pi_\gamma(m_1)|_{V^\gamma_{(-,*,*)}} &= -1, \\ \pi_\gamma(m_2)|_{V^\gamma_{(*,+,*)}} &= 1, & \pi_\gamma(m_2)|_{V^\gamma_{(*,-,*)}} &= -1, \\ \pi_\gamma(m_3)|_{V^\gamma_{(*,*,+)}} &= 1, & \pi_\gamma(m_3)|_{V^\gamma_{(*,*,-)}} &= -1, \end{aligned}$$

$*$ denoting $+$ or $-$.

LEMMA 8.1. *Suppose γ is in $\hat{K}$, m, n are nonnegative integers given in Lemma 7.2. Then there is an M-intertwining correspondence between* (8.1) *and* (8.5), *that is, whenever $(m-n)/2$ and n are even*

$$\begin{gathered} V^{(m,n)}_{(0,+,+)} + V^{(m,n)}_{(0,-,-)} \to V^\gamma_{(+,+,+)}, \\ V^{(m,n)}_{(0,+,-)} + V^{(m,n)}_{(0,-,+)} \to V^\gamma_{(+,-,+)}, \\ V^{(m,n)}_{(1,+,+)} + V^{(m,n)}_{(1,-,-)} \to V^\gamma_{(-,+,-)}, \\ V^{(m,n)}_{(1,+,-)} + V^{(m,n)}_{(1,-,+)} \to V^\gamma_{(-,-,-)}, \\ V^\gamma_{(+,+,-)} = V^\gamma_{(+,-,-)} = V^\gamma_{(-,+,+)} = V^\gamma_{(-,-,+)} = \{0\}; \end{gathered}$$

whenever $(m-n)/2$ is odd and n even

$$V_{(0,+,+)}^{(m,n)} + V_{(0,-,-)}^{(m,n)} \to V_{(+,-,+)}^{\gamma},$$
$$V_{(0,+,-)}^{(m,n)} + V_{(0,-,+)}^{(m,n)} \to V_{(+,+,+)}^{\gamma},$$
$$V_{(1,+,+)}^{(m,n)} + V_{(1,+,+)}^{(m,n)} \to V_{(-,-,-)}^{\gamma},$$
$$V_{(1,+,-)}^{(m,n)} + V_{(1,-,+)}^{(m,n)} \to V_{(-,+,-)}^{\gamma},$$
$$V_{(+,+,-)}^{\gamma} = V_{(+,-,-)}^{\gamma} = V_{(-,+,+)}^{\gamma} = V_{(-,-,+)}^{\gamma} = \{0\};$$

whenever $(m-n)/2$ *is even and* n *odd*

$$V_{(0,+,+)}^{(m,n)} + V_{(0,-,-)}^{(m,n)} \to V_{(+,-,-)}^{\gamma},$$
$$V_{(0,+,-)}^{(m,n)} + V_{(0,-,+)}^{(m,n)} \to V_{(+,+,-)}^{\gamma},$$
$$V_{(1,+,+)}^{(m,n)} + V_{(1,+,+)}^{(m,n)} \to V_{(-,-,+)}^{\gamma},$$
$$V_{(1,+,-)}^{(m,n)} + V_{(1,-,+)}^{(m,n)} \to V_{(-,+,+)}^{\gamma},$$
$$V_{(+,+,+)}^{\gamma} = V_{(+,-,+)}^{\gamma} = V_{(-,+,-)}^{\gamma} = V_{(-,-,-)}^{\gamma} = \{0\};$$

whenever $(m-n)/2$ *and* n *are odd*

$$V_{(0,+,+)}^{(m,n)} + V_{(0,-,-)}^{(m,n)} \to V_{(+,+,-)}^{\gamma},$$
$$V_{(0,+,-)}^{(m,n)} + V_{(0,-,+)}^{(m,n)} \to V_{(+,-,-)}^{\gamma},$$
$$V_{(1,+,+)}^{(m,n)} + V_{(1,+,+)}^{(m,n)} \to V_{(-,+,+)}^{\gamma},$$
$$V_{(1,+,-)}^{(m,n)} + V_{(1,-,+)}^{(m,n)} \to V_{(-,-,+)}^{\gamma},$$
$$V_{(+,+,+)}^{\gamma} = V_{(+,-,+)}^{\gamma} = V_{(-,+,-)}^{\gamma} = V_{(-,-,-)}^{\gamma} = \{0\}.$$

PROOF. Let m, n be as in the lemma and i, j integers such that $0 \le i \le m$, $0 \le j \le n$. According to Lemma 7.2 and (8.2), we have

$$\begin{aligned}
&\pi_\gamma(m_1)(z_1^{m-i}z_2^i \otimes z_1^{n-j}z_2^j) \\
&= \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}\right)(z_1^{m-i}z_2^i \otimes z_1^{n-j}z_2^j) \\
&= (\sqrt{-1}z_1)^{m-i}(-\sqrt{-1}z_2)^i \otimes (-\sqrt{-1}z_1)^{n-j}(\sqrt{-1}z_2)^j \\
&= (-1)^{(m-n)/2+(i-j)} z_1^{m-i}z_2^i \otimes z_1^{n-j}z_2^j, \qquad (0 \le i \le m, 0 \le j \le n)
\end{aligned}$$

