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On loosely self-similar sets

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1. Introduction

In [7], J. E. Hutchinson set up a theory of strictly self-similar set, which is defined as the unique compact set satisfying the following equality;

$$K = \bigcup_{i=1}^{m} f_i(K)$$

for a given finite set $\{f_i\}_{i=1}^{m}$ of contraction affine maps on a compact subset X of $\mathbb{R}^N$ ($m \geq 2$). Let r_i be the contraction rate of f_i, that is, $|f_i(x) - f_i(y)| = r_i|x - y|$ for $x, y \in X$, $i = 1, 2, \cdots, m$, and let α be the unique solution of $\sum_{i=1}^{m} r_i^{\alpha} = 1$. In his theory, a Borel probability measure ν on $\mathbb{R}^N$ satisfying $\nu(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(X)) = \prod_{j=1}^{n} r_{i_j}^{\alpha}$ coincides with the α-dimensional Hausdorff measure on K up to constant, that is, there exists a positive constant C such that $\nu(A) = C\mathrm{H}^{\alpha}(A)$ for any Borel set $A \subseteq K$. Here H^{α} denotes the α-dimensional Hausdorff measure.

We now explain his result from the standpoint of Tricot. Tricot [13] showed that for any Borel set $E \subset \mathbb{R}^N$,

$$\text{H-dim}(E) = \sup_{\mu \in \mathscr{M}_E} \{\inf_{x \in E} \phi(\mu; x)\}. \tag{1.1}$$

Where $\mathscr{M}_E = \{\mu;$ positive finite Borel measure on $\mathbb{R}^N$ with $\mu(E) > 0\}$ and for $\mu \in \mathscr{M}_E$

$$\phi(\mu; x) = \liminf_{r \downarrow 0} \frac{\log \mu(E \cap B(x, r))}{\log r}. \tag{1.2}$$

H-dim(E) denotes the Hausdorff dimension of E, $B(x, r)$ denotes the closed ball with radius r and center at x. We can easily see that the α-dimensional Hausdorff measure itself attains the supreme in the righthand side of (1.1) in Hutchinson's case. Let

$$K(P_1, P_2, \cdots, P_m) =$$
$$\{x \in \textstyle\bigcap_{n=1}^{\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(X); \#\{j; i_j = k, j \leq n\}/n \to P_k \text{ as } n \to \infty\},$$

$\beta(P_1, P_2, \cdots, P_m)$ denote the Hausdorff dimension of $K(P_1, P_2, \cdots, P_m)$ and $\nu_{(P_1, P_2, \cdots, P_m)}$ be the Borel probability measure satisfying

$$\nu_{(P_1,P_2,\cdots,P_m)}(f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_n}(X))=\prod_{j=1}^{n}P_{i_j}.$$

Billingsley [1] treated $K(P_1, P_2,\cdots,P_m)$ instead of K. Since in Billingsley's cases, $\nu_{(P_1,P_2,\cdots,P_m)}$ attains the supreme in the righthand side of (1.1), we analogically guess that $\nu_{(P_1,P_2,\cdots,P_m)}$ is equivalent to $\beta(P_1, P_2,\cdots,P_m)$-dimensional Hausdorff measure like ν for K. In this paper, however, readers will know that it is not so.

In this paper, we will introduce *a loosely self-similar set* K (see (2.4)) which is a Cantor set topologically isomorphic to $\{1, 2,\cdots,m\}^{\mathbb{N}}$ but does not have strict self-similarity in the sense of Hutchinson's. We construct a Borel probability measure ν (similar to the case of strictly self-similar set) and show that ν and the α-dimensional Hausdorff measure are absolutely continuous to each other on K (see THEOREM 1 (A)) but they are not necessarily coincident up to constant (see section 4). Nevertheless in Hutchinson's case, they are coincident up to constant.

Moreover, we show that a Borel probability measure $\nu_{(P_1,P_2,\cdots,P_m)}$ and $\beta(P_1, P_2,\cdots,P_m)$-dimensional Hausdorff measure are absolutely continuous to each other on $K(P_1, P_2,\cdots,P_m)$ if and only if $(P_1, P_2,\cdots,P_m)=(r_1^\alpha, r_2^\alpha,\cdots,r_m^\alpha)$ (see REMARK of THEOREM 1).

Finally in this paper, we show that $K(P_1, P_2,\cdots,P_m)$ and K are equivalent in the view of the box dimension (see THEOREM 4) but not so in the view of the Hausdorff dimension (see THEOREM 3 (G)(I)). More precisely if $(P_1, P_2,\cdots, P_m)\neq(r_1^\alpha, r_2^\alpha,\cdots,r_m^\alpha)$ then the Hausdorff dimension of $K(P_1, P_2,\cdots,P_m)$ is less than α (see THEOREM 3 (I)). The α-dimensional Hausdorff measure of $K\setminus K(r_1^\alpha, r_2^\alpha,\cdots,r_m^\alpha)$ equals to 0 (see THEOREM 2 (E)). However $K\setminus K(r_1^\alpha, r_2^\alpha,\cdots, r_m^\alpha)$ and K are equivalent in the view of the Hausdorff dimension (see THEOREM 3 (H)).

In section 2, we introduce *a loosely self-similar set* and claim the results in this paper. In section 3, we prove them. In section 4, we introduce two examples.

