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On random Clarkson inequalities

Dedicated to Professor Satoru Igari on his sixtieth birthday

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ABSTRACT. We prove Tonge's random Clarkson inequalities given for L_p in a generalized setting, where the *unknown* absolute constant appearing in his original inequalities is taken to be 1. As a corollary these inequalities for fairly many other Banach spaces such as $L_p(L_q)$, $W_p^k(\Omega)$ and c_p, etc. are immediately obtained.

Introduction

In connection with generalized Clarkson's inequalities (Kato [5]; see also the recent book [10]), a high-dimensional version of Clarkson-Boas-Koskela-type inequalities (cf. [3], [2], [8]), Tonge [12] presented random Clarkson inequalities for L_p.

In this article we prove the random Clarkson inequalities for a Banach space satisfying (p, p')-Clarkson's inequality ($1 \leqq p \leqq 2$); further the *unknown* absolute constant included in Tonge's original inequalities is replaced here by one. This enables us to obtain these inequalities for fairly many other Banach spaces, e.g., L_q-valued L_p-spaces $L_p(L_q)$, Sobolev spaces $W_p^k(\Omega)$ and the spaces c_p of p-Schatten class operators, etc (cf. [2], [4], [6], [9]; see also [10]).

In what follows, let p', q', ... denote the conjugate exponents of p, q, ... Let us first recall the generalized Clarkson inequalities: Let $A_n = (\varepsilon_{ij})$ be the Littlewood matrices, that is,

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \qquad (n = 1, 2, \ldots).$$

GENERALIZED CLARKSON'S INEQUALITIES (Kato [5], Theorem 1; cf. [12], [11], [10]). Let $1 < p < \infty$ and $1 \leqq r, s \leqq \infty$. Then, for an arbitrary positive

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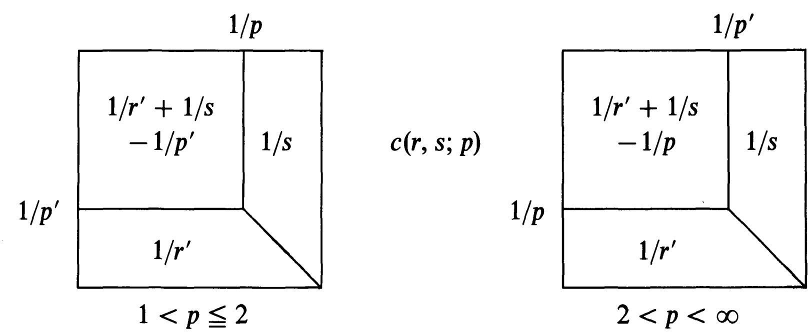
integer n and all $f_1, f_2, \dots, f_{2^n} \in L_p$,

$$\text{(GCI)} \qquad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^s \right\}^{1/s} \leqq 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \|f_j\|_p^r \right\}^{1/r},$$

where

$$c(r, s; p) = \begin{cases} \dfrac{1}{r'} + \dfrac{1}{s} - \min\left(\dfrac{1}{p}, \dfrac{1}{p'}\right) & \text{if } \min(p, p') \leqq r \leqq \infty, \\ & \quad 1 \leqq s \leqq \max(p, p'), \\ \dfrac{1}{s} & \text{if } 1 \leqq r \leqq \min(p, p'), \\ & \quad 1 \leqq s \leqq r', \\ \dfrac{1}{r'} & \text{if } s' \leqq r \leqq \infty, \\ & \quad \max(p, p') \leqq s \leqq \infty. \end{cases}$$

The constant $c(r, s; p)$ is well expressed visually in the following unit squares with axes $1/r$ (horizontal) and $1/s$ (vertical):



Tonge's random Clarkson inequalities are stated as

RANDOM CLARKSON INEQUALITIES (Tonge [12], Theorem 3). Let $1 \leqq p, r, s \leqq \infty$. Let n be an arbitrary positive integer and let $A = (a_{ij})$ be an $n \times n$ matrix whose coefficients are independent indentically distributed random variables taking the values ± 1 with equal probability. Then, $\mathbf{E}$ denoting mathematical expectation, for any $f_1, f_2, \dots, f_n$ in $L_p(\mu)$,

$$\text{(RCI)} \qquad \mathbf{E}\left(\sum_{i=1}^{n} \left\| \sum_{j=1}^{n} a_{ij} f_j \right\|_p^s \right)^{1/s} \leqq K n^{c(r,s;p)} \left(\sum_{j=1}^{n} \|f_j\|_p^r \right)^{1/r},$$

where $c(r, s; p)$ is as in (GCI) and K is a positive absolute constant.