and the signature of $\pi_\gamma(m_1)$ is determined by $(m-n)/2-(i-j)$. Furthermore, by Lemma 7.2 and (8.4) we have

$$\pi_\gamma(m_3)(z_1^{m-i}z_2^i \otimes z_1^{n-j}z_2^j)$$
$$= \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}\right)(z_1^{m-i}z_2^i \otimes z_1^{n-j}z_2^j)$$
$$= \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}, (-1) \times \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}\right)(z_1^{m-i}z_2^i \otimes z_1^{n-j}z_2^j)$$
$$= (-1)^n(-1)^{(m-n)/2+(i-j)}z_1^{m-i}z_2^i \otimes z_1^{n-j}z_2^j, \qquad (0 \le i \le m, 0 \le j \le n)$$

and the signature of $\pi_\gamma(m_3)$ is determined by that of $\pi_\gamma(m_1)$ and n. On the other hand, by Lemma 7.2 and (8.3) we have

$$\pi_\gamma(m_2)((z_1^{m-i}z_2^i + z_1^i z_2^{m-i}) \otimes (z_1^{n-j}z_2^j + z_1^j z_2^{m-j}))$$
$$= \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C, C)\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}\right)$$
$$\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C, C)^{-1}((z_1^{m-i}z_2^i + z_1^i z_2^{m-i}) \otimes (z_1^{n-j}z_2^j + z_1^j z_2^{n-j})).$$

We may assume $m - i \ge i$, $n - j \ge j$ without any loss of generality. Thus the last expression equals

$$\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C, C)\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}\right)$$
$$((2^{-1/2}\cdot\sqrt{-1})^{m+n}((z_1 + z_2)^{m-i}(z_1 - z_2)^i + (z_1 + z_2)^i(z_1 - z_2)^{m-i})$$
$$\otimes ((z_1 + z_2)^{n-j}(z_1 - z_2)^j + (z_1 + z_2)^j(z_1 - z_2)^{n-j}))$$
$$= \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C, C)\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}\right)$$
$$((2^{-1/2}\cdot\sqrt{-1})^{m+n}((z_1^2 - z_2^2)^i((z_1 + z_2)^{m-2i} + (z_1 - z_2)^{m-2i})$$
$$\otimes (z_1^2 - z_2^2)^j((z_1 + z_2)^{n-2j} + (z_1 - z_2)^{n-2j})),$$

by (5.2) and (5.3)

$$= \tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n(C, C)\tilde{\pi}_m \hat{\otimes} \tilde{\pi}_n\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}\right)$$
$$\left((-2^{-1/2}\cdot\sqrt{-1})^{m+n}\cdot\left((z_1^2 - z_2^2)^i \sum_{\substack{0 \le p \le m-2i \\ p \equiv 0}} 2\binom{m-2i}{p} z_1^{m-2i-p}z_2^p\right)\right.$$

$$\otimes\left((z_1^2 - z_2^2)^j \sum_{\substack{0\le q\le n-2j\\ q\equiv 0}} 2\binom{n-2j}{q} z_1^{n-2j-q} z_2^q\right)\Bigg)$$

$$= (-1)^{(m+n)/2}\tilde{\pi}_m \hat{\otimes}\, \tilde{\pi}_n(C, C)\Bigg((-2^{-1/2}\cdot\sqrt{-1})^{m+n}\cdot(z_1^2 - z_2^2)^{i+j}$$

$$\cdot\left(\sum_{\substack{0\le p\le m-2i\\ p\equiv 0}} 2\binom{m-2i}{p} z_1^{m-2i-p} z_2^p\right)\otimes\left(\sum_{\substack{0\le q\le n-2j\\ q\equiv 0}} 2\binom{n-2j}{q} z_1^{n-2j-q} z_2^q\right)\Bigg)$$

$$= (-1)^{(m+n)/2}((z_1^{m-i}z_2^i + z_1^i z_2^{m-i})\otimes(z_1^{n-j}z_2^j + z_1^j z_2^{n-j}))\,.$$

In the same way, we obtain

$$\pi_\gamma(m_2)((z_1^{m-i}z_2^i - z_1^i z_2^{m-i})\otimes(z_1^{n-j}z_2^j - z_1^j z_2^{n-j}))$$
$$= (-1)^{(m+n)/2}((z_1^{m-i}z_2^i - z_1^i z_2^{m-i})\otimes(z_1^{n-j}z_2^j - z_1^j z_2^{n-j}))$$

and

$$\pi_\gamma(m_2)((z_1^{m-i}z_2^i \pm z_1^i z_2^{m-i})\otimes(z_1^{n-j}z_2^j \pm(-1)z_1^j z_2^{n-j}))$$
$$= (-1)^{(m+n)/2-1}((z_1^{m-i}z_2^i \pm z_1^i z_2^{m-i})\otimes(z_1^{n-j}z_2^j \pm(-1)z_1^j z_2^{n-j}))\,.$$

These formulae lead us to the assertion of the lemma.