2. Results

Through the whole paper, H^α and λ_N denote the α-dimensional Hausdorff measure and the N-dimensional Lebesgue measure, respectively ($\alpha\geq 0$, $N\in\mathbb{N}$). H-dim, $\underline{\text{M-dim}}$ and $\overline{\text{M-dim}}$ denote the Hausdorff dimension, the lower and the upper box dimensions, respectively, which are defined on the Euclidean space $(\mathbb{R}^N, d)$ as follows; for any bounded set $E\subset\mathbb{R}^N$

$$\text{H-dim}(E)=\inf\{\alpha;\ \mathrm{H}^\alpha(E)=0\}=\sup\{\alpha;\ \mathrm{H}^\alpha(E)=\infty\},$$

$$\underline{\text{M-dim}}\ (E) = \liminf_{\varepsilon \downarrow 0} \frac{\log(\mathrm{N}_\varepsilon(E))}{\log 1/\varepsilon}, \quad \overline{\text{M-dim}}\ (E) = \limsup_{\varepsilon \downarrow 0} \frac{\log(\mathrm{N}_\varepsilon(E))}{\log 1/\varepsilon},$$

where

$$\mathrm{H}^\alpha(E) = \lim_{\varepsilon \downarrow 0} \inf\{\sum_i |U_i|^\alpha;\ E \subseteq \bigcup_i U_i,\ |U_i| \le \varepsilon\},$$

$$\mathrm{N}_\varepsilon(E) = \inf \#\{U_i;\ E \subseteq \bigcup_i U_i,\ |U_i| \le \varepsilon\}$$

and $|U| = \sup_{x,y \in U} |x-y|$. We know that H-dim $(E) \le \underline{\text{M-dim}}\ (E) \le \overline{\text{M-dim}}\ (E)$ in general.

Suppose that $\{\varphi_{i_1 i_2 \cdots i_k}: (i_1, i_2, \cdots, i_k) \in \{1, 2, \cdots, m\}^k,\ k = 1, 2, \cdots\}$ $(m \ge 2)$ is a sequence of mappings on a compact subset X of $\mathbb{R}^N$ with $\lambda_N(X) > 0$ such that

$$\varphi_{i_1 i_2 \cdots i_k}: X \to X, \qquad i_j \in \{1, 2, \cdots, m\}, \tag{2.1}$$

$$r_{i_k}|x - y| = |\varphi_{i_1 i_2 \cdots i_k}(x) - \varphi_{i_1 i_2 \cdots i_k}(y)| \quad \text{for all } x, y \in X,\ 0 < r_{i_k} < 1, \tag{2.2}$$

$$\varphi_{i_1 i_2 \cdots i_{k-1} i_k}(X) \cap \varphi_{i_1 i_2 \cdots i_{k-1} i'_k}(X) = \emptyset \quad (i_k \ne i'_k). \tag{2.3}$$

Put

$$[i_1, i_2, \cdots, i_n] = \varphi_{i_1} \circ \varphi_{i_1 i_2} \circ \cdots \circ \varphi_{i_1 i_2 \cdots i_n}(X)$$

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, i_2, \cdots, i_n) \in \{1, 2, \cdots, m\}^n} [i_1, i_2, \cdots, i_n]. \tag{2.4}$$

We say that K is *a loosely self-similar set generated by* $\{\varphi_{i_1 i_2 \cdots i_k}\}$.

Since $\bigcap_{n=1}^{\infty} [\omega_1, \omega_2, \cdots, \omega_n]$ consists of a single point for any $\omega = (\omega_1, \omega_2, \cdots) \in \{1, 2, \cdots, m\}^{\mathbb{N}}$, we denote it by $\bigcap_{n=1}^{\infty} [\omega_1, \omega_2, \cdots, \omega_n]$. Then we can define a bijection map φ from $\{1, 2, \cdots, m\}^{\mathbb{N}}$ to K by

$$\varphi: \omega = (\omega_1, \omega_2, \cdots) \in \{1, 2, \cdots, m\}^{\mathbb{N}} \to \varphi(\omega) = \bigcap_{n=1}^{\infty} [\omega_1, \omega_2, \cdots, \omega_n]. \tag{2.5}$$

Through the whole paper, we assume that $\{P_i\}_{i=1}^m$ satisfies the conditions

$$\sum_{i=1}^m P_i = 1, \qquad 0 < P_i < 1, \tag{2.6}$$

and set

$$K(P_1, P_2, \cdots, P_m) = \left\{\varphi(\omega);\ \frac{N_i(\omega, n)}{n} \to P_i \text{ as } n \to \infty\right\},$$

where

$$N_i(\omega, n) = \#\{k; 1 \le k \le n, \omega_k = i\} \quad \text{for} \quad \omega = (\omega_1, \omega_2, \cdots) \in \{1, 2, \cdots, m\}^{\mathbb{N}}.$$

$K(P_1, P_2, \cdots, P_m)$ is a Borel set but not a compact set and hence it is not a Cantor set. Let $\nu_{(P_1, P_2, \cdots, P_m)}$ be the Borel probability measure on $\mathbb{R}^N$ such that $\nu_{(P_1, P_2, \cdots, P_m)}([\omega_1, \omega_2, \cdots, \omega_n]) = \prod_{j=1}^{n} P_{\omega_j}$ for any $n, \omega_1, \omega_2, \cdots, \omega_n$. Since

$$\nu_{(P_1, P_2, \cdots, P_m)}(K) = \nu_{(P_1, P_2, \cdots, P_m)}(K(P_1, P_2, \cdots, P_m)) = 1,$$

the probability measure $\nu_{(P_1, P_2, \cdots, P_m)}$ is called *the* $(P_1, P_2, \cdots, P_m)$*-Bernoulli measure on* K.

