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Complex structures on $L(p, q) \times S^1$

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0. Introduction

Let M^3 be a compact orientable 3-manifold. Then, $M^3 \times S^1$ has an almost complex structure, because the tangent bundle of M^3 is trivial. However by [8] Theorem 3.1, $M^3 \times S^1$ cannot have any complex structure unless M^3 admits a Seifert fibering structure. Moreover these complex structures are deformation equivalent except the case that M^3 is homeomorphic to a lens space by [8] Theorem 3.2 and [11] Theorems C-1 and C-2. In this note, we determine the deformation types of all the complex structures on the product manifold $L(p, q) \times S^1$. We begin with the precise definition of deformation types or deformation equivalence.

DEFINITION 0.1. ([6] p. 71 Definition 2.9) *When there exists a complex analytic family (M, B, π) such that B is a connected complex manifold and the Jacobian of π has the maximal rank at any point, any two fibers of π are called deformations of each other.*

DEFINITION 0.2. *Complex manifolds X and Y are called deformation equivalent or have the same deformation type if there exists a series of connected complex manifolds X_i for $i = 1, 2, \cdots, n$ such that $X_1 = X$ and $X_n = Y$ and X_{i+1} is a deformation of X_i for $i = 1, \cdots, n-1$.*

REMARK: This definition of deformation equivalence is equivalent to Definition 1.1 in [3].

The purpose of this paper is to prove the following main Theorem 2.1. Let $n(N)$ denote the number of deformation types of the complex manifolds which are diffeomorphic to the manifold N.

THEOREM 2.1. *Let p and q be positive integers with $p > 1$ and $(p, q) = 1$ and $L(p, q)$ a 3-dimensional lens space. Then,*

$$n(L(p, q) \times S^1) = \begin{cases} 1 & \text{if } q^2 \equiv -1 \pmod p \\ 2 & \text{if } q^2 \not\equiv -1 \pmod p \end{cases}$$

The latter case is characterized as the case that $L(p, q)$ and $L(p, -q)$ are isomorphic only by an orientation reversing diffeomorphism. Theorem 2.1 is a generalization of Dabrowsky's Theorem ([2] Corollary 4.4) about complex structures on $S^3 \times S^1$. To prove this theorem we list up all the complex structures on $L(p, q) \times S^1$ and define two types so that any two complex structures in the same type are deformation equivalent. Then it suffices to show that these types are invariant under infinitesimal deformation, or equivalently to construct local versal families which contain only one type of complex structures in §2.

1. Complex structures on $L(p, q) \times S^1$ and associated covering transformations

Any complex surfaces homeomorphic to $L(p, q) \times S^1$ have p-fold coverings homeomorphic to $S^3 \times S^1$ which are primary Hopf surfaces by Theorem 1 in [5]. So they are secondary Hopf surfaces and their universal covering space is $W = \mathbf{C}^2 - \{(0, 0)\}$. By taking conjugate the generators of commuting covering transformations of W with finite and infinite order are given as follows according to [4] pp. 230–231 and [7] pp. 1566–1568 and Theorem 6 in [10]:

$$g(r): (z_1, z_2) \longmapsto (\zeta^r z_1, \zeta z_2) \quad \text{with} \quad \zeta = e^{2\pi i/p}$$

and

$$f(\alpha, \beta, \lambda, r): (z_1, z_2) \longmapsto (\alpha z_1 + \lambda z_2^r, \beta z_2) \quad \text{with} \quad \alpha, \beta, \lambda \in \mathbf{C} \text{ and } 0 < |\alpha|, |\beta| < 1.$$

By taking conjugate again we may assume $\lambda = 0$ and $|\alpha| \le |\beta|$ if $\alpha \ne \beta^r$ and $\lambda = 1$ if $\alpha = \beta^r$.