LEMMA 8.2. *Suppose γ is in $\hat{K}$ and C as above. Then we have the following relations,*

(1) $$\pi_\gamma(\iota(C, C))\pi_\gamma(m_1)\pi_\gamma(\iota(C, C))^{-1} = \pi_\gamma(m_1 m_2 m_3)\,,$$

(2) $$\pi_\gamma(\iota(C, C))\pi_\gamma(m_2)\pi_\gamma(\iota(C, C))^{-1} = \pi_\gamma(m_3)\,,$$

(3) $$\pi_\gamma(\iota(C, C))\pi_\gamma(m_3)\pi_\gamma(\iota(C, C))^{-1} = \pi_\gamma(m_2)\,.$$

PROOF. We shall prove (1). From (8.2) we have

$$\pi_\gamma(\iota(C, C))\pi_\gamma(m_1)\pi_\gamma(\iota(C, C))^{-1} = \pi_\gamma(\iota(C, C)\iota(\exp \pi X_3, \exp -\pi X_3)\iota(C, C)^{-1})\,.$$

By (7.7) and the relation

$$\iota(\exp -tX_1, \exp -tX_1) = \begin{bmatrix} \cos t & & & -\sin t \\ & 1 & & \\ & & 1 & \\ \sin t & & & \cos t \end{bmatrix}, \qquad (t \in \boldsymbol{R})$$

the last expression equals

$$= \pi_\gamma(m_1 m_2 m_3)\,.$$

Next we shall prove (2) and (3). From (8.4) we have

$$\pi_\gamma(\iota(C, C))\pi_\gamma(m_3)\pi_\gamma(\iota(C, C))^{-1}$$
$$= \pi_\gamma\left(\iota(C, C)\iota\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}\right)\iota(C, C)^{-1}\right).$$

Since $\left(\begin{pmatrix}-1 & \\ & -1\end{pmatrix}, \begin{pmatrix}-1 & \\ & -1\end{pmatrix}\right)$ is in the kernel of ι, the last expression is equal to

$$\pi_\gamma\left(\iota(C, C)\iota\left(\begin{pmatrix}-\sqrt{-1} & \\ & \sqrt{-1}\end{pmatrix}, \begin{pmatrix}-\sqrt{-1} & \\ & \sqrt{-1}\end{pmatrix}\right)\iota(C, C)^{-1}\right)$$
$$= \pi_\gamma\left(\iota(C, C)\iota\left(\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}, \begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix}^{-1}\right)\iota(C, C)^{-1}\right),$$

by (8.3)

$$= \pi_\gamma(m_2).$$

This proves (2). We have (3) from (2).

For simplicity, we denote $V^\gamma_{(*,*,*)}$ by $(*, *, *)$.

COROLLARY 8.3. *Suppose γ is in $\hat{K}$ and C as above. Then we have the following diagram.*

$$\begin{aligned}
(+, +, +) &\xrightarrow{\pi_\gamma(\iota(C,C))} (+, +, +)\\
(+, -, +) &\longrightarrow (-, +, -)\\
(-, +, -) &\longrightarrow (+, -, +)\\
(-, -, -) &\longrightarrow (-, -, -)\\
(+, +, -) &\longrightarrow (-, -, +)\\
(+, -, -) &\longrightarrow (+, -, -)\\
(-, +, +) &\longrightarrow (-, +, +)\\
(-, -, +) &\longrightarrow (+, +, -)
\end{aligned}$$

LEMMA 8.4. *Let γ be in $\hat{K}$ and w_i $(1 \leq i \leq 3)$ be as in §7, then $\pi_\gamma(w_i)$ satisfy the following diagram.*

$$\begin{aligned}
(+, +, +) &\xrightarrow{\pi_\gamma(w_1)} (+, +, +)\\
(+, -, +) &\longrightarrow (+, -, +)
\end{aligned}$$

$$(-,+,-) \longrightarrow (-,-,-)$$
$$(-,-,-) \longrightarrow (-,+,-)$$
$$(+,+,-) \longrightarrow (+,+,-)$$
$$(+,-,-) \longrightarrow (+,-,-)$$
$$(-,+,+) \longrightarrow (-,-,+)$$
$$(-,-,+) \longrightarrow (-,+,+)$$

$$\begin{array}{lll}
(+,+,+) \xrightarrow{\pi_\gamma(w_2)} (+,+,+) & \quad & (+,+,+) \xrightarrow{\pi_\gamma(w_3)} (+,+,+) \\
(+,-,+) \longrightarrow (-,-,-) & & (+,-,+) \longrightarrow (+,-,+) \\
(-,+,-) \longrightarrow (-,+,-) & & (-,+,-) \longrightarrow (-,-,-) \\
(-,-,-) \longrightarrow (+,-,+) & & (-,-,-) \longrightarrow (-,+,-) \\
(+,+,-) \longrightarrow (+,+,-) & & (+,+,-) \longrightarrow (+,-,-) \\
(+,-,-) \longrightarrow (-,-,+) & & (+,-,-) \longrightarrow (+,+,-) \\
(-,+,+) \longrightarrow (-,+,+) & & (-,+,+) \longrightarrow (-,+,+) \\
(-,-,+) \longrightarrow (+,-,-) & & (-,-,+) \longrightarrow (-,-,+)
\end{array}$$

Since the proof is simple, it is left to the reader.