We say that a (an outer) measure μ on $\mathbb{R}^N$ is a Borel (outer) measure if any Borel set is μ-measurable. It is well-known that β-dimensional Hausdorff measure H^β is a Borel outer measure, since it is a metric outer measure [5]. Two Borel (outer) measures ν and μ on $\mathbb{R}^N$ are said to be absolutely continuous to each other on a given Borel set F if $\nu(B) = 0 \Leftrightarrow \mu(B) = 0$ for any Borel set $B \subseteq F$.

THEOREM 1. *Assume that* $(P_1, P_2, \cdots, P_m)$ *satisfies* (2.6). *Let* $\beta(P_1, P_2, \cdots, P_m) = H\text{-}dim\,(K(P_1, P_2, \cdots, P_m))$ *and* α *be the unique solution of* $\sum_{i=1}^{m} r_i^\alpha = 1$. *Then*

(A) $\nu_{(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)}$ *and the* α*-dimensional Hausdorff measure are absolutely continuous to each other on* K.

(B) There exists a Borel subset M *of* $K(P_1, P_2, \cdots, P_m)$ *such that* $\nu_{(P_1, P_2, \cdots, P_m)}(M) = 1$ *and* $H^{\beta(P_1, P_2, \cdots, P_m)}(M) = 0$ *unless* $(P_1, P_2, \cdots, P_m) = (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$.

REMARK. Taking THEOREM 2 (D) and the fact $\nu(K(P_1, P_2, \cdots, P_m)) = 1$ into consideration, by THEOREM 1 we see that the Hausdorff measure $H^{\beta(P_1, P_2, \cdots, P_m)}$ and $\nu_{(P_1, P_2, \cdots, P_m)}$ are absolutely continuous to each other on $K(P_1, P_2, \cdots, P_m)$ if and only if $(P_1, P_2, \cdots, P_m) = (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$. On the other hand, by using Bowen's results [3], K. Handa [6] has already acquired a similar result to (A) on $\mathbb{R}^1$ under a different setting. Our idea of proof in this paper is different from his. It seems to us difficult that we generalize his proof to $\mathbb{R}^N$. Moreover we add the result (B) to his results in this paper.

The second theorem claims that the α-dimensional Hausdorff measure on K concentrates in $K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$.

THEOREM 2. *For* α *in* THEOREM 1,

(C) $H\text{-}dim\,(K) = \alpha$,

(D) $H^\alpha(K) = H^\alpha(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha))$,

(E) $H^\alpha(K \setminus K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) = 0$,

(F) $0 < H^\alpha(K) < \infty$.

The third theorem claims that the Hausdorff dimension of $K \setminus K(r_1^\alpha, r_2^\alpha, \cdots,$

r_m^α) equals to the Hausdorff dimension of K itself. For a similar example, the Hausdorff dimensions of simply normal numbers and simply non-normal numbers on [0, 1] both equal to 1 (c.f. [12]). Nevertheless the one-dimensional Hausdorff measure of simply non-normal numbers equals to 0. This is clear from the law of large number and the fact that the one-dimensional Hausdorff measure on [0, 1] coincides with the one-dimensional Lebesgue measure.

THEOREM 3. *For* α *in* THEOREM 1,
(*G*) $H\text{-}dim\,(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) = H\text{-}dim\,(K) = \alpha$,
(*H*) $H\text{-}dim\,(K \setminus K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) = H\text{-}dim\,(K)$,
(*I*) $H\text{-}dim(K(P_1, P_2, \cdots, P_m)) = \dfrac{\sum_{i=1}^m P_i \log P_i}{\sum_{i=1}^m P_i \log r_i} \leq \alpha$ *for any* $(P_1, P_2, \cdots, P_m)$ *satisfying* (2.6) *and the equality is attained only in the case of* $(P_1, P_2, \cdots, P_m) = (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$.

The fourth theorem claims that the box dimension of $K(P_1, P_2, \cdots, P_m)$ equals to α for any $(P_1, P_2, \cdots, P_m)$. Together with (G), this fact implies that there is a gap between the Hausdorff dimension and the box dimension of $K(P_1, P_2, \cdots, P_m)$ if $(P_1, P_2, \cdots, P_m) \neq (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$.

THEOREM 4. *For* α *in* THEOREM 1 *and for any* $(P_1, P_2, \cdots, P_m)$ *satisfying* (2.6),
(*J*) $\underline{M\text{-}dim}\,(K(P_1, P_2, \cdots, P_m)) = \overline{M\text{-}dim}\,(K(P_1, P_2, \cdots, P_m)) = \alpha$.

3. Proofs

For the proof of THEOREM 1, the result of THEOREM 3 (I) is needed. Therefore we will prove THEOREM 3 (I) at first. Put

$$\mathscr{R}_n = \{[\omega_1, \omega_2, \cdots, \omega_n]; (\omega_1, \omega_2, \cdots, \omega_n) \in \{1, 2, \cdots, m\}^n\}, \ \mathscr{R} = \bigcup_{n=1}^{\infty} \mathscr{R}_n.$$

The following two propositions are proved under more general conditions [8]. PROPOSITION 3.1 can be proved in accordance with Billingsley's method [1]. In this paper, we will give a brief proof of PROPOSITION 3.2 for readers' convenience.