The quotient space is naturally diffeomorphic to $L(p, r) \times S^1$. If $L(p, r) \times S^1$ is diffeomorphic to $L(p, q) \times S^1$, then $L(p, r)$ is h-cobordant and hence diffeomorphic to $L(p, q)$ by Theorem 7.27 in [1]. So, by the classification theorem of lens space due to [9] the integer r satisfies

$$(*) \ q^{\varepsilon} \cdot r \equiv \varepsilon' \pmod p \text{ with } \varepsilon = \pm 1 \text{ and } \varepsilon' = \pm 1.$$

So, we see that all the complex structures on $L(p, q) \times S^1$ are given as the quotient space $W/\langle f(\alpha, \beta, \lambda, r), g(r) \rangle$ where r satisfies the condition $(*)$. Omitting $\bmod p$ hereafter, we have the following four types depending on r by defining q' with $q \cdot q' \equiv 1$:

type Ia: $r \equiv q$, type Ib: $r \equiv q'$, type IIa: $r \equiv -q$ and type IIb: $r \equiv -q'$.

When $p = 2$ there is no difference between the four types. For $p \ge 3$ we have the following three cases:

(1) $q \equiv q'$ (i.e. $q^2 \equiv 1$),

(2)-(i) $q \not\equiv q'$ (i.e. $q^2 \not\equiv 1$) and $-q \equiv q'$ (i.e. $q^2 \equiv -1$), and

(2)-(ii) $q \not\equiv q'$ (i.e. $q^2 \not\equiv 1$) and $-q \not\equiv q'$ (i.e. $q^2 \not\equiv -1$).

In the case (1) type Ia = type Ib and type IIa = type IIb. Moreover by the following lemma 1.1, we see that $\{\text{type Ia}\} \cap \{\text{type Ib}\} \neq \phi$ and $\{\text{type IIa}\} \cap \{\text{type IIb}\} \neq \phi$. Here $\{\cdot\}$ stands for the set of complex structures coming from type $\cdot$. So, we should call type I and type II by uniting type Ia with Ib and IIa with IIb for the complex structures. We see also that type Ia = type IIb and type Ib = type IIa in the case (2)(i). On the other hand, we see that $\{\text{type I}\} \cap \{\text{type II}\} = \phi$ in the case (1) or (2)(ii) by the following Lemma 1.2.

LEMMA 1.1. *In the case* (2), *let* $r \equiv q$, $s \equiv q'$ (*or* $r \equiv -q$, $s \equiv -q'$). *Then there exists a biholomorphic map*

$$\varphi: W/\langle f(\alpha, \beta, \lambda, r), g(r)\rangle \longrightarrow W/\langle f(\gamma, \delta, \mu, s), g(s)\rangle$$

if and only if $\lambda = \mu = 0$, $\alpha = \delta$ *and* $\beta = \gamma$.

PROOF. If φ exists, the lifting of φ on W can be extended to a holomorphic map $\tilde{\varphi}$ from $\mathbf{C}^2$ to $\mathbf{C}^2$ by Hartogs' theorem, that is, $\tilde{\varphi}(z_1, z_2) = (\sum_{i,j\geq 0} a_{i,j} z_1^i z_2^j, \sum_{i,j\geq 0} b_{i,j} z_1^i z_2^j)$. From the compatibility of $\tilde{\varphi}$ and the covering transformations, we get $\lambda = \mu = 0$, $\alpha = \delta$, $\beta = \gamma$ and $\tilde{\varphi}(z_1, z_2) = (z_2, z_1)$. The converse is clear.

LEMMA 1.2. ([8] Lemma 3.5.5) *If* $q^2 \not\equiv -1$, *then there exist no biholomorphic map*

$$\varphi: W/\langle f(\alpha, \beta, \lambda, r), g(r)\rangle \longrightarrow W/\langle f(\gamma, \delta, \mu, s), g(s)\rangle$$

for $r \equiv q$ *and* $s \equiv -q$, $r \equiv q$ *and* $s \equiv -q'$, $r \equiv q'$ *and* $s \equiv -q$ *or* $r \equiv q'$ *and* $s \equiv -q'$.