§ 9. The determinant of the C-function

In this section we shall give an explicit formula of the determinant of $B_\gamma^\sigma(\bar{P}:P:\nu)$. We define the functions $\alpha^\sigma(\nu,(m,n))$ and $\beta^\sigma(\nu,(m,n))$ in ν ($\nu \in \mathfrak{a}_C^*$) as follows:

if $(m-n)/2$ and n are even,

$$\alpha^{+,+,+}(\nu,(m,n)) = \prod_{\substack{(i,j)\equiv 0 \\ 0\le i\le [m/2] \\ 0\le j\le [n/2]}} \alpha_{i,j}(\nu_1-\nu_2,(m,n)) \prod_{\substack{(k,l)\equiv 0 \\ 0\le k<[m/2] \\ 0\le l<[n/2]}} \beta_{k,l}(\nu_1-\nu_2,(m,n)),$$

$$\alpha^{+,-,+}(\nu,(m,n)) = \prod_{\substack{(i,j)\equiv 0 \\ 0\le i<[m/2] \\ 0\le j<[n/2]}} \alpha_{i,j}(\nu_1-\nu_2,(m,n)) \prod_{\substack{(k,l)\equiv 0 \\ 0\le k<[m/2] \\ 0\le l<[n/2]}} \beta_{k,l}(\nu_1-\nu_2,(m,n)),$$

$$\alpha^{-,+,-}(\nu,(m,n)) = \prod_{\substack{(i,j)\equiv 1 \\ 0\le i\le [m/2] \\ 0\le j\le [n/2]}} \alpha_{i,j}(\nu_1-\nu_2,(m,n)) \prod_{\substack{(k,l)\equiv 1 \\ 0\le k<[m/2] \\ 0\le l<[n/2]}} \beta_{k,l}(\nu_1-\nu_2,(m,n)),$$

$$\alpha^{-,-,-}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i<[m/2]\\ 0\le j<[n/2]}} \alpha_{i,j}(v_1-v_2,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \beta_{k,l}(v_1-v_2,(m,n)),$$

$$\beta^{+,+,+}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{+,-,+}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i<[m/2]\\ 0\le j<[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{-,+,-}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{-,-,-}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i<[m/2]\\ 0\le j<[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

the others are equal to 1;

if $(m-n)/2$ is odd and n even,

$$\alpha^{+,-,+}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}} \alpha_{i,j}(v_1-v_2,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{+,+,+}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i<[m/2]\\ 0\le j<[n/2]}} \alpha_{i,j}(v_1-v_2,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{-,-,-}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}} \alpha_{i,j}(v_1-v_2,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{-,+,-}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i<[m/2]\\ 0\le j<[n/2]}} \alpha_{i,j}(v_1-v_2,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \beta_{k,l}(v_1-v_2,(m,n)),$$

$$\beta^{+,-,+}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{+,+,+}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i<[m/2]\\ 0\le j<[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{-,-,-}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{-,+,-}(v,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i<[m/2]\\ 0\le j<[n/2]}} \beta_{i,j}(v_3-v_4,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(v_3-v_4,(m,n)),$$

the others are equal to 1;

if $(m-n)/2$ is even and n odd,

$$\alpha^{+,-,-}(v,(m,n))=\prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{+,+,-}(v,(m,n))=\prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{-,-,+}(v,(m,n))=\prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{-,+,+}(v,(m,n))=\prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\beta^{+,-,-}(v,(m,n))=\prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\beta_{i,j}(v_3-v_4,(m,n))\prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{+,+,-}(v,(m,n))=\prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\beta_{i,j}(v_3-v_4,(m,n))\prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{-,-,+}(v,(m,n))=\prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\beta_{i,j}(v_3-v_4,(m,n))\prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\alpha_{k,l}(v_3-v_4,(m,n)),$$

$$\beta^{-,+,+}(v,(m,n))=\prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\beta_{i,j}(v_3-v_4,(m,n))\prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\alpha_{k,l}(v_3-v_4,(m,n)),$$

the others are equal to 1;

if $(m-n)/2$ and n are odd,

$$\alpha^{+,+,-}(v,(m,n))=\prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{+,-,-}(v,(m,n))=\prod_{\substack{(i,j)\equiv 0\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{-,+,+}(v,(m,n))=\prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\alpha^{-,-,+}(v,(m,n))=\prod_{\substack{(i,j)\equiv 1\\ 0\le i\le[m/2]\\ 0\le j\le[n/2]}}\alpha_{i,j}(v_1-v_2,(m,n))\prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}}\beta_{k,l}(v_1-v_2,(m,n)),$$

$$\beta^{+,+,-}(\nu,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i\le [m/2]\\ 0\le j\le [n/2]}} \beta_{i,j}(\nu_3-\nu_4,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(\nu_3-\nu_4,(m,n)),$$

$$\beta^{+,-,-}(\nu,(m,n)) = \prod_{\substack{(i,j)\equiv 0\\ 0\le i\le [m/2]\\ 0\le j\le [n/2]}} \beta_{i,j}(\nu_3-\nu_4,(m,n)) \prod_{\substack{(k,l)\equiv 0\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(\nu_3-\nu_4,(m,n)),$$

$$\beta^{-,+,+}(\nu,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i\le [m/2]\\ 0\le j\le [n/2]}} \beta_{i,j}(\nu_3-\nu_4,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(\nu_3-\nu_4,(m,n)),$$

$$\beta^{-,-,+}(\nu,(m,n)) = \prod_{\substack{(i,j)\equiv 1\\ 0\le i\le [m/2]\\ 0\le j\le [n/2]}} \beta_{i,j}(\nu_3-\nu_4,(m,n)) \prod_{\substack{(k,l)\equiv 1\\ 0\le k<[m/2]\\ 0\le l<[n/2]}} \alpha_{k,l}(\nu_3-\nu_4,(m,n)),$$

the others are equal to 1.