PROPOSITION 3.1. *Assume that* μ *is a positive finite Borel measure on* $\mathbb{R}^N$ *such that*

$$\mu([\omega_1, \omega_2, \cdots, \omega_n]) > 0 \qquad \textit{for any} \ \ (\omega_1, \omega_2, \cdots, \omega_n). \tag{3.1}$$

If $E \subseteq K$ *with* $\mu^*(E) > 0$ *satisfies*

$$a \leq \liminf_{n\to\infty} \frac{\log(\mu([\omega_1, \omega_2, \cdots, \omega_n]))}{\log(|[\omega_1, \omega_2, \cdots, \omega_n]|)} \leq \limsup_{n\to\infty} \frac{\log(\mu([\omega_1, \omega_2, \cdots, \omega_n]))}{\log(|[\omega_1, \omega_2, \cdots, \omega_n]|)} \leq b$$

for any $\bigcap_{n=1}^{\infty}[\omega_1, \omega_2, \cdots, \omega_n] \in E$, *then*

$$a \leq H\text{-}dim(E) \leq b,$$

where μ^* *is the outer measure induced from the measure* μ.

PROPOSITION 3.2. *Assume that* μ *is a positive finite Borel measure on* $\mathbb{R}^N$ *satisfying the condition* (3.1). *If*

$$a \leq \liminf_{n\to\infty} \frac{\mu([\omega_1, \omega_2, \cdots, \omega_n])}{|[\omega_1, \omega_2, \cdots, \omega_n]|^\delta} \leq \limsup_{n\to\infty} \frac{\mu([\omega_1, \omega_2, \cdots, \omega_n])}{|[\omega_1, \omega_2, \cdots, \omega_n]|^\delta} \leq b$$

hold for any $\bigcap_{n=1}^{\infty}[\omega_1, \omega_2, \cdots, \omega_n] \in E$, *then there exists a positive constant* L *depending only on* N, X, $\lambda = 1/\min_i r_i > 1$ *such that*

$$b^{-1}\lambda^{-\delta}L^{-1}\mu^*(E) \leq H^\delta(E) \leq a^{-1}\mu^*(E).$$

PROOF OF PROPOSITION 3.2. For $\rho > 0$, $\varepsilon > 0$, set

$$E_{\rho,\varepsilon} = \{x \in E; (a-\varepsilon)|R|^\delta \leq \mu(R) \leq (b+\varepsilon)|R|^\delta \text{ or } |R| \geq \lambda\rho$$
$$\text{for any } R \in \mathscr{R} \text{ such that } x \in R\}.$$

Firstly we prove the lefthand side inequality of the proposition. Put $C = \lambda_N(X)/|X|^N$, $L = (2\lambda)^N \Omega_N C^{-1}$ and $\Omega_N = \pi^{\frac{1}{2}N}/\Gamma(N/2+1)$. Then $0 < L < \infty$, since $0 < \lambda_N(X) < \infty$. For a given $U \subset \mathbb{R}^N$ and the integer n with $\lambda^{-n} < |U| \leq \lambda^{-n+1}$, $U \cap K$ can be covered by R's less than L such that $R \in \mathscr{R}$, $\lambda^{-n} < |R| \leq \lambda^{-n+1}$. For any $\gamma > 0$, $(0 <)\rho' < \rho$, let $\{U_i\}_i$ be a ρ'-covering of $E_{\rho,\varepsilon}$ such that $\mathrm{H}^\delta_{\rho'}(E_{\rho,\varepsilon}) \geq \sum_i |U_i|^\delta - \gamma$. Then we can find $\{R_{ij}\}_{j=1}^{m_i} \subset \mathscr{R}$ such that

$$m_i \leq L,\ R_{ij} \cap E_{\rho,\varepsilon} \neq \emptyset,\ U_i \cap E_{\rho,\varepsilon} \subseteq \bigcup_{j=1}^{m_i} R_{ij},$$
$$\lambda^{-1}|U_i| \leq |R_{ij}| \leq \lambda|U_i| \qquad \text{for any } i, j.$$

Then

$$\sum_{i,j} |R_{ij}|^\delta \leq \lambda^\delta L \sum_i |U_i|^\delta \leq \lambda^\delta L(\mathrm{H}^\delta_{\rho'}(E_{\rho,\varepsilon}) + \gamma).$$

By the definition of $E_{\rho,\varepsilon}$ and $|R_{ij}| < \lambda\rho$, we have $\mu(R_{ij}) \leq (b+\varepsilon)|R_{ij}|^\delta$ for any i, j. Therefore we have the following estimate

$$\lambda^\delta L(\mathrm{H}^\delta(E) + \gamma) \geq \lambda^\delta L(\mathrm{H}^\delta_{\rho'}(E_{\rho,\varepsilon}) + \gamma) \geq \sum_{i,j} |R_{ij}|^\delta$$

$$\geq (b+\varepsilon)^{-1}\sum_{i,j}\mu(R_{ij}) \geq (b+\varepsilon)^{-1}\mu^*(E_{\rho,\varepsilon}).$$

By letting $\gamma \downarrow 0$, we have

$$(b+\varepsilon)^{-1}\mu^*(E_{\rho,\varepsilon}) \leq \lambda^\delta \mathrm{LH}^\delta(E).$$

Since μ^* is an outer measure and $E_{\rho,\varepsilon} \uparrow E$ as $\rho \downarrow 0$, we have

$$(b+\varepsilon)^{-1}\mu^*(E) \leq \lambda^\delta \mathrm{LH}^\delta(E).$$

Since $\varepsilon > 0$ is arbitrary, we have the lefthand side inequality.

