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On a fractal set with a gap between its Hausdorff dimension and box dimension

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1. Introduction

In this paper, we study a property of some fractal set K satisfying the condition

$$\text{H-dim}\,(K) < \underline{\text{M-dim}}\,(K),$$

by examining the density with respect to the Hausdorff measure.

We claim that, roughly speaking, for arbitrary small $\delta > 0$ we can find an essential subset of K on which the lower density of K is less than δ. Here, a subset K_{ess} of K is called *an essential subset*, if K_{ess} satisfies the conditions

$$\text{H-dim}\,(K_{ess}) < \underline{\text{M-dim}}\,(K_{ess}) = \underline{\text{M-dim}}\,(K)$$

and

$$\text{H-dim}\,(K) = \text{H-dim}\,(K \setminus K_{ess}) = \underline{\text{M-dim}}\,(K \setminus K_{ess}).$$

The key point of this paper is the fact that the cause of the gap between the Hausdorff dimension and the lower box dimension arises in a "neighborhood" of the subset of K on which the lower density of K equals to 0.

2. Results and proofs

On the Euclidean space $(\mathbb{R}^N, d)$, the upper and the lower box dimension $\overline{\text{M-dim}}$, $\underline{\text{M-dim}}$ and the Hausdorff dimension H-dim are defined as follows; for any bounded set $E \subset \mathbb{R}^N$

$$\overline{\text{M-dim}}\,(E) = \limsup_{\varepsilon \downarrow 0} \frac{\log\,(\mathrm{N}_\varepsilon(E))}{\log 1/\varepsilon}, \quad \underline{\text{M-dim}}\,(E) = \liminf_{\varepsilon \downarrow 0} \frac{\log\,(\mathrm{N}_\varepsilon(E))}{\log 1/\varepsilon},$$

$$\text{H-dim}\,(E) = \inf\{\alpha;\, \mathrm{H}^\alpha(E) = 0\} = \sup\{\alpha;\, \mathrm{H}^\alpha(E) = \infty\},$$

where

$$\mathrm{N}_\varepsilon(E) = \inf{}^{\#}\{U_i;\, E \subseteq \bigcup_i U_i,\, |U_i| \leq \varepsilon\},$$

$$\mathrm{H}^{\alpha}(E) = \lim_{\varepsilon \downarrow 0} \{\inf \{\sum_i |U_i|^{\alpha};\ E \subseteq \bigcup_i U_i,\ |U_i| \le \varepsilon\}\}.$$

and $|U| = \sup_{x,y \in U} d(x, y)$. We know that $\text{H-dim}\,(E) \le \underline{\text{M-dim}}\,(E) \le \overline{\text{M-dim}}\,(E)$ in general.

Now we introduce the upper and the lower spherical densities of K at x as

$$\bar{D}^{\alpha}(K, x) = \limsup_{r \downarrow 0} \frac{\mathrm{H}^{\alpha}(B(x, r) \cap K)}{|B(x, r)|^{\alpha}},\quad \underline{D}^{\alpha}(K, x) = \liminf_{r \downarrow 0} \frac{\mathrm{H}^{\alpha}(B(x, r) \cap K)}{|B(x, r)|^{\alpha}}$$

where $B(x, r)$ denotes the closed ball with radius r and center at x. It is a well-known fact that if $0 < \mathrm{H}^{\alpha}(K) < \infty$ then

$$\bar{D}^{\alpha}(K, x) \le 1 \qquad \mathrm{H}^{\alpha}\text{-a.e. } x \in K \tag{2.1}$$

(see Corollary 2.5 in [4]). The following theorem is the main result in this paper.

THEOREM 2.1. *Let K be a bounded Borel-measurable subset of $\mathbb{R}^N$ and put*

$$K^0 = \{x \in K;\ \underline{D}^{\alpha}(K, x) = 0\}. \tag{2.2}$$

If

$$H\text{-}dim\,(K) < \underline{M\text{-}dim}\,(K) \tag{2.3}$$

and

$$H^{\alpha}(K^0) < H^{\alpha}(K) < \infty, \tag{2.4}$$

then for any $\varepsilon > 0$, there exists a subset K_{ε} of K such that

$$K^0 \subseteq K_{\varepsilon} \subseteq K,\ H^{\alpha}(K_{\varepsilon} \setminus K^0) \le \varepsilon,$$

$$\overline{M\text{-}dim}\,(K_{\varepsilon}) = \overline{M\text{-}dim}\,(K),\ \underline{M\text{-}dim}\,(K_{\varepsilon}) = \underline{M\text{-}dim}\,(K)$$

and

$$\alpha = H\text{-}dim\,(K \setminus K_{\varepsilon}) = \underline{M\text{-}dim}\,(K \setminus K_{\varepsilon}) = \overline{M\text{-}dim}\,(K \setminus K_{\varepsilon}).$$