LEMMA 9.1. *Suppose γ in $\hat{K}$ and σ in $\hat{M}$ satisfy $V_\sigma^\gamma \neq \{0\}$. Then we have*

$$\det(B_\gamma^\sigma(w_1,\nu)) = \alpha^\sigma(\nu,(m,n)),$$

$$\det(B_\gamma^\sigma(w_3,\nu)) = \beta^\sigma(\nu,(m,n)).$$

PROOF. We shall prove only in the case that σ is $(+,+,+)$ and $(n-m)/2$, n are even. The proof of the other cases is similar to the above one and left to the reader. Let u_i, v_j $(1\le i\le m, 1\le j\le n)$ be as in Lemma 7.3. From Lemma 8.1, $\{u_i\otimes v_j+u_{m-i}\otimes v_{n-j},\ u_i\otimes v_{n-j}+u_{m-i}\otimes v_j\}_{\substack{1\le i\le m\\ 1\le j\le n}}$ is the basis of V_γ^σ. Furthermore, by Lemma 7.5 we have

$$\begin{aligned}
&B_\gamma(w_1,\nu)(u_i\otimes v_j+u_{m-i}\otimes v_{n-j})\\
&\quad = \alpha_{i,j}(\nu_1-\nu_2,(m,n))(u_i\otimes v_j+u_{m-i}\otimes v_{n-j}),\\
&B_\gamma(w_1,\nu)(u_i\otimes v_{n-j}+u_{m-i}\otimes v_j)\\
&\quad = \beta_{i,j}(\nu_1-\nu_2,(m,n))(u_i\otimes v_{n-j}+u_{m-i}\otimes v_j), \qquad (1\le i\le m, 1\le j\le n).
\end{aligned}$$

Therefore, we obtain (1). Similarly, we can prove (2).

THEOREM 9.2. *Let γ be in $\hat{K}$ and σ in $\hat{M}$ such that $V_\sigma^\gamma\neq\{0\}$. Then we have the following relations.*

(1) *If* $\sigma=(+,+,+)$,

$$\begin{aligned}
&\det(B_\gamma^\sigma(\bar{P}:P:\nu))\\
&\quad = \mathrm{Const}\cdot\alpha^{+,+,+}(w_2w_1w_3w_2w_1\nu,\ (m,n))\beta^{+,+,+}(-\iota(C,C)\cdot w_1w_3w_2w_1\nu,\ (m,n))\\
&\qquad \cdot\alpha^{+,+,+}(w_3w_2w_1\nu,\ (m,n))\beta^{+,+,+}(w_2w_1\nu,\ (m,n))\\
&\qquad \cdot\beta^{+,+,+}(-\iota(C,C)\cdot w_1\nu,\ (m,n))\alpha^{+,+,+}(\nu,\ (m,n)).
\end{aligned}$$

(2) *If* $\sigma=(+,+,-)$,

$$\det(B^{\sigma}_{\gamma}(\bar{P}:P:v))$$

$$= \mathrm{Const}\cdot\alpha^{-,-,+}(w_2w_1w_3w_2w_1v,\ (m,n))\beta^{+,-,-}(-\iota(C,C)\cdot w_1w_3w_2w_1v,\ (m,n))$$

$$\cdot\alpha^{+,-,-}(w_3w_2w_1v,\ (m,n))\beta^{+,+,-}(w_2w_1v,\ (m,n))$$

$$\cdot\beta^{-,-,+}(-\iota(C,C)\cdot w_1v,\ (m,n))\alpha^{+,+,-}(v,\ (m,n)).$$

(3) *If* $\sigma=(+,-,+)$,

$$\det(B^{\sigma}_{\gamma}(\bar{P}:P:v))$$

$$= \mathrm{Const}\cdot\alpha^{+,-,+}(w_2w_1w_3w_2w_1v,\ (m,n))\beta^{-,-,-}(-\iota(C,C)\cdot w_1w_3w_2w_1v,\ (m,n))$$

$$\cdot\alpha^{-,+,-}(w_3w_2w_1v,\ (m,n))\beta^{-,-,-}(w_2w_1v,\ (m,n))$$

$$\cdot\beta^{-,+,-}(-\iota(C,C)\cdot w_1v,\ (m,n))\alpha^{+,-,+}(v,\ (m,n)).$$

(4) *If* $\sigma=(+,-,-)$,

$$\det(B^{\sigma}_{\gamma}(\bar{P}:P:v))$$

$$= \mathrm{Const}\cdot\alpha^{-,+,+}(w_2w_1w_3w_2w_1v,\ (m,n))\beta^{-,+,+}(-\iota(C,C)\cdot w_1w_3w_2w_1v,\ (m,n))$$