Secondly we prove the righthand side inequality. For $\gamma > 0$, $(0<)\rho' < \rho$, we can find $\{R_i\}_i \subset \mathscr{R}$ such that

$$|R_i| < \rho',\ E_{\rho,\varepsilon} \subseteq \bigcup_i R_i,\ R_i \cap R_j = \emptyset\,(i \neq j),\ R_i \cap E_{\rho,\varepsilon} \neq \emptyset,$$

$$0 \leq \sum_i \mu(R_i) - \mu^*(E_{\rho,\varepsilon}) < \gamma.$$

Since $(a-\varepsilon)|R_i|^\delta \leq \mu(R_i)$ by the definition of $E_{\rho,\varepsilon}$ and $|R_i| < \lambda\rho$,

$$\mu^*(E) \geq \mu^*(E_{\rho,\varepsilon}) \geq \sum_i \mu(R_i) - \gamma$$

$$\geq (a-\varepsilon)\sum_i |R_i|^\delta - \gamma \geq (a-\varepsilon)\mathrm{H}^\delta_{\rho'}(E_{\rho,\varepsilon}) - \gamma.$$

By letting $\rho', \gamma \downarrow 0$, we have

$$\mathrm{H}^\delta(E_{\rho,\varepsilon}) \leq (a-\varepsilon)^{-1}\mu^*(E).$$

Therefore we have the righthand side inequality. □

PROOF OF THEOREM 3 (I). By the definition of $K(P_1, P_2, \cdots, P_m)$, for all $(\omega_1, \omega_2, \cdots)$ such that $\bigcap_{n=1}^\infty [\omega_1, \omega_2, \cdots, \omega_n] \in K(P_1, P_2, \cdots, P_m)$,

$$\lim_{n\to\infty} \frac{\log \nu_{(P_1,P_2,\cdots,P_m)}([\omega_1, \omega_2, \cdots, \omega_n])}{\log |[\omega_1, \omega_2, \cdots, \omega_n]|} = \lim_{n\to\infty} \frac{\sum_{i=1}^m N_i(\omega, n)\log P_i}{\sum_{i=1}^m N_i(\omega, n)\log r_i}$$
$$= \frac{\sum_{i=1}^m P_i \log P_i}{\sum_{i=1}^m P_i \log r_i}.$$

Since $\nu_{(P_1,P_2,\cdots,P_m)}(K(P_1, P_2, \cdots, P_m)) = 1$, we have

$$\text{H-dim}\,(K(P_1, P_2, \cdots, P_m)) = \frac{\sum_{i=1}^m P_i \log P_i}{\sum_{i=1}^m P_i \log r_i} \quad \text{for any } (P_1, P_2, \cdots, P_m)$$

by PROPOSITION 3.1. Since $\dfrac{\sum_{i=1}^m P_i \log P_i}{\sum_{i=1}^m P_i \log r_i} \leq \alpha$ and the equality holds if and only if $P_i = r_i^\alpha$, $i = 1, 2, \cdots, m$, we have (I). $\square$

PROPOSITION 3.3. *Assume that* $(P_1, P_2, \cdots, P_m)$ *satisfies* (2.6). *Put* $H\text{-}dim(K(P_1, P_2, \cdots, P_m)) = \beta(P_1, P_2, \cdots, P_m)$. *For* $\omega = (\omega_1, \omega_2, \cdots) \in \{1, 2, \cdots, m\}^{\mathbb{N}}$, *set*

$$d_n(\omega) = \frac{\nu_{(P_1, P_2, \cdots, P_m)}([\omega_1, \omega_2, \cdots, \omega_n])}{|[\omega_1, \omega_2, \cdots, \omega_n]|^{\beta(P_1, P_2, \cdots, P_m)}}$$

and define

$$B = \left\{\varphi(\omega);\ \limsup_{n\to\infty} d_n(\omega) = \infty,\ \lim_{n\to\infty} \frac{N_i(\omega, n)}{n} = P_i\ \ i = 1, 2, \cdots, m\right\}.$$

Then we see that

(*a*) $\nu_{(P_1, P_2, \cdots, P_m)}(B) = 1$ *unless* $(P_1, P_2, \cdots, P_m) = (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$,

(*b*) $H^\beta(B) = 0$ *unless* $(P_1, P_2, \cdots, P_m) = (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$.

PROOF OF (a). Put $P = \nu_{(P_1, P_2, \cdots, P_m)} \circ \varphi$, $\beta = \beta(P_1, P_2, \cdots, P_m)$ and $X_n(\omega) = \log \frac{P_{\omega_n}}{r_{\omega_n}^\beta}$, then $\{X_n\}$ is independent, identically distributed random variables with respect to P. Since $\beta = \frac{\sum_{i=1}^m P_i \log P_i}{\sum_{i=1}^m P_i \log r_i}$ by (I), we see that

$$E_P[X_n] = \sum_{i=1}^m P_i(\log P_i - \log r_i^\beta) = 0.$$

By the uniqueness of α, $P_i = r_i^\beta$ for $i = 1, 2, \cdots, m$ if and only if $(P_1, P_2, \cdots, P_m) = (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$. Since $(P_1, P_2, \cdots, P_m) \neq (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$ by the assumption, we have

$$0 < E_P[X_n^2] = \sum_{i=1}^m P_i(\log P_i - \log r_i^\beta)^2 < \infty.$$

Since $\log d_n(\omega) = \sum_{j=1}^n X_j(\omega)$, by the law of iterated logarithm [2], we see that

$$\nu_{(P_1, P_2, \cdots, P_m)}(\{\varphi(\omega);\ \limsup_{n\to\infty} \log d_n(\omega) = \infty\}) = 1.$$