PROOF. If φ exists, the lifting of φ on W can be extended to a holomorphic map $\tilde{\varphi}: \mathbf{C}^2 \to \mathbf{C}^2$ satisfying $\tilde{\varphi}(0, 0) = (0, 0)$. The induced map $\varphi_1: \mathbf{C}^2/\langle g(r)\rangle \to \mathbf{C}^2/\langle g(s)\rangle$ is a homeomorphism from the cone of $L(p, r)$ into the cone of $L(p, s)$ which restricts to a diffeomorphism outside the cone point. For a $t > 0$ big enough, $L(p, s) \times \{1\} \cap \varphi_1(L(p, r) \times \{t\}) = \phi$. So, we have an h-cobordism between $L(p, s) \times \{1\}$ and $\varphi_1(L(p, r) \times \{t\})$, and they should be isomorphic by orientation preserving diffeomorphism by Theorem 7.27 in [1]. Note that $L(p, r)$ is isomorphic to $L(p, s)$ by orientation preserving diffeomorphism if and only if $r^{\pm 1} \cdot s \equiv 1$, and by orientation reversing diffeomorphism if and only if $r^{\pm 1} \cdot s \equiv -1$ due to [9]. So, if $q^2 \not\equiv -1$, $L(p, r)$ and $L(p, s)$ are isomorphic only by orientation reversing diffeomorphism. Therefore φ_1 is an orientation preserving diffeomorphism but reverses the orientation of the component of lens space, and hence so φ_1 reverses the orientation of the cone parameter. But this contradicts the fact that φ_1 preserves the cone point. Therefore φ does not exist.

2. Deformations of complex structures on $L(p, q) \times S^1$

First we will construct complex analytic families which connect all the complex structures of type Ia on $L(p, q) \times S^1$. Take a region $K = \{(\alpha, \beta) \in \mathbf{C}^2 \mid 0 < |\alpha|, |\beta| < 1\}$ and put $B = K \times \mathbf{C}$. In this paragraph $m > 0$ and $n \geq 0$ denote integers such that $m \equiv n \equiv q$.

Holomorphic maps $\tilde{f}_n, \tilde{g}: W \times B \to W \times B$ are defined by

$$\tilde{f}_n(z_1, z_2, \alpha, \beta, \lambda) = \begin{cases} (\alpha z_1 + \lambda z_2^n, \beta z_2, \alpha, \beta, \lambda) & \text{if } n \neq 0 \\ (\alpha z_1, \beta z_2, \alpha, \beta, \lambda) & \text{if } n = 0 \end{cases}$$

and

$$\tilde{g}(z_1, z_2, \alpha, \beta, \lambda) = (\zeta^q z_1, \zeta z_2, \alpha, \beta, \lambda).$$

An analytic family (V_n, B, π_n) is defined by $V_n = W \times B / \langle \tilde{f}_n, \tilde{g} \rangle$ and a projection map $\pi_n: V_n \to B$. We define $K_m = \{(\alpha, \beta) \in \mathbf{C}^2 \mid \alpha = \beta^m\}$. Define $C = (K - \bigcup_m K_m) \times \mathbf{C} \subset B$ and a holomorphic map $\Phi_n: W \times C \to W \times C\,(n > 0)$ by

$$\Phi_n(z_1, z_2, \alpha, \beta, \lambda) = (z_1 - \lambda z_2^n/(\alpha - \beta^n), z_2, \alpha, \beta, \lambda).$$

Then we have a fiber preserving biholomorphic map $\varphi_n: \pi_0^{-1}(C) \to \pi_n^{-1}(C)$ induced by Φ_n (See [2] Lemma 4.3). Since C is not empty, we see that any complex structures of type Ia are deformation equivalent.

In the same way we can connect all the complex structures of type Ib, IIa and IIb respectively. This means that we have estimated $n(L(p, q) \times S^1)$ for each case as

$$n(L(p, q) \times S^1) \begin{cases} = 1 & \text{for the case } p = 2 \text{ or (2)-(i)} \\ \leq 2 & \text{for the case (1) or (2)-(ii).} \end{cases}$$

Moreover we will show that the equality holds in the latter case. Then, we have Theorem 2.1 which is presented in the introduction. Let X be a complex surface diffeomorphic to $L(p, q) \times S^1$. Because $c_1^2(X) = c_2(X) = 0$, we have $h^0(X, \Theta) = h^1(X, \Theta)$ and $h^2(X, \Theta) = 0$ by Riemann-Roch theorem, where Θ is the sheaf of germs of holomorphic vector fields over X. We will give some basis of $H^1(X, \Theta)$ to examine the versality for the deformations of complex structures. When r is fixed, the complex structures $X = W/\langle f(\alpha, \beta, \varepsilon, r), g(r) \rangle$ with $\varepsilon = 0$ or 1 and $0 < |\alpha| \leq |\beta| < 1$ can be divided into the following cases.