We now present the random Clarkson inequalities in a generalized setting; here it is worth stating that the absolute constant K in (RCI) can be removed:

THEOREM. *Let* $1 \leqq p \leqq 2$ and $1 \leqq r, s \leqq \infty$. *Let n be an arbitrary positive integer and let $A = (a_{ij})$ be an $n \times n$ matrix whose coefficients are independent identically distributed random variables taking the values ± 1 with equal probability. Let X be a Banach space which satisfies* (p, p')-Clarkson's inequality

$$(\mathrm{CI}_p) \qquad (\|x+y\|^{p'} + \|x-y\|^{p'})^{1/p'} \leqq 2^{1/p'}(\|x\|^p + \|y\|^p)^{1/p}.$$

Then, for any $x_1, x_2, \ldots, x_n$ in X,

$$(\mathrm{RCI}^*) \qquad \mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^s\right)^{1/s} \leqq n^{c(r,s;p)}\left(\sum_{j=1}^{n}\|x_j\|^r\right)^{1/r},$$

where

$$c(r, s; p) = \begin{cases} \dfrac{1}{r'} + \dfrac{1}{s} - \dfrac{1}{p'} & \text{if (i) } p \leqq r \leqq \infty, \quad 1 \leqq s \leqq p', \\[2ex] \dfrac{1}{s} & \text{if (ii) } 1 \leqq r \leqq p, \quad 1 \leqq s \leqq r', \\[2ex] \dfrac{1}{r'} & \text{if (iii) } s' \leqq r \leqq \infty, \quad p' \leqq s \leqq \infty. \end{cases}$$

For the proof we need the following recent result of the authors [7]:

LEMMA (Kato and Takahashi [7], Theorem 1). *Let* $1 < p \leqq 2$. *Then, a Banach space X satisfies (p, p')-Clarkson's inequality* (CI_p) *if and only if X satisfies the type p inequality*

$$(\mathrm{TI}_p) \quad \left(\frac{1}{2^n}\sum_{\theta_j = \pm 1}\left\|\sum_{j=1}^{n} \theta_j x_j\right\|^{p'}\right)^{1/p'} \leqq \left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p} \qquad (\forall x_1, x_2, \ldots, x_n \in X)$$

for any n.

PROOF OF THEOREM. Let $1 < p \leqq 2$. Let us first prove (RCI*) for the case $(r, s) = (p, p')$, i.e.,

$$(\mathrm{RCI}_p^*) \qquad \mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^{p'}\right)^{1/p'} \leqq n^{1/p'}\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p}.$$

By Lemma, we have

$$(1) \qquad \left(\mathbf{E}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^{p'}\right)^{1/p'} = \left(\frac{1}{2^n}\sum_{\theta_j=\pm 1}\left\|\sum_{j=1}^{n}\theta_j x_j\right\|^{p'}\right)^{1/p'} \leqq \left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p}$$

for each i. Hence, we obtain

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^{p'}\right)^{1/p'} \leqq \left(\mathbf{E}\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^{p'}\right)^{1/p'} \leqq \left\{n\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{p'/p}\right\}^{1/p'} \leqq n^{1/p'}\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p}.$$

We next show that for any t with $1 < t < p \leqq 2$,

$$(\mathrm{RCI}_t^*) \qquad \mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^{t'}\right)^{1/t'} \leqq n^{1/t'}\left(\sum_{j=1}^{n}\|x_j\|^t\right)^{1/t}.$$

To see this, we have only to note that the inequality (CI_p) follows

$$(\mathrm{CI}_t) \qquad (\|x+y\|^{t'} + \|x-y\|^{t'})^{1/t'} \leqq 2^{1/t'}(\|x\|^t + \|y\|^t)^{1/t}$$

for $1 < t < p \leqq 2$ (the above argument works for t instead of p). Indeed, put $\theta = p'/t'$ $(0 < \theta < 1)$. Then clearly

$$(2) \qquad M_1 = \|A_1 : l_1^2(X) \to l_\infty^2(X)\| = 1$$

and by (CI_p)

$$M_2 = \|A_1 : l_p^2(X) \to l_{p'}^2(X)\| = 2^{1/p'}$$

(note that equality is attained in (CI_p) by putting $y = 0$). Hence, by interpolation (cf. [1], Theorems 5.1.2, 4.2.1 and 4.1.2) we have

$$\|A_1 : l_t^2(X) \to l_{t'}^2(X)\| \leqq M_1^{1-\theta}M_2^{\theta} = 2^{1/t'},$$

which implies the inequality (CI_t).