This implies $\nu_{(P_1, P_2, \cdots, P_m)}(B) = 1$. $\square$

PROOF OF (b). Put
$\mathscr{R}_n(\eta) = \{[\omega_1, \omega_2, \cdots, \omega_n];\ d_n(\omega) > \eta,\ \ \omega = (\omega_1, \omega_2, \cdots) \in \{1, 2, \cdots, m\}^{\mathbb{N}}\}$, $\mathscr{R}(\eta) = \bigcup_{n=1}^\infty \mathscr{R}_n(\eta)$. Then we can choose $\{R_i^\eta\}_i$ for any $\rho > 0$ and $\eta > 0$, such that

$$B \subseteq \bigcup_i R_i^\eta,\ |R_i^\eta| < \rho,\ R_i^\eta \in \mathscr{R}(\eta),\ \nu_{(P_1, P_2, \cdots, P_m)}(R_i^\eta) > \eta\, |R_i^\eta|^\beta,\ R_i^\eta \cap R_j^\eta = \emptyset\,(i \neq j).$$

By the definition of H^β, we have

$$H_\rho^\beta(B) \le \sum_i |R_i^\eta|^\beta < \frac{1}{\eta}\sum_i \nu_{(P_1,P_2,\cdots,P_m)}(R_i^\eta) \le \frac{1}{\eta} \qquad \text{for any } \rho > 0.$$

Therefore, by letting $\eta \to \infty$, we see

$$H_\rho^\beta(B) = 0 \qquad \text{for any } \rho > 0.$$

This implies $H^\beta(B) = 0$. □

Now we prove THEOREM 1.

PROOF OF THEOREM 1. (B) is clear from PROPOSITION 3.3. Let α be the positive number which satisfies $\sum_{i=1}^m r_i^\alpha = 1$. Let us assume that $(P_1, P_2, \cdots, P_m) = (r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)$. Then we have for all $\omega \in \{1, 2, \cdots, m\}^{\mathbb{N}}$ and $n \in \mathbb{N}$,

$$\frac{\nu_{(r_1^\alpha,r_2^\alpha,\cdots,r_m^\alpha)}([\omega_1, \omega_2, \cdots, \omega_n])}{|[\omega_1, \omega_2, \cdots, \omega_n]|^\alpha} = \frac{\prod_{i=1}^m r_i^{\alpha N_i(\omega,n)}}{\prod_{i=1}^m r_i^{\alpha N_i(\omega,n)} |X|^\alpha} = |X|^{-\alpha}. \tag{3.2}$$

Since $\nu_{(r_1^\alpha,r_2^\alpha,\cdots,r_m^\alpha)}([\omega_1, \omega_2, \cdots, \omega_n]) > 0$ for any $\omega_1, \omega_2, \cdots, \omega_n$, n by the condition (2.6), the condition (3.1) of PROPOSITION 3.2 is satisfied. Therefore by PROPOSITION 3.2, we have

$$\lambda^{-\alpha} L_{N,\lambda,C}^{-1} |X|^\alpha \nu_{(r_1^\alpha,r_2^\alpha,\cdots,r_m^\alpha)}(B) \le H^\alpha(K \cap B) \le |X|^\alpha \nu_{(r_1^\alpha,r_2^\alpha,\cdots,r_m^\alpha)}(B) \tag{3.3}$$

for any Borel set $B \subseteq K$. Therefore we have THEOREM 1 (A). □

PROOF OF THEOREM 2. By the definition of $K(P_1, P_2, \cdots, P_m)$, we see that $K(P_1, P_2, \cdots, P_m)$ is a Borel set for any $(P_1, P_2, \cdots, P_m)$. Since

$$\nu_{(P_1,P_2,\cdots,P_m)}(K \setminus K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) = 0,$$

we see (E) $H^\alpha(K \setminus K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) = 0$ by (3.3). Since H^α is an outer measure,

$$\begin{aligned} H^\alpha(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) &\le H^\alpha(K) \\ &\le H^\alpha(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) + H^\alpha(K \setminus K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) \\ &= H^\alpha(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)). \end{aligned} \tag{3.4}$$

Therefore we have (D) $H^\alpha(K) = H^\alpha(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha))$. On the other hand, by (3.3) and $\nu_{(r_1^\alpha,r_2^\alpha,\cdots,r_m^\alpha)}(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) = 1$, we see that (F) $0 < H^\alpha(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) < \infty$. Therefore together with (D), we have (G) $\text{H-dim}(K) = \text{H-dim}(K(r_1^\alpha, r_2^\alpha, \cdots, r_m^\alpha)) = \alpha$. □

(I) has been already proved and (C) is clear from THEOREM 3 (G). Therefore we have to prove only (H) and (J).

PROOF OF THEOREM 3 (H). Suppose that $\{P_{i,k}\}_{i=1}^m$, $k = 1, 2, 3, \cdots$ is a sequence of probability vectors such that

$$0 < P_{i,k} < 1, \ \sum_{i=1}^{m} P_{i,k} = 1, \ \lim_{k \to \infty} P_{i,k} = r_i^{\alpha}, \ (P_{1,k}, P_{2,k}, \cdots, P_{m,k}) \neq (r_1^{\alpha}, r_2^{\alpha}, \cdots, r_m^{\alpha}).$$

Then by (I), we see

$$\alpha \geq \text{H-dim}\,(K \setminus K(r_1^{\alpha}, r_2^{\alpha}, \cdots, r_m^{\alpha})) \geq \text{H-dim}\,(K(P_{1,k}, P_{2,k}, \cdots, P_{m,k})) = \frac{\sum_{i=1}^{m} P_{i,k} \log P_{i,k}}{\sum_{i=1}^{m} P_{i,k} \log r_i}$$

for any k. Letting $k \to \infty$, we have

$$\alpha = \text{H-dim}\,(K \setminus K(r_1^{\alpha}, r_2^{\alpha}, \cdots, r_m^{\alpha})).$$

Therefore we have (H). □

Finally, we will prove THEOREM 4 (J). It is showed by the next proposition.