(i) $\varepsilon = 0$ and $\alpha = \beta^n$ for some positive integer n with $n \equiv r$, and (a) $n = 1$ or (b) $n \geq 2$

(ii) $\varepsilon = 0$ and $\alpha \neq \beta^n$ for any positive integer n with $n \equiv r$,

(iii) $\varepsilon = 1$ and $\alpha = \beta^r$ with (a) $r = 1$ or (b) $r \geq 2$.

LEMMA 2.2. *In the above subdivided cases*
(i) (a) $H^0(X, \Theta) \cong \mathbf{C}\{z_1\partial/\partial z_1, z_2\partial/\partial z_1, z_1\partial/\partial z_2, z_2\partial/\partial z_2\}$,
(b) $H^0(X, \Theta) \cong \mathbf{C}\{z_1\partial/\partial z_1, z_2^n\partial/\partial z_1, z_2\partial/\partial z_2\}$,
(ii) $H^0(X, \Theta) \cong \mathbf{C}\{z_1\partial/\partial z_1, z_2\partial/\partial z_2\}$ *and*
(iii) $H^0(X, \Theta) \cong \mathbf{C}\{rz_1\partial/\partial z_1 + z_2\partial/\partial z_2, z_2^r\partial/\partial z_1\}$.

PROOF. Let G be the group $\langle f, g\rangle$ generated by $f = f(\alpha, \beta, \varepsilon, r)$ and $g = g(r)$. For the case (i) and (ii) we simply consider G-invariant holomorphic vector fields on W. By Hartogs' theorem we can extend any holomorphic vector field over W to that over $\mathbf{C}^2$. So any element θ of $H^0(W, \Theta)$ can be written as

$$\theta = \sum_{i,j\geq 0} a_{i,j} z_1^i z_2^j \partial/\partial z_1 + \sum_{i',j'\geq 0} b_{i',j'} z_1^{i'} z_2^{j'} \partial/\partial z_2.$$

Now $H^0(X, \Theta) = H^0(W, \Theta)^G$ is obtained as mentioned above, because θ should satisfy

$$\begin{aligned}(id - g_*)\theta = &\sum_{i,j\geq 0} a_{i,j}(1 - \zeta^{r(1-i)-j}) z_1^i z_2^j \partial/\partial z_1 \\ &+ \sum_{i',j'\geq 0} b_{i',j'}(1 - \zeta^{1-ri'-j'}) z_1^{i'} z_2^{j'} \partial/\partial z_2 = 0\end{aligned}$$

and

$$\begin{aligned}(id - f_*)\theta = &\sum_{i,j\geq 0} a_{i,j}(1 - \alpha^{1-i}\beta^{-j}) z_1^i z_2^j \partial/\partial z_1 \\ &+ \sum_{i',j'\geq 0} b_{i',j'}(1 - \alpha^{-i'}\beta^{1-j'}) z_1^{i'} z_2^{j'} \partial/\partial z_2 = 0\end{aligned}$$

for $f(\alpha, \beta, 0, r)(z_1, z_2) = (\alpha z_1, \beta z_2)$.

For the case (iii), $rz_1\partial/\partial z_1 + z_2\partial/\partial z_2$ and $z_2^r\partial/\partial z_1$ are clearly G-invariant vector fields on W. Wehler showed that $\langle f\rangle$-invariant vector fields on W are generated by the above two vector fields in [12] p. 24 Remark 2. So the result holds.

Next we are concerned with $H^1(X, \Theta)$. Let $\mathscr{U} = \{U_i \mid i \in I\}$ be a locally finite open Stein covering of $X = W/\langle f, g\rangle$. Taking open subsets U_i' of $W' = W/\langle g\rangle$ homeomorphic to U_i by the canonical projection, we define $\tilde{\mathscr{U}} = \{\tilde{U}_i \mid i \in I\}$ by $\tilde{U}_i = \coprod_{m\in\mathbf{Z}} f^m(U_i')$ and we get a short exact sequence of cochain complexes as in [12] p. 26,

$$0 \longrightarrow C^\cdot(\mathscr{U}, \Theta_X) \longrightarrow C^\cdot(\tilde{\mathscr{U}}, \Theta_{W'}) \xrightarrow{id - f_*} C^\cdot(\tilde{\mathscr{U}}, \Theta_{W'}) \longrightarrow 0.$$

