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On reduced Q-functions

To Kiyosato Okamoto on his sixtieth birthday, with affection and admiration

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ABSTRACT. Schur's Q-functions with reduced variables are discussed by employing a combinatorics of strict partitions. They are called reduced Q-functions. We give a description of the linear relations among reduced Q-functions.

0. Introduction

Q-functions were introduced by Schur in his study of projective representations of symmetric groups. They are symmetric functions and, if we express them in terms of the power sum symmetric functions, each coefficient essentially gives the spin character of the symmetric group. This note deals with the *r-reduced* Q-functions which are defined by putting $p_{jr} = 0$ for $j = 1, 3, 5, \dots$ in the power sum expression. When $r = p$ is a prime number, they play a role in p-modular projective representations of symmetric groups.

In a previous work [8] we showed that r-reduced Q-functions are weight vectors of the basic representation of the affine Lie algebra $A_{2t}^{(2)}$ $(r = 2t + 1)$ and chose a proper basis for each weight space.

To be more precise, let α_i (resp. $\alpha_i^{\vee}$) $(0 \le i \le t)$ be the simple roots (resp. coroots) of the affine Lie algebra $A_{2t}^{(2)}$ and $\delta = 2\sum_{i=0}^{t-1}\alpha_i + \alpha_t$ be its fundamental imaginary root. The irreducible representation with highest weight Λ_0 is called the basic representation, where $\Lambda_0(\alpha_i^{\vee}) = \delta_{i0}$. The set of weights is described by

$$P = \{w\Lambda_0 - n\delta;\ w \in W, n \in N\},$$

where W is the Weyl group. This basic representation can be realized on the polynomial ring $C[t_j; j \ge 1,$ odd and $j \not\equiv 0 \pmod r]$. In this realization each r-reduced Q-function turns out to be a weight vector. We answered in

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[8] the question to which weight the given r-reduced Q-function belongs, by using a combinatorics of strict partitions, i.e., bar cores and bar quotients.

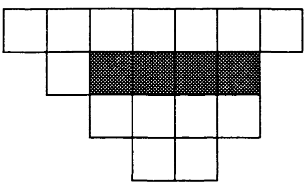
By virtue of the above result we can fully investigate the r-reduced Q-functions themselves. The result in this note is an explicit description of the linear relations satisfied by r-reduced Q-functions. We remark that such a description has been obtained for *r-reduced Schur functions* [1, 2].

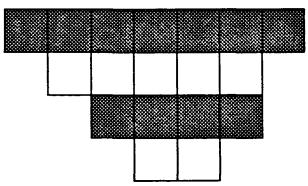
1. Combinatorics of strict partitions

We first present a collection of definitions and results concerning strict partitions and Schur's Q-functions, mainly referring to Macdonald's book [5].

We denote the set of all partitions of n by $\mathscr{P}_n$ and the set of all partitions by $\mathscr{P}$. A partition $\lambda = (\lambda_1, \dots, \lambda_l)$ is said to be *strict* if $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. We denote the set of all strict partitions of n by $\mathscr{S}\mathscr{P}_n$ and the set of all strict partitions by $\mathscr{S}\mathscr{P}$.

To each element $\lambda \in \mathscr{S}\mathscr{P}$ we can associate the shifted diagram $S(\lambda)$, which is obtained from the ordinary Young diagram of λ by shifting the i-th row $(i-1)$-positions to the right. The j-th cell in the i-th row will be called the (i, j)-cell. Let $(i, j) \in S(\lambda)$ and $k \geq i$. The set of cells $(a, b) \in S(\lambda)$ is called an *$(i, j)_k$-bar* of λ if it satisfies that $a = i$ or k and $b \geq j$, and that the diagram obtained by the removal of these cells has no two rows of equal length. Then we have two types of $(i, j)_k$-bars: (i) if $k > i$, there is an $(i, j)_k$-bar only when $j = 1$, in which case it consists of the i-th and k-th rows, and (ii) if $k = i$, there is an $(i, j)_k$-bar only when $j - 1 \neq \lambda_{a'}$ for $a' \geq i + 1$. For instance, we illustrate $(2, 2)_2$-bar and $(1, 1)_3$-bar of $\lambda = (7, 5, 4, 2)$, which are shaded, respectively:





The length of a bar is the number of cells it contains. A bar of length r will be often called an r-bar. Throughout this note we always assume that r is an odd number.