We now proceed in the proof of the whole part of (RCI*): (i) Let $p \leqq r \leqq \infty$, $1 \leqq s \leqq p'$. Then, by (TI_p) or (1), for each i

$$\left(\mathbf{E}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^s\right)^{1/s} \leqq \left(\mathbf{E}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^{p'}\right)^{1/p'}$$
$$\leqq \left(\sum_{j=1}^{n} \|x_j\|^p\right)^{1/p}$$
$$\leqq n^{1/p-1/r}\left(\sum_{j=1}^{n} \|x_j\|^r\right)^{1/r}.$$

Consequently,

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^s\right)^{1/s} \leqq \left(\mathbf{E}\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^s\right)^{1/s}$$
$$\leqq \left[n\left\{n^{1/p-1/r}\left(\sum_{j=1}^{n} \|x_j\|^r\right)^{1/r}\right\}^s\right]^{1/s}$$
$$= n^{1/s+1/p-1/r}\left(\sum_{j=1}^{n} \|x_j\|^r\right)^{1/r}.$$

(ii) Let $1 \leqq r \leqq p$ and $1 \leqq s \leqq r'$. Then, by (RCI_t^*) with $t = r$,

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^s\right)^{1/s} \leqq n^{1/s-1/r'}\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^{r'}\right)^{1/r'}$$
$$\leqq n^{1/s}\left(\sum_{j=1}^{n} \|x_j\|^r\right)^{1/r}.$$

(iii) Let $s' \leqq r \leqq \infty$ and $p' \leqq s \leqq \infty$. Then, by (RCI_t^*) with $t = s'$,

$$\mathbf{E}\left(\sum_{i=1}^{n}\left\|\sum_{j=1}^{n} a_{ij}x_j\right\|^s\right)^{1/s} \leqq n^{1/s}\left(\sum_{j=1}^{n} \|x_j\|^{s'}\right)^{1/s'}$$
$$\leqq n^{1/s}n^{1/s'-1/r}\left(\sum_{j=1}^{n} \|x_j\|^r\right)^{1/r}$$
$$\leqq n^{1/r'}\left(\sum_{j=1}^{n} \|x_j\|^r\right)^{1/r}.$$

In the case where $p = 1$, the inequalities (CI_1) and (RCI^*) are both valid for *any* Banach space. (Indeed, (CI_1) is equivalent to (2); for (RCI^*) use the argument in the proof of the case (i).) This completes the proof.

REMARKS. (i) A Banach space X satisfies (p, p')-Clarkson's inequality (CI_p) if and only if its dual space X' does. (This is easy to see; cf. [7], Theorem 3.)

(ii) The inequalities (RCI*), stated for $1 \leqq p \leqq 2$, include Tonge's original inequalities (RCI) for the cases $1 \leqq p \leqq 2$ and $2 \leqq p \leqq \infty$ unifyingly. Indeed, if $1 \leqq p \leqq 2$, L_p satisfies (CI_p) ([3]) and if $2 \leqq p \leqq \infty$, $L_p = (L_{p'})'$ satisfies ($CI_{p'}$) ([3]; or by (i)).

COROLLARY. (i) *Let* $1 \leqq p \leqq \infty$ *and let* $t = \min\{p, p'\}$. *Then, in the space* $l_p(L_p)$ (L_p*-valued* l_p*-space; in particular* L_p), $W_p^k(\Omega)$ *and* c_p, *the random Clarkson inequalities* (RCI*) *hold with the constant* $c(r, s; t)$.

(ii) *Let* $1 \leqq p$, $q \leqq \infty$ *and let* $t = \min\{p, p', q, q'\}$. *Then, the random Clarkson inequalities* (RCI*) *hold in* $L_p(L_q)$ *with the constant* $c(r, s; t)$.

Indeed, the spaces in (i) resp. $L_p(L_q)$ in (ii) satisfy (CI_t) for $t = \min\{p, p'\}$ ([2], [6]; [4], [6]; [9]) resp. for $t = \min\{p, p', q, q'\}$ ([2], [6]).

For further examples of Banach spaces satisfying Clarkson's inquality (CI_p), and hence (RCI*), we refer the reader to [10].

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