The associated long cohomology exact sequence

$$0 \longrightarrow H^0(X, \Theta) \longrightarrow H^0(W', \Theta) \xrightarrow{id - f_*} H^0(W', \Theta) \xrightarrow{\sigma} H^1(X, \Theta) \longrightarrow \cdots$$

has a connecting morphism σ. So,

$$H^1(X, \Theta) \supseteq \mathrm{Im}\,\sigma \cong \mathrm{Coker}\,(id - f_*\colon H^0(W', \Theta) \longrightarrow H^0(W', \Theta)).$$

LEMMA 2.3. $H^1(X, \Theta)$ coinsides with $\mathrm{Im}\,\sigma$.

PROOF. Any elemnt θ of $H^0(W', \Theta) = H^0(W, \Theta)^{\langle g\rangle}$ is written as

$$\theta = \sum'_{i,j\geq 0} a_{i,j} z_1^i z_2^j \partial/\partial z_1 + \sum'_{i',j'\geq 0} b_{i',j'} z_1^{i'} z_2^{j'} \partial/\partial z_2.$$

where the summation $\sum'$ is taken for i, j, i', j' satisfying $r(i-1)+j \equiv 0$ and $ri'+j'-1 \equiv 0$.

If we call $a_{i,j}$, $b_{i',j'}$ the coefficients of θ with respect to $z_1^i z_2^j \partial/\partial z_1$, $z_1^{i'} z_2^{j'} \partial/\partial z_2$, we can define complex numbers $\overline{a_{i,j}}$, $\overline{b_{i',j'}}$ by the coefficients of $(id + f_*)\theta$ with respect to $z_1^i z_2^j \partial/\partial z_1$, $z_1^{i'} z_2^{j'} \partial/\partial z_2$. Then, $z_1^i z_2^j \partial/\partial z_1$, $z_1^{i'} z_2^{j'} \partial/\partial z_2$ with the pairs (i, j) and (i', j') for which $\overline{a_{i,j}} = \overline{b_{i',j'}} = 0$ for any $a_{i,j}$ and $b_{i',j'}$ represent a set of generators of Coker $(id - f_*)$.

In the cases (i) and (ii), we have $\overline{a_{i,j}} = (1 - \alpha^{1-i}\beta^{-j})a_{i,j}$, $\overline{b_{i',j'}} = (1 - \alpha^{-i'}\beta^{1-j'})b_{i',j'}$ and Coker $(id - f_*)$ is generated by the basis corresponding to the basis of Ker $(id - f_*\colon H^0(W', \Theta) \to H^0(W', \Theta)) = H^0(X, \Theta)$. Because $h^0(X, \Theta) = h^1(X, \Theta)$, we obtain the results.

In the case (iii),

$$\begin{aligned} f_*\theta = {} & \sum'_{i,j\geq 0} a_{i,j}(\beta^{-r}z_1 - \beta^{-2r}z_2^r)^i(\beta^{-1}z_2)^j\beta^r\partial/\partial z_1 \\ & + \sum'_{i',j'\geq 0} b_{i',j'}(\beta^{-r}z_1 - \beta^{-2r}z_2^r)^{i'}(\beta^{-1}z_2)^{j'}(r\beta^{1-r}z_2^{r-1}\partial/\partial z_1 + \beta\partial/\partial z_2). \end{aligned}$$

So, if $r(k-1)+l \not\equiv 0$ then $\overline{a_{k,l}} = 0$ and otherwise

$$\begin{aligned} \overline{a_{k,l}} = a_{k,l} & - \sum_{i\geq k}\binom{i}{k}(-1)^{i-k}\beta^{r(1-i)-l}a_{i,l-r(i-k)} \\ & - r\sum_{i'\geq k}\binom{i'}{k}(-1)^{i'-k}\beta^{-ri'-l}b_{i',l+1+r(k-i'-1)}. \end{aligned}$$

In the same way, if $rk+l-1 \not\equiv 0$ then $\overline{b_{k,l}} = 0$ and otherwise

$$\overline{b_{k,l}} = b_{k,l} - \sum_{i\geq k}\binom{i}{k}(-1)^{i-k}\beta^{-ri-l+1}b_{i,l-r(i-k)}.$$