PROPOSITION 3.4. *Assume that M is a Borel subset of K and μ is a positive finite Borel measure on $\mathbb{R}^N$. Put $\mathscr{R}^{(n)} = \{R \in \mathscr{R};\ \lambda^{-n} < |R| \leq \lambda^{-n+1}\}$ and*

$$C_1(\alpha, n) = \min_{R \in \mathscr{R}^{(n)},\, R \cap M \neq \emptyset} \frac{\mu(R)}{|R|^{\alpha}}, \quad C_2(\alpha, n) = \max_{R \in \mathscr{R}^{(n)},\, R \cap M \neq \emptyset} \frac{\mu(R)}{|R|^{\alpha}}.$$

If

$$\mu\Big(\bigcup_{R \in \mathscr{R}^{(n)},\, R \cap M \neq \emptyset} R\Big) = \mu(\mathbb{R}^N) \qquad \textit{for all } n \in \mathbb{N} \tag{3.5}$$

and

$$\lim_{n \to \infty} \frac{\log (C_1(\alpha, n))}{n} = \lim_{n \to \infty} \frac{\log (C_2(\alpha, n))}{n} = 0, \tag{3.6}$$

then we have

$$\underline{\textit{M-dim}}\,(M) = \overline{\textit{M-dim}}\,(M) = \alpha.$$

PROOF OF PROPOSITION 3.4. For any $\varepsilon > 0$, we can find n such that $\lambda^{-n} < \varepsilon \leq \lambda^{-n+1}$. Let $\{U_i\}_i$ be an ε-covering of M such that $\#\{U_i\} = \mathrm{N}_{\varepsilon}(M)$. Here $\mathrm{N}_{\varepsilon}(M) = \min_{\{U_i\}} \#\{U_i;\ M \subseteq \bigcup_i U_i,\ |U_i| \leq \varepsilon\}$. Then there exists a positive constant L' not depending on $\varepsilon > 0$ such that

$$1 \leq \#\{R_{i,j}^n \in \mathscr{R}^{(n)};\ R_{i,j}^n \cap U_i \neq \emptyset,\ U_i \cap M \subseteq \textstyle\bigcup_j R_{i,j}^n,\ R_{i,j}^n \cap R_{i,j'}^n \neq \emptyset\ (j \neq j')\} \leq L'$$

for any i. Therefore we have

$$L'^{-1}\mathrm{N}_\varepsilon^{\mathscr{R}}(M) \le \mathrm{N}_\varepsilon(M) \le \mathrm{N}_\varepsilon^{\mathscr{R}}(M) \tag{3.7}$$

where $\mathrm{N}_\varepsilon^{\mathscr{R}}(M) = \min_{\{R_i\}} \#\{R_i;\ M \subseteq \bigcup_i R_i,\ |R_i| \le \varepsilon,\ R_i \in \mathscr{R}\}$. Since

$$\frac{\mu(\mathbb{R}^N)}{\max_{R\in\mathscr{R}^{(n)},\,R\cap M\neq\emptyset}\mu(R)} \le \mathrm{N}_\varepsilon^{\mathscr{R}}(M) \le \frac{\mu(\mathbb{R}^N)}{\min_{R\in\mathscr{R}^{(n)},\,R\cap M\neq\emptyset}\mu(R)}$$

by (3.5), we see by (3.7) that

$$\mu(\mathbb{R}^N)\lambda^{-\alpha}L'^{-1}C_2^{-1}(\alpha, n)\lambda^{\alpha n} \le \mathrm{N}_\varepsilon(M) \le \mu(\mathbb{R}^N)C_1^{-1}(\alpha, n)\lambda^{\alpha n}.$$

By (3.6), we have

$$\lim_{\varepsilon\downarrow 0} \frac{\log(\mathrm{N}_\varepsilon(M))}{\log 1/\varepsilon} = \alpha.$$

This implies that

$$\underline{\text{M-dim}}\,(M) = \overline{\text{M-dim}}\,(M) = \alpha.$$

PROOF OF THEOREM 4 (J). In Proposition 3.4, put $M = K(P_1, P_2,\cdots, P_m)$ and $\mu = \nu_{(r_1^\alpha, r_2^\alpha,\cdots, r_m^\alpha)}$. Then we see by (3.2) that

$$C_1(\alpha, n) = C_2(\alpha, n) = |X|^{-\alpha}.$$

Therefore we can easily see that $\mu = \nu_{(r_1^\alpha, r_2^\alpha,\cdots, r_m^\alpha)}$ and $M = K(P_1, P_2,\cdots, P_m)$ satisfy the conditions (3.5) and (3.6). Therefore we have

$$\underline{\text{M-dim}}\,(M) = \overline{\text{M-dim}}\,(M) = \alpha \qquad \text{for any } (P_1, P_2,\cdots, P_m).$$

4. Examples

EXAMPLE 4.1. Let us define two sequences of contraction maps $\{\varphi_{i_1 i_2\cdots i_n}\}$ and $\{\psi_{i_1 i_2\cdots i_n}\}$ for $(i_1, i_2,\cdots, i_n)\in\{1, 2\}^n$, $n = 1, 2,\cdots$. Put $X = [0, 1]^2$. Suppose that

$$\varphi_i, \psi_i : X \to X,\ i = 1, 2,$$

$$\varphi_1 = \psi_1 : (x, y) \to \left(\frac{1}{3}x, \frac{1}{3}y\right),$$

$$\varphi_2 : (x, y) \to \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right),$$

$$\psi_2 : (x, y) \to \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3}\right).$$