COROLLARY. *Let K be a bounded Borel-measurable subset of $\mathbb{R}^N$. If K satisfies the condition* (2.3), *then either $H^{\alpha}(K) = \infty$ holds or for any $\varepsilon > 0$ there exists K_{ε} such that*

$$K^0 \subseteq K_{\varepsilon} \subseteq K,\ H^{\alpha}(K_{\varepsilon} \setminus K^0) \le \varepsilon$$

$$\overline{M\text{-}dim}\,(K_{\varepsilon}) = \overline{M\text{-}dim}\,(K),\ \underline{M\text{-}dim}\,(K_{\varepsilon}) = \underline{M\text{-}dim}\,(K),$$

and

$$\alpha \geq H\text{-}dim\,(K \setminus K_\varepsilon) = \underline{M\text{-}dim}\,(K \setminus K_\varepsilon) = \overline{M\text{-}dim}\,(K \setminus K_\varepsilon).$$

From now on, we assume that K is a bounded Borel-measurable set satisfying $\mathrm{H}^\alpha(K) < \infty$. For the proof of THEOREM 2.1, we will introduce several notations and show lemmas.

Put

$$\mathscr{R}^{(n)} = \{\textstyle\prod_{k=1}^{N} [i_k 2^{-n},\ (i_k + 1)2^{-n}),\ i_k \in \mathbb{Z}\}, \quad \mathscr{R} = \bigcup_{i=1}^{\infty} \mathscr{R}^{(n)},$$

and for each $R = \prod_{k=1}^{N} [i_k 2^{-n},\ (i_k + 1)2^{-n}) \in \mathscr{R}^{(n)}$, we put

$$\mathscr{S}_R = \textstyle\prod_{k=1}^{N} [(i_k - 1)2^{-n},\ (i_k + 2)2^{-n}).$$

For $x \in K$, put

$$\mathscr{R}_{\mathscr{S}}\text{-}\bar{D}^\alpha(K, x) = \limsup_{n \to 0,\, x \in R \in \mathscr{R}^{(n)}} \frac{\mathrm{H}^\alpha(\mathscr{S}_R \cap K)}{|\mathscr{S}_R|^\alpha},$$

$$\mathscr{R}_{\mathscr{S}}\text{-}\underline{D}^\alpha(K, x) = \liminf_{n \to 0,\, x \in R \in \mathscr{R}^{(n)}} \frac{\mathrm{H}^\alpha(\mathscr{S}_R \cap K)}{|\mathscr{S}_R|^\alpha}.$$

LEMMA 2.2. *We have the following inequalities*

$$\mathscr{R}_{\mathscr{S}}\text{-}\bar{D}^\alpha(K, x) \leq 2^\alpha \bar{D}^\alpha(K, x) \qquad \textit{for any } x \in K, \tag{2.5}$$

$$\underline{D}^\alpha(K, x) \leq (3N^{\frac{1}{2}})^\alpha \mathscr{R}_{\mathscr{S}}\text{-}\underline{D}^\alpha(K, x) \qquad \textit{for any } x \in K. \tag{2.6}$$

PROOF OF LEMMA 2.2. If $2^{-(n+1)} \leq r < 2^{-n}$, $x \in R \in \mathscr{R}^{(n)}$ then

$$\begin{aligned}\frac{\mathrm{H}^\alpha(B(x, r) \cap K)}{|B(x, r)|^\alpha} &\leq \frac{\mathrm{H}^\alpha(\mathscr{S}_R \cap K)}{|B(x, r)|^\alpha} \\ &= \frac{|\mathscr{S}_R|^\alpha}{|2r|^\alpha} \frac{\mathrm{H}^\alpha(\mathscr{S}_R \cap K)}{|\mathscr{S}_R|^\alpha} \\ &\leq (3N^{\frac{1}{2}})^\alpha \frac{\mathrm{H}^\alpha(\mathscr{S}_R \cap K)}{|\mathscr{S}_R|^\alpha}.\end{aligned}$$

Therefore we have (2.6). We can show (2.5) similarly. □

For any $m \in \mathbb{N}$, $\delta > 0$, put

$$M_\delta^{(m)} = \left\{ x \in K\,;\, \delta \leq \frac{\mathrm{H}^\alpha(\mathscr{S}_R \cap K)}{|\mathscr{S}_R|^\alpha} \leq 2^{\alpha+1},\ x \in R \in \mathscr{R}^{(n)} \text{ for any } n \geq m \right\}.$$

The following lemma is easily seen (c.f. Theorem 1.5, Lemma 2.1 etc. in [4]).