In the case (a), we have $\overline{a_{1,0}} = -\beta^{-1}b_{1,0}$, $\overline{b_{0,1}} = -\overline{a_{1,0}}$ and $\overline{b_{1,0}} = 0$. Therefore, the images of $z_1\partial/\partial z_1 + z_2\partial/\partial z_2$ and $z_1\partial/\partial z_2$ are linearly independent in Coker $(id - f_*)$. Because $h^0(X, \Theta) = h^1(X, \Theta) = 2$, the result holds. In the case (b), if $c(z_1\partial/\partial z_1) + d(z_2\partial/\partial z_2) \in \mathrm{Im}\,(id - f_*)$, then $c = d = 0$ because $\overline{a_{1,0}} = \overline{b_{0,1}} = 0$. Therefore, the images of $z_1\partial/\partial z_1$ and $z_2\partial/\partial z_2$ are linearly independent in Coker $(id - f_*)$. Because $h^0(X, \Theta) = h^1(X, \Theta) = 2$, the result holds.

Let $W' = W/\langle g(r)\rangle$. If $F: W' \times S \to W'$ is a holomorphic map which satisfies $F(x, s_0) = f(x)$ and $F \mid W' \times \{s\}$ is biholomorphic for any $s \in S$, we can define a biholomorphic map $H: W' \times S \to W' \times S$ by $H(x, s) = (F(x, s), s)$ where $x = (z_1, z_2)$. Let $Y = (W' \times S)/H$ and $\pi: Y \to S$ a projection map. Then (Y, S, π) is a complex analytic family containing $\pi^{-1}(s_0) = X = W/\langle f, g\rangle$.

We will give examples of S and F which induce complex analytic families versal at s_0 as in [12] Theorem 2.

(i)(a)

$$S = \{s \in GL(2, \mathbf{C}) \mid |\text{eigenvalues of } s| < 1\}$$

$$F(x, s) = s \cdot {}^t x, \ s_0 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

(b)

$$S = \{(\alpha, \beta, \lambda) \in \mathbf{C}^3 \mid 0 < |\alpha|, |\beta| < 1\}$$

$$F(z_1, z_2, \alpha, \beta, \lambda) = (\alpha z_1 + \lambda z_2^n, \beta z_2), \ s_0 = (\beta^n, \beta, 0) \text{ with } n \equiv r$$

(ii)

$$S = \{(\alpha, \beta) \in \mathbf{C}^2 \mid |\alpha|, |\beta| < 1\}$$

$$F(z_1, z_2, \alpha, \beta) = (\alpha z_1, \beta z_2), \ s_0 = (\alpha, \beta)$$

(iii)(a)

$$S = \{(\alpha, \gamma) \in \mathbf{C}^2 \mid 0 < |\alpha|, |\gamma| < 1\}$$

$$F(z_1, z_2, \alpha, \gamma) = (\alpha z_1 + z_2, \gamma z_1 + \alpha z_2), \ s_0 = (\alpha, 0)$$

(b)

$$S = \{(\alpha, \beta) \in \mathbf{C}^2 \mid 0 < |\alpha|, |\beta| < 1\}$$

$$F(z_1, z_2, \alpha, \beta) = (\alpha z_1 + z_2^r, \beta z_2), \ s_0 = (\beta^r, \beta)$$

To show the above families versal at s, it suffices to show the infinitesimal deformation map

$$\rho: T_s S \longrightarrow H^1(X, \Theta)$$

is an isomorphism. We note that the infinitesimal deformation map can be decomposed as follows.

Lemma 2.4. *In the above examples a linear map $\tau: T_s S \to H^0(W', \Theta)$ is defined by*

$$\tau\left(\textstyle\sum_{k=1}^n a_k(\partial/\partial s_k)_s\right) = \left(\textstyle\sum_{k=1}^n a_k a_{1k}\right)\partial/\partial z_1 + \left(\textstyle\sum_{k=1}^n a_k a_{2k}\right)\partial/\partial z_2,$$

where $\partial F/\partial s(x, s) = (a_{ik})_{\substack{i=1,2\\ 1 \le k \le n}}$. Then $\rho = \sigma \circ \tau$.

PROOF. This lemma follows from [12] Lemma 3 by changing W by W'.

THEOREM 2.5. *Each family* (Y, S, π) *given above is a versal deformation at* s_0.