$$\cdot\alpha^{-,-,+}(w_3w_2w_1v,\ (m,n))\beta^{-,-,+}(w_2w_1v,\ (m,n))$$

$$\cdot\beta^{+,-,-}(-\iota(C,C)\cdot w_1v,\ (m,n))\alpha^{+,-,-}(v,\ (m,n)).$$

(5) *If* $\sigma=(-,+,+)$,

$$\det(B^{\sigma}_{\gamma}(\bar{P}:P:v))$$

$$= \mathrm{Const}\cdot\alpha^{+,+,-}(w_2w_1w_3w_2w_1v,\ (m,n))\beta^{-,-,+}(-\iota(C,C)\cdot w_1w_3w_2w_1v,\ (m,n))$$

$$\cdot\alpha^{+,+,-}(w_3w_2w_1v,\ (m,n))\beta^{+,-,-}(w_2w_1v,\ (m,n))$$

$$\cdot\beta^{+,+,-}(-\iota(C,C)\cdot w_1v,\ (m,n))\alpha^{-,+,+}(v,\ (m,n)).$$

(6) *If* $\sigma=(-,+,-)$,

$$\det(B^{\sigma}_{\gamma}(\bar{P}:P:v))$$

$$= \mathrm{Const}\cdot\alpha^{-,-,-}(w_2w_1w_3w_2w_1v,\ (m,n))\beta^{-,+,-}(-\iota(C,C)\cdot w_1w_3w_2w_1v,\ (m,n))$$

$$\cdot\alpha^{+,-,+}(w_3w_2w_1v,\ (m,n))\beta^{+,-,+}(w_2w_1v,\ (m,n))$$

$$\cdot\beta^{-,-,-}(-\iota(C,C)\cdot w_1v,\ (m,n))\alpha^{-,+,-}(v,\ (m,n)).$$

(7) *If* $\sigma=(-,-,+)$,

$$\det(B^{\sigma}_{\gamma}(\bar{P}:P:v))$$

$$= \mathrm{Const}\cdot \alpha^{+,-,-}(w_2 w_1 w_3 w_2 w_1 \nu,\ (m, n))\beta^{+,+,-}(-\iota(C, C)\cdot w_1 w_3 w_2 w_1 \nu,\ (m, n))$$
$$\cdot \alpha^{-,+,+}(w_3 w_2 w_1 \nu,\ (m, n))\beta^{-,+,+}(w_2 w_1 \nu,\ (m, n))$$
$$\cdot \beta^{-,+,+}(-\iota(C, C)\cdot w_1 \nu,\ (m, n))\alpha^{-,-,+}(\nu,\ (m, n)).$$

(8) *If* $\sigma = (-, -, -)$,

$$\det(B_\gamma^\sigma(\bar{P} : P : \nu))$$
$$= \mathrm{Const}\cdot \alpha^{-,+,-}(w_2 w_1 w_3 w_2 w_1 \nu,\ (m, n))\beta^{+,-,+}(-\iota(C, C)\cdot w_1 w_3 w_2 w_1 \nu,\ (m, n))$$
$$\cdot \alpha^{-,-,-}(w_3 w_2 w_1 \nu,\ (m, n))\beta^{-,+,-}(w_2 w_1 \nu,\ (m, n))$$
$$\cdot \beta^{+,-,+}(-\iota(C, C)\cdot w_1 \nu,\ (m, n))\alpha^{-,-,-}(\nu,\ (m, n)).$$

PROOF. We shall prove (2). Let

$$\pi_\gamma^\sigma(w) = \pi_\gamma(w)|_{V_\sigma^\gamma}, \qquad (w \in M')$$

By (7.1) and Lemma 8.4 we have

$$B_\gamma^\sigma(\bar{P} : P : \nu) = B_\gamma^{(+,+,-)}(w_1, \nu)\pi_\gamma^{(+,+,-)}(w_1)B_\gamma^{(+,+,-)}(w_2, w_1 \nu)\pi_\gamma^{(+,+,-)}(w_2)$$
$$\cdot B_\gamma^{(+,+,-)}(w_3, w_2 w_1 \nu)\pi_\gamma^{(+,-,-)}(w_3)B_\gamma^{(+,-,-)}(w_1, w_3 w_2 w_1 \nu)\pi_\gamma^{(+,-,-)}(w_1)$$
$$\cdot B_\gamma^{(+,-,-)}(w_2, w_1 w_3 w_2 w_1 \nu)\pi_\gamma^{(-,-,+)}(w_2)B_\gamma^{(-,-,+)}(w_1, w_2 w_1 w_3 w_2 w_1 \nu)$$
$$\cdot \pi_\gamma^{(-,+,+)}(w_1)\pi_\gamma^{(+,+,-)}(w_0). \tag{9.1}$$

By (7.11) and Corollary 8.3, we obtain

$$B_\gamma^{(+,+,-)}(w_2, w_1 \nu) = \pi_\gamma^{(-,-,+)}(\iota(C, C))B_\gamma^{(-,-,+)}(w_3, -(\iota(C, C)\cdot w_1 \nu))\pi_\gamma^{(+,+,-)}$$
$$\cdot(\iota(C, C)^{-1})$$

and

$$B_\gamma^{(+,-,-)}(w_2, w_1 w_3 w_2 w_1 \nu) = \pi_\gamma^{(+,-,-)}(\iota(C, C))B_\gamma^{(+,-,-)}(w_3, -(\iota(C, C)\cdot w_1 w_3 w_2 w_1 \nu))$$
$$\cdot \pi_\gamma^{(+,-,-)}(\iota(C, C)^{-1}).$$