LEMMA 2.3. *The functions* $\mathscr{R}_{\mathscr{S}}$-$\bar{D}^{\alpha}(K, x)$ *and* $\mathscr{R}_{\mathscr{S}}$-$\underline{D}^{\alpha}(K, x)$ *on* K *are Borel measurable, and for any* $m \in \mathbb{N}$, $\delta > 0$, $M_{\delta}^{(m)}$ *is a Borel measurable set.*

LEMMA 2.4. *For any* $m \in \mathbb{N}$, $\delta > 0$, *we have the following evaluations*

$$\overline{M\text{-}dim}\,(M_{\delta}^{(m)}) \leq \alpha, \quad \overline{M\text{-}dim}\,(K \setminus M_{\delta}^{(m)}) = \overline{M\text{-}dim}\,(K),$$

$$\underline{M\text{-}dim}\,(K \setminus M_{\delta}^{(m)}) = \underline{M\text{-}dim}\,(K).$$

Especially, if $H^{\alpha}(M_{\delta}^{(m)}) > 0$ *then*

$$\overline{M\text{-}dim}\,(M_{\delta}^{(m)}) = \underline{M\text{-}dim}\,(M_{\delta}^{(m)}) = H\text{-}dim\,(M_{\delta}^{(m)}) = \alpha.$$

PROOF OF LEMMA 2.4. Let $\{U_i\}_i$ be a minimal ε-covering of $M_{\delta}^{(m)}$, that is $^{\#}\{U_i\}_i = \mathrm{N}_{\varepsilon}(M_{\delta}^{(m)})$. Let $\{R_i\}_i$ be a minimal covering of $M_{\delta}^{(m)}$ by $\mathscr{R}^{(l)}$ (where $l = l(\varepsilon) = [-\log_2(\varepsilon N^{-\frac{1}{2}})] + 1$ and $[x]$ denotes the integer part of x). Let L be a natural number such that $L \geq (4N^{\frac{1}{2}})^N \pi^{\frac{1}{2}N} / \Gamma(N/2 + 1)$. Then for any j there exist $\{R_{j,k}\}_{j,k} \subseteq \{R_i\}$ such that

$$U_j \cap M_{\delta}^{(m)} \subseteq \bigcup_{k=1}^{P_j} R_{j,k}, \qquad P_j \leq L.$$

Since $|R_i| < \varepsilon$, we see that

$$\mathrm{N}_{\varepsilon}(M_{\delta}^{(m)}) = {}^{\#}\{U_i\}_i \leq {}^{\#}\{R_i\}_i \leq L \cdot {}^{\#}\{U_i\}_i = L \cdot \mathrm{N}_{\varepsilon}(M_{\delta}^{(m)}). \tag{2.7}$$

By the definition of $M_{\delta}^{(m)}$, we have that for any $R \in \mathscr{R}^{(n)}$, $R \cap M_{\delta}^{(m)} \neq \emptyset$, $n \geq m$

$$\delta\,|\mathscr{S}_R|^{\alpha} \leq H^{\alpha}(\mathscr{S}_R \cap K) \leq 2^{\alpha+1}\,|\mathscr{S}_R|^{\alpha}. \tag{2.8}$$

Let ε be an arbitrary positive number satisfying $l(\varepsilon) \geq m$. Taking the multiplicity of $\mathscr{S}_{R_i}$'s and the measurability of $\mathscr{S}_{R_j} \cap K$ into consideration, together with (2.7), (2.8), we see the following inequalities

$$\mathrm{N}_{\varepsilon}(M_{\delta}^{(m)}) \leq {}^{\#}\{R_i\}_i = {}^{\#}\{\mathscr{S}_{R_i}\}_i \leq \sup_j \frac{\mathrm{H}^{\alpha}(K)}{\mathrm{H}^{\alpha}(\mathscr{S}_{R_j} \cap K)} \cdot 3^N$$

$$< 3^{N-\alpha}\delta^{-1} N^{-\frac{1}{2}\alpha} H^{\alpha}(K) 2^{\alpha l}.$$

Therefore we see

$$\frac{\log(\mathrm{N}_{\varepsilon}(M_{\delta}^{(m)}))}{\log 1/\varepsilon} < \frac{\log(3^{N-\alpha}\delta^{-1} N^{-\frac{1}{2}\alpha} H^{\alpha}(K)) + \alpha l \log 2}{-\frac{1}{2}\log N + l \log 2} \longrightarrow \alpha \quad \text{as} \quad \varepsilon \downarrow 0.$$