PROOF. Case (i)-(a): Because $F(z_1, z_2, \alpha, \beta, \gamma, \delta) = (\alpha z_1 + \beta z_2, \gamma z_1 + \delta z_2)$ where we rewrite a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ as a row vector $(\alpha, \beta, \gamma, \delta)$,

$$\partial F/\partial s(H^{-1}(z_1, z_2, \alpha, 0, \alpha, 0)) = \begin{pmatrix} \alpha^{-1}z_1 & \alpha^{-1}z_2 & 0 & 0 \\ 0 & 0 & \alpha^{-1}z_1 & \alpha^{-1}z_2 \end{pmatrix}.$$

Therefore for a tangent vector

$$v = a_1(\partial/\partial\alpha)_{s_0} + a_2(\partial/\partial\beta)_{s_0} + a_3(\partial/\partial\gamma)_{s_0} + a_4(\partial/\partial\delta)_{s_0},$$
$$\tau(v) = a_1\alpha^{-1}z_1\partial/\partial z_1 + a_2\alpha^{-1}z_2\partial/\partial z_1 + a_3\alpha^{-1}z_1\partial/\partial z_2 + a_4\alpha^{-1}z_2\partial/\partial z_2.$$

As is mentioned in the proof of Lemma 2.3, Coker $(id - f_*)$ is generated by $z_1\partial/\partial z_1$, $z_2\partial/\partial z_1$, $z_1\partial/\partial z_2$ and $z_2\partial/\partial z_2$ which are contained in the images of τ. So, τ and ρ are surjections by Lemma 2.3 and hence ρ is an isomorphism because $\dim_{\mathbf{C}} T_{s_0}S = h^1(X, \Theta) = 4$. In the cases (i)-(b) and (ii) are shown in the same way.

Case (iii)-(b): Because $F(z_1, z_2, \alpha, \beta) = (\alpha z_1 + z_2^r, \beta z_2)$,

$$\partial F/\partial s(H^{-1}(z_1, z_2, \beta^r, \beta)) = \begin{pmatrix} \beta^{-r}z_1 - \beta^{-2r}z_2^r & 0 \\ 0 & \beta^{-1}z_2 \end{pmatrix}.$$

So, for a tangent vector $v = a_1(\partial/\partial\alpha)_{s_0} + a_2(\partial\beta)_{s_0}$,

$$\tau(v) = a_1(\beta^{-r}z_1 - \beta^{-2r}z_2^r)\partial/\partial z_1 + a_2\beta^{-1}z_2\partial/\partial z_2.$$

Because $\overline{a_{0,r}} = \beta^{-r}a_{1,0} - r\beta^{-r}b_{0,1}$ and $a_{1,0}$ appears only in $\overline{a_{0,r}}$, we see that $z_2^r\partial/\partial z_1 = (id - f_*)(\beta^r z_1\partial/\partial z_1)$. So, $\tau(v) = a_1\beta^{-r}z_1\partial/\partial z_1 + a_2\beta^{-1}z_2\partial/\partial z_2$ mod $\mathrm{Im}(id - f_*)$. As is mentioned in the proof of Lemma 2.3, Coker $(id - f_*)$ is generated by $z_1\partial/\partial z_1$ and $z_2\partial/\partial z_2$ contained in the images of τ, so ρ is an isomorphism by Lemma 2.3 as before. In the case (iii)-(a) are shown in the same way.

COROLLARY 2.6. *The types I and II of the complex structures on* $L(p, q) \times S^1$ *are invariant under deformation.*

PROOF. Note first that any complex structure is biholomorphic to the fiber at s_0 for some family in Theorem 2.5. By the versality any of its local deformation belongs to the family of quotient spaces given in Theorem 2.5. Since the parameter r for the quotient space is invariant in the family given in Theorem 2.5, the type defined in the introduction is invariant under the local deformation and hence under the deformation.

The proof of Theorem 2.1 follows immediately from Corollary 2.6.

REMARK: The invariance mentioned in Corrollary 2.6 is obtained also in Corollary 7.23 in [3]. However, the statement of Corollary 7.25 in [3] seems incorrect: $q \not\equiv \pm q^{-1}$ in the first line should be $q \not\equiv -q^{-1}$ and $q \equiv \pm q^{-1}$ in the third line should be $q \equiv -q^{-1}$.

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