Therefore, (9.1) is equal to

$$B_\gamma^{(+,+,-)}(w_1, \nu)\pi_\gamma^{(+,+,-)}(w_1)\pi_\gamma^{(-,-,+)}(\iota(C, C))$$
$$\cdot B_\gamma^{(-,-,+)}(w_3, -(\iota(C, C)\cdot w_1 \nu))\pi_\gamma^{(+,+,-)}(\iota(C, C)^{-1})\pi_\gamma^{(+,+,-)}(w_2)$$
$$\cdot B_\gamma^{(+,+,-)}(w_3, w_2 w_1 \nu)\pi_\gamma^{(+,-,-)}(w_3)B_\gamma^{(+,-,-)}(w_1, w_3 w_2 w_1 \nu)\pi_\gamma^{(+,-,-)}(w_1)$$
$$\cdot \pi_\gamma^{(+,-,-)}(\iota(C, C))B_\gamma^{(+,-,-)}(w_3, -(\iota(C, C)\cdot w_1 w_3 w_2 w_1 \nu))$$
$$\cdot \pi_\gamma^{(+,-,-)}(\iota(C, C)^{-1})\pi_\gamma^{(-,-,+)}(w_2)B_\gamma^{(-,-,+)}(w_1, w_2 w_1 w_3 w_2 w_1 \nu)$$

$$\cdot \pi_\gamma^{(-,+,+)}(w_1)\pi_\gamma^{(+,+,-)}(w_0)\,. \tag{9.2}$$

Let i be an integer such that $1 \leq i \leq n-1$ and σ in M'. We extend $B_\gamma^{\sigma'}(w_i, \cdot)$ to an operator $B_\gamma^{\sigma'}(w_i, \cdot)$ of V^γ by

$$B_\gamma^{\sigma'}(w_i, \cdot) = \begin{cases} B_\gamma^{\sigma'}(w_i, \cdot) & \text{on } V_\sigma^\gamma\,, \\ \text{identity} & \text{otherwise} \end{cases} \tag{9.3}$$

and define

$$\begin{aligned} \tilde{B}_\gamma^\sigma(\bar{P} : P : v) = \tilde{B}_\gamma^{(+,+,-)}&(w_1, v)\pi_\gamma(w_1)\pi_\gamma(\iota(C, C))\tilde{B}_\gamma^{(-,-,+)}(w_3, -(\iota(C, C)\cdot w_1 v)) \\ &\cdot \pi_\gamma(\iota(C, C)^{-1})\pi_\gamma(w_2)\tilde{B}_\gamma^{(+,+,-)}(w_3, w_2 w_1 v)\pi_\gamma(w_3) \\ &\cdot \tilde{B}_\gamma^{(+,-,-)}(w_1, w_3 w_2 w_1 v)\pi_\gamma(w_1)\pi_\gamma(\iota(C, C)) \\ &\cdot \tilde{B}_\gamma^{(+,-,-)}(w_3, -(\iota(C, C)\cdot w_1 w_3 w_2 w_1 v))\pi_\gamma(\iota(C, C)^{-1})\pi_\gamma(w_2) \\ &\cdot \tilde{B}_\gamma^{(-,-,+)}(w_1, w_2 w_1 w_3 w_2 w_1 v)\pi_\gamma(w_1)\pi_\gamma(w_0)\,. \end{aligned} \tag{9.4}$$

Then from (9.2) we have

$$\tilde{B}_\gamma^\sigma(\bar{P} : P : v)|_{V_\gamma^\sigma} = B_\gamma^\sigma(\bar{P} : P : v) \tag{9.5}$$

and

$$\det(\tilde{B}_\gamma^\sigma(\bar{P} : P : v)) = d_1 \cdot \det(B_\gamma^\sigma(\bar{P} : P : v))\,, \tag{9.6}$$

where d_1 is a nonzero constant which is independent of v. On the other hand, from (9.3) and (9.4) we have

$$\begin{aligned} \det(\tilde{B}_\gamma^\sigma(\bar{P} : P : v)) = d_2 \cdot \det(B_\gamma^{(+,+,-)}&(w_1, v)) \det(B_\gamma^{(-,-,+)}(w_3, -(\iota(C, C)\cdot w_1 v)) \\ &\cdot \det(B_\gamma^{(+,+,-)}(w_3, w_2 w_1 v)) \det(B_\gamma^{(+,-,-)}(w_1, w_3 w_2 w_1 v)) \\ &\cdot \det(B_\gamma^{(+,-,-)}(w_3, -(\iota(C, C)\cdot w_1 w_3 w_2 w_1 v)) \\ &\cdot \det(B_\gamma^{(-,-,+)}(w_1, w_2 w_1 w_3 w_2 w_1 v))\,, \end{aligned} \tag{9.7}$$

where d_2 is a constant such that $|d_2| = 1$. By Lemma 9.1 and (9.6), (9.7), we can prove (2). Similarly, we can prove the others.

Appendix

A.1. *Suppose that q is a positive integer and* $\operatorname{Re} z > q/2$. *Then* $\int_0^\infty t^{q-1}(1+t^2)^{-z}\,dt$ *converges absolutely and is equal to* $\frac{1}{2}B(q/2, z-q/2)$ (see [11], p. 262).