Then define

$$\begin{cases} \varphi_{i_1 i_2 \cdots i_n} = \varphi_{i_n}, \\ \psi_{i_1 i_2 \cdots i_n} = \begin{cases} \psi_{i_n} & n = 1, \\ \varphi_{i_n} & i_1 = 1, n \geq 2, \\ \psi_{i_n} & i_1 = 2, n \geq 2. \end{cases} \end{cases}$$

Put

$$K_\varphi = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, i_2, \cdots, i_n) \in \{1,2\}^n} \varphi_{i_1} \circ \varphi_{i_1 i_2} \circ \cdots \circ \varphi_{i_1 i_2 \cdots i_n}(X),$$

$$K_\psi = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, i_2, \cdots, i_n) \in \{1,2\}^n} \psi_{i_1} \circ \psi_{i_1 i_2} \circ \cdots \circ \psi_{i_1 i_2 \cdots i_n}(X).$$

Then we see that K_φ is Cantor's ternary set C on $[0, 1]$ and $K_\psi = \{(x, f(x));\ x \in C\}$. Here f: $[0, 1] \to [0, 1]$ such that

$$f(x) = \begin{cases} 0 & 0 \leq x \leq 1/2, \\ x & 1/2 < x \leq 1. \end{cases}$$

By the THEOREM 1, H-dim (K_φ) = H-dim $(K_\psi) = \log 2/\log 3 = \alpha$, and H^α on K_φ (resp. K_ψ) the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure on K_φ (resp. K_ψ) are absolutely continuous to each other.

In fact, for any Borel set B,

$$\mathrm{H}^\alpha(B \cap K_\varphi) = \int_B d\nu^\varphi_{(\frac{1}{2}, \frac{1}{2})}(\omega),$$

$$\mathrm{H}^\alpha(B \cap K_\psi) = \int_B (I_{[1]_\psi}(\omega) + I_{[2]_\psi}(\omega) \cdot 2^{\frac{1}{2}\alpha}) d\nu^\psi_{(\frac{1}{2}, \frac{1}{2})}(\omega),$$

where I_A is the indicator function of A, $[1]_\psi = \psi_1(X)$ and $[2]_\psi = \psi_2(X)$. $\nu^\varphi_{(\frac{1}{2}, \frac{1}{2})}$ and $\nu^\psi_{(\frac{1}{2}, \frac{1}{2})}$ denote $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure on K_φ and K_ψ, respectively. That is to say, H^α and $\nu^\varphi_{(\frac{1}{2}, \frac{1}{2})}$ are coincident but H^α and $\nu^\psi_{(\frac{1}{2}, \frac{1}{2})}$ are not coincident up to constant.

EXAMPLE 4.2. Let f: $[0, 1] \to \mathbb{R}$ be a Lipschitz continuous function such that

$$|f(x) - f(y)| \leq Q|x - y| \qquad \text{for any } x, y \in [0, 1] \text{ and } f(0) = 0 \tag{4.1}$$

with some positive constant Q. Now we will construct a Cantor set on $\{(x, f(x));\ x \in [0, 1]\}$ by our method. Put $X = [0, 1] \times [-Q, Q]$. Define a sequence of functions $\{\varphi_{i_1 i_2 \cdots i_n};\ (i_1, i_2, \cdots, i_n) \in \{1, 2\}^n,\ n = 1, 2, \cdots\}$ such that for any $i_1, i_2, \cdots, i_n$, n,

$$\varphi_{i_1 i_2 \cdots i_n} \colon X \to X$$

$$\varphi_{i_1} \circ \varphi_{i_1 i_2} \circ \cdots \circ \varphi_{i_1 i_2 \cdots i_n} : (x, y) \to (x/3^n + \sum_{j=1}^{n} \varepsilon(i_j)/3^j,\ y/3^n + f(\sum_{j=1}^{n} \varepsilon(i_j)/3^j))$$

where $\varepsilon: \{1, 2\} \to \{0, 2\}$ such that $\varepsilon(1) = 0$, $\varepsilon(2) = 2$. Then we can see that

$$|\varphi_{i_1 i_2 \cdots i_n}(x) - \varphi_{i_1 i_2 \cdots i_n}(y)| = \frac{1}{3}|x - y| \qquad \text{for any } x, y \in X \text{ and } i_1, i_2, \cdots, i_n, n.$$

Put

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, i_2, \cdots, i_n) \in \{1,2\}^n} \varphi_{i_1} \circ \varphi_{i_1 i_2} \circ \cdots \circ \varphi_{i_1 i_2 \cdots i_n}(X).$$

Then we see that $K = \{(x, f(x));\, x \in C\}$, where C is Cantor's ternary set, H-dim$(K) = \log 2/\log 3$ $(= \alpha)$ and that by THEOREM 1, H^{α} and $\nu_{(\frac{1}{2}, \frac{1}{2})}$ are absolutely continuous to each other on K. Furthermore if f is differentiable on $(0, 1)$, then we can easily see that

$$\mathrm{H}^{\alpha}(B \cap K) = \int_B (1 + (f' \circ \pi(\omega))^2)^{\frac{1}{2}\alpha} d\nu_{(\frac{1}{2}, \frac{1}{2})}(\omega) \qquad \text{for any Borel set } B \subseteq K. \tag{4.2}$$

Here π is the projection, that is, $\pi((x, y)) = x$.

For any Cantor set $C' \subseteq [0, 1]$ constructed by Hutchinson or our method, we can construct $\{(x, f(x));\, x \in C'\}$ by using our method and have a similar formula to (4.2).

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