A strict partition is called an *r-bar core* if it contains no r-bars. Given a strict partition λ we obtain another strict partition by removing an r-bar and rearranging the rows in descending order. By repeating this process as long as possible we shall end up with an r-bar core λ^c, which is called the r-bar core of λ.

For $0 \leq i \leq r-1$ we define sequences

$$X^i(\lambda) = \{x \in N;\ \lambda_k = rx + i \text{ for some } k\}.$$

The *r-bar quotient* $\lambda^q = (\lambda^0, \lambda^1, \ldots, \lambda^t)$ $(t = (r-1)/2)$ is a $(t+1)$-tuple of partitions given by $\lambda^0 = X^0(\lambda)$ and $\lambda^i = (X^i(\lambda)|X^{r-i}(\lambda))$ $(1 \leq i \leq t)$ in *Frobenius notation*. Note that λ^0 is a strict partition. We remark that any r-bar core and r-bar quotient uniquely determine a strict partition [9]. Since this is a one-to-one correspondence, we often write (λ^c, λ^q) instead of λ, e.g., $Q_{(\lambda^c, \lambda^q)}(x)$, etc.

We introduce the r-bar sign for strict partitions. Let $\overline{H}$ be the $(i,j)_k$-bar of length r in $\lambda \in \mathscr{SP}$. The leglength of $\overline{H}$ is defined by

$$b(\overline{H}) = \begin{cases} \lambda_k + \#\{a;\ \lambda_i > \lambda_a > \lambda_k\} & \text{if } k > i \\ \#\{a;\ \lambda_i > \lambda_a > \lambda_i - r\} & \text{if } k = i. \end{cases}$$

If the strict partition μ is obtained from λ by removing r-bars H_1, H_2, $\ldots$, H_q successively, then the r-bar sign of λ relative to μ is defined by

$$\delta_{\bar{r}}(\lambda, \mu) = \prod_{m=1}^{q} (-1)^{b(\overline{H}_m)}.$$

We set $\delta_{\bar{r}}(\lambda) = \delta_{\bar{r}}(\lambda, \lambda^c)$ and call it the r-bar sign of λ. We should remark that $\delta_{\bar{r}}(\lambda, \mu)$ does not depend on the choices of r-bars being removed in going from λ to μ [6].

Let $\Lambda_{\boldsymbol{Q}}$ denote the graded $\boldsymbol{Q}$-algebra of symmetric functions in countably many independent variables $x = (x_1, x_2, \ldots)$. Among various bases of $\Lambda_{\boldsymbol{Q}}$, the Schur functions $s_\lambda(x)$ $(\lambda \in \mathscr{P})$ are convenient for our purpose. Let $\Gamma_{\boldsymbol{Q}}$ be the subalgebra of $\Lambda_{\boldsymbol{Q}}$ generated by $p_1, p_3, p_5, \ldots$;

$$\Gamma_{\boldsymbol{Q}} = \boldsymbol{Q}[p_1, p_3, p_5, \ldots],$$

where $p_j = x_1^j + x_2^j + \cdots$ is the j-th power sum.

Schur's Q-functions $Q_\lambda(x)$ $(\lambda \in \mathscr{P})$ are defined by specializing to $t = -1$ in the Hall-Littlewood symmetric functions $Q_\lambda(x; t)$. This specialization implies that $Q_\lambda(x) = 0$ unless $\lambda \in \mathscr{SP}$. It is known that $\{Q_\lambda(x);\ \lambda \in \mathscr{SP}\}$ gives a basis of $\Gamma_{\boldsymbol{Q}}$. When $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a strict partition and $\mu = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(l)})$ for $\sigma \in \mathfrak{S}_l$, then we define $Q_\mu(x) = \operatorname{sgn}(\mu) Q_\lambda(x)$, where we define $\operatorname{sgn}(\mu) = \operatorname{sgn}(\sigma)$.