This implies

$$\overline{\text{M-dim}}\,(M_\delta^{(m)}) \leq \alpha \qquad \text{for any } \delta > 0. \tag{2.9}$$

Furthermore if $H^\alpha(M_\delta^{(m)}) > 0$, then

$$\alpha = \text{H-dim}\,(M_\delta^{(m)}) \leq \underline{\text{M-dim}}\,(M_\delta^{(m)}) \leq \overline{\text{M-dim}}\,(M_\delta^{(m)}). \tag{2.10}$$

Together with (2.4), (2.9) and (2.10), we have

$$\underline{\text{M-dim}}\,(M_\delta^{(m)}) = \overline{\text{M-dim}}\,(M_\delta^{(m)}) = \text{H-dim}\,(K) = \alpha \qquad \text{if } \text{H}^\alpha(M_\delta^{(m)}) > 0. \tag{2.11}$$

Lastly, we will prove that

$$\overline{\text{M-dim}}\,(K \setminus M_\delta^{(m)}) = \overline{\text{M-dim}}\,(K), \ \underline{\text{M-dim}}\,(K \setminus M_\delta^{(m)}) = \underline{\text{M-dim}}\,(K). \tag{2.12}$$

From (2.9) and the condition (2.3), we see

$$0 \leq \limsup_{\varepsilon \downarrow 0} \frac{\log(\text{N}_\varepsilon(M_\delta^{(m)}))}{\log(\text{N}_\varepsilon(K))} < 1 \qquad \text{for any } \delta > 0.$$

In addition, since $\text{N}_\varepsilon(K) \to \infty$ as $\varepsilon \downarrow 0$, we have

$$\liminf_{\varepsilon \downarrow 0} \frac{\text{N}_\varepsilon(M_\delta^{(m)})}{\text{N}_\varepsilon(K)} = \limsup_{\varepsilon \downarrow 0} \frac{\text{N}_\varepsilon(M_\delta^{(m)})}{\text{N}_\varepsilon(K)} = 0. \tag{2.13}$$

On the other hand, since $M_\delta^{(m)} \subseteq K$, we have the following inequalities

$$\text{N}_\varepsilon(K) - \text{N}_\varepsilon(M_\delta^{(m)}) \leq \text{N}_\varepsilon(K \setminus M_\delta^{(m)})) \leq \text{N}_\varepsilon(K).$$

Therefore we see

$$\begin{aligned}
\liminf_{\varepsilon \downarrow 0} \frac{\log\left(N_\varepsilon(K)\left(1 - \dfrac{\text{N}_\varepsilon(M_\delta^{(m)})}{\text{N}_\varepsilon(K)}\right)\right)}{\log 1/\varepsilon} &\leq \liminf_{\varepsilon \downarrow 0} \frac{\log(\text{N}_\varepsilon(K \setminus M_\delta^{(m)}))}{\log 1/\varepsilon} \\
&\leq \liminf_{\varepsilon \downarrow 0} \frac{\log(\text{N}_\varepsilon(K))}{\log 1/\varepsilon}.
\end{aligned}$$

Together with (2.13), we see

$$\liminf_{\varepsilon \downarrow 0} \frac{\log(\text{N}_\varepsilon(K \setminus M_\delta^{(m)}))}{\log 1/\varepsilon} = \liminf_{\varepsilon \downarrow 0} \frac{\log(\text{N}_\varepsilon(K))}{\log 1/\varepsilon}.$$

This implies

$$\underline{\text{M-dim}}\,(K \setminus M_\delta^{(m)}) = \underline{\text{M-dim}}\,(K).$$

Similarly, we see

$$\overline{\text{M-dim}}\,(K \setminus M_\delta^{(m)}) = \overline{\text{M-dim}}\,(K) \quad \square$$

Now we will prove THEOREM 2.1.