A.2. *Suppose that λ is an element in* $\boldsymbol{C}$ *such that* $\operatorname{Re}\lambda < -1$ *and l is an integer. Then we have*

$$(*)\quad \int_{-\infty}^{\infty} (1+\sqrt{-1}\,x)^{\lambda+l/2}(1-\sqrt{-1}\,x)^{\lambda-l/2}dx = \frac{2^{\lambda+2}\pi\Gamma(-\lambda-1)}{\Gamma\left(-\dfrac{\lambda+l}{2}\right)\Gamma\left(-\dfrac{\lambda-l}{2}\right)}.$$

PROOF. We shall first prove inductively that $(*)$ holds for all nonnegative integers l. If $l=0$, then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+\sqrt{-1}\,x)^{\lambda/2}(1-\sqrt{-1}\,x)^{\lambda/2}dx \\
&\quad = 2\int_{0}^{\infty}(1+x^2)^{\lambda/2}dx = B\left(\frac{1}{2},\ -\frac{\lambda}{2}-\frac{1}{2}\right) \\
&\quad = \frac{2^{\lambda+2}\pi\Gamma(-\lambda-1)}{\Gamma\left(-\dfrac{\lambda}{2}\right)^2}.
\end{aligned}$$

If $l=1$, then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+\sqrt{-1}\,x)^{(\lambda+1)/2}(1-\sqrt{-1}\,x)^{(\lambda-1)/2}\,dx \\
&\quad = \int_{-\infty}^{\infty}(1+x^2)^{(\lambda-1)/2}(1+\sqrt{-1}\,x)dx \\
&\quad = \int_{-\infty}^{\infty}(1+x^2)^{(\lambda-1)/2}dx + \sqrt{-1}\int_{-\infty}^{\infty}x(1+x^2)^{(\lambda-1)/2}dx.
\end{aligned}$$

Since the second term is equal to 0, the last expression equals

$$\begin{aligned}
B\left(\frac{1}{2},-\frac{\lambda}{2}\right) &= \frac{\Gamma\left(\dfrac{1}{2}\right)\Gamma\left(-\dfrac{\lambda}{2}\right)\Gamma\left(-\dfrac{\lambda+1}{2}\right)}{\Gamma\left(-\dfrac{\lambda-1}{2}\right)\Gamma\left(-\dfrac{\lambda+1}{2}\right)} \\
&= \frac{2^{\lambda+2}\pi\Gamma(-\lambda-1)}{\Gamma\left(-\dfrac{\lambda-1}{2}\right)\Gamma\left(-\dfrac{\lambda+1}{2}\right)}.
\end{aligned}$$

Let l be an integer such that $l\geq 2$ and put

$$I_l(\lambda) = \int_{-\infty}^{\infty}(1+\sqrt{-1}\,x)^{(\lambda+1)/2}(1-\sqrt{-1}\,x)^{(\lambda-1)/2}dx.$$

Then it is not difficult to see that the following recurrence formula holds

$$I_l(\lambda) = 2I_{l-1}(\lambda - 1) - I_{l-2}(\lambda), \qquad (l \geq 2).$$

Suppose that (*) is true in $l-1$, $l-2$, then an easy computation gives that (*) is true for $l \geq 0$. By the relation

$$\int_{-\infty}^{\infty} (1 + \sqrt{-1}\, x)^{(\lambda-1)/2}(1 - \sqrt{-1}\, x)^{(\lambda+1)/2} dx$$
$$= \int_{-\infty}^{\infty} (1 + \sqrt{-1}\, x)^{(\lambda+1)/2}(1 - \sqrt{-1}\, x)^{(\lambda-1)/2} dx,$$

(*) is also true for $l < 0$.

A.3. *Let s be a complex number such that* $\operatorname{Re} s > 0$ *and i, n nonnegative integers such that $0 \leq i \leq n$. Then we have*

$$\int_{-\infty}^{\infty} (1 + x^2)^{-(s+1)/2}\left(\frac{1 - \sqrt{-1}\, x}{(1 + x^2)^{1/2}}\right)^{n/2-i} dx$$
$$= \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-(n/2-i)}{2}\right)\Gamma\left(\frac{s+1+(n/2-i)}{2}\right)}.$$

PROOF. By A.2, we have

$$\int_{-\infty}^{\infty} (1 + x^2)^{\lambda/2}\left(\frac{1 - \sqrt{-1}\, x}{(1 + x^2)^{1/2}}\right)^{l} dx$$
$$= \int_{-\infty}^{\infty} (1 + \sqrt{-1}\, x)^{(\lambda+1)/2}(1 - \sqrt{-1}\, x)^{(\lambda-1)/2} dx = \frac{2^{\lambda+2}\pi\Gamma(-\lambda-1)}{\Gamma\left(-\frac{\lambda+l}{2}\right)\Gamma\left(-\frac{\lambda-l}{2}\right)},$$

putting $\lambda = -s-1$, $l = n/2 - i$, we have

$$= \frac{2^{-s+1}\pi\Gamma(s)}{\Gamma\left(\frac{s+1-l}{2}\right)\Gamma\left(\frac{s+1+l}{2}\right)}$$
$$= \frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1-l}{2}\right)\Gamma\left(\frac{s+1+l}{2}\right)}.$$

This proves the assertion of A.3.

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