PROOF OF THEOREM 2.1. Let ε be any positive number such that

$$\mathrm{H}^\alpha(K) - \mathrm{H}^\alpha(K^0) > 2\varepsilon. \tag{2.14}$$

Put

$$N_\delta = \{x \in K;\ \delta < \mathscr{R}_{\mathscr{S}}\text{-}\underline{D}^\alpha(K, x) \le \mathscr{R}_{\mathscr{S}}\text{-}\bar{D}^\alpha(K, x) < 2^{\alpha+1}\}.$$

By the inequalities (2.1), (2.5) and LEMMA 2.3, we see that there exists $\delta > 0$ such that

$$\mathrm{H}^\alpha(K \setminus (N_\delta \cup K^0)) < \varepsilon/2. \tag{2.15}$$

Furthermore we see that $\bigcup_{k=1}^\infty M_\delta^{(k)} \supseteq N_\delta$. Since $M_\delta^{(m)}$ is increasing in monotone as $m \to \infty$, there exists $m_0 \in \mathbb{N}$ such that

$$\mathrm{H}^\alpha(N_\delta \setminus M_\delta^{(m_0)}) < \varepsilon/2. \tag{2.16}$$

Put $K_\varepsilon = K \setminus M_\delta^{(m_0)}$. Then by (2.15), (2.16), we see

$$\begin{aligned}
&\mathrm{H}^\alpha(K_\varepsilon \setminus K^0)\\
&\quad = \mathrm{H}^\alpha((K_\varepsilon \cap N_\delta) \setminus K^0) + \mathrm{H}^\alpha((K_\varepsilon \setminus N_\delta) \setminus K^0)\\
&\quad = \mathrm{H}^\alpha((K \cap N_\delta) \setminus (K^0 \cup M_\delta^{(m_0)})) + \mathrm{H}^\alpha((K \setminus (M_\delta^{(m_0)} \cup N_\delta \cup K^0))\\
&\quad \le \mathrm{H}^\alpha(N_\delta \setminus M_\delta^{(m_0)}) + \mathrm{H}^\alpha(K \setminus (N_\delta \cup K^0)) < \varepsilon.
\end{aligned}$$

By (2.14), (2.15) and (2.16), $\mathrm{H}^\alpha(M_\delta^{(m_0)}) > 0$ holds. Hence by Lemma 2.4, we see

$$\overline{\text{M-dim}}\,(K_\varepsilon) = \overline{\text{M-dim}}\,(K), \quad \underline{\text{M-dim}}\,(K_\varepsilon) = \underline{\text{M-dim}}\,(K)$$

and

$$\overline{\text{M-dim}}\,(K \setminus K_\varepsilon) = \underline{\text{M-dim}}\,(K \setminus K_\varepsilon) = \text{H-dim}\,(K) = \alpha. \quad \square$$

From the proof of THEOREM 2.1, we can see that K_ε is an essential subset of K in §1. For the case $\mathrm{H}^\alpha(K) = 0$ or $\mathrm{H}^\alpha(K) = \mathrm{H}^\alpha(K^0)$, $K_\varepsilon = K$ satisfies the assertion in the corollary. Therefore COROLLARY is obvious by THEOREM 2.1.

Here we observe a simple example. Put $K = \mathbb{Q}_{[0,1]} \cup C$ and $\alpha = \log 2/\log 3$ where $\mathbb{Q}_{[0,1]} = \mathbb{Q} \cap [0, 1]$ and C is the Cantor set. Then

$$\text{H-dim}\,(K) = \alpha < 1 = \underline{\text{M-dim}}\,(K),$$

$$0 < \mathrm{H}^\alpha(K) = 1 < \infty$$

and

$$\underline{D}^{\alpha}(K, x) \geq 6^{-\alpha} \qquad \text{for } x \in C$$

hold. The last inequality means $N_{1/3} = C$, $K^0 = \mathbb{Q}_{[0,1]} \setminus C$ and $\mathrm{H}^{\alpha}(K^0) = 0$. In this case, $K_{\varepsilon} = \mathbb{Q}_{[0,1]}$ satisfies the conditions in THEOREM 2.1. Actually,

$$\mathrm{H}^{\alpha}(\mathbb{Q}_{[0,1]}) = 0, \ \underline{\text{M-dim}}\,(\mathbb{Q}_{[0,1]}) = \overline{\text{M-dim}}\,(\mathbb{Q}_{[0,1]}) = 1$$

and

$$\alpha = \text{H-dim}\,(C \setminus \mathbb{Q}_{[0,1]}) = \underline{\text{M-dim}}\,(C \setminus \mathbb{Q}_{[0,1]}) = \overline{\text{M-dim}}\,(C \setminus \mathbb{Q}_{[0,1]})$$

hold as well known.

For more complicated cases, we can not expect that THEOREM 2.1 is valid for $\varepsilon = 0$, because the box dimension is not stable. Complicated examples will be discussed separately.

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