The Q-functions are related to the power sums via the character values of the symmetric group. An explicit expression is given as follows. Let $\zeta_\lambda(\pi)$ $(\lambda \in \mathscr{SP}_n)$ be the value of the irreducible projective character ζ_λ of the symmetric group $\mathfrak{S}_n$ at the class $\pi = (1^{\pi_1} 3^{\pi_3} 5^{\pi_5} \cdots)$ and set $z_\pi = \prod_{j \geq 1} \pi_j! j^{\pi_j}$. Then we have

$$Q_\lambda(x) = \sum_{\pi} 2^{[(l(\lambda)+l(\pi)+1)/2]} z_\pi^{-1} \zeta_\lambda(\pi) p_1^{\pi_1} p_3^{\pi_3} p_5^{\pi_5} \cdots,$$

where the summation runs over all odd partitions $\pi = (1^{\pi_1} 3^{\pi_3} 5^{\pi_5} \cdots)$ of size n, and $l(\lambda)$ denotes the length of the partition λ.

The following formula is of importance for our purpose.

THEOREM 1.1. *Let $j \geq 1$ be odd and $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathscr{S}\mathscr{P}$. Then*

$$p_j Q_\lambda(x) = \frac{1}{2} \sum_{i=1}^{(j-1)/2} (-1)^i Q_{(\lambda; j-i, i)}(x) + \frac{1}{2} Q_{(\lambda; j)}(x) + \sum_{i=1}^{l} Q_{\lambda + j\varepsilon_i}(x),$$

where $(\lambda; j-i, i) = (\lambda_1, \ldots, \lambda_l, j-i, i)$, $(\lambda; j) = (\lambda_1, \ldots, \lambda_l, j)$ and $\varepsilon_i = (\delta_{ik})_{1 \leq k \leq l}$.

This is a direct consequence of [4, Theorem 2.4] by looking at the correspondence between removal of *strips* and/or *double strips* [4] and that of j-bars.

2. Reduced Q-functions and their linear relations

Let $\Gamma_Q^{(r)} = \boldsymbol{Q}[p_j; j \geq 1$, odd and $j \not\equiv 0 \pmod r]$ and define the *r-reduced Q-function* indexed by the strict partition λ by

$$Q_\lambda^{(r)}(x) = Q_\lambda(x)|_{p_r = p_{3r} = p_{5r} = \cdots = 0} \in \Gamma_Q^{(r)}.$$

PROPOSITION 2.1. *Let λ be a strict partition and $\lambda^q = (\lambda^0, \ldots, \lambda^t)$ $(t = (r-1)/2)$ be the r-bar quotient of λ. Then the set $\{Q_\lambda^{(r)}(x); \lambda^0 = \varnothing\}$ gives a basis of $\Gamma_Q^{(r)}$.*

To prove this proposition we need the following lemma.

LEMMA 2.2. *Let V be an infinite dimensional $\boldsymbol{Q}$-vector space with basis $\{v_\lambda; \lambda \in \mathscr{S}\mathscr{P}\}$ and define $v_{\lambda\sigma} = \operatorname{sgn}(\sigma) v_\lambda$ for permutations σ; $v_\lambda = 0$ if $\lambda \notin \mathscr{S}\mathscr{P}$.*

(1) *If we set*

$$V_1 := V \Big/ \sum_{\lambda \in \mathscr{S}\mathscr{P}} \boldsymbol{Q}\left(\sum_{j \geq 1, \text{odd}} \left(\frac{1}{2} \sum_{i=1}^{(j-1)/2} (-1)^i v_{(\lambda; j-i, i)} + \frac{1}{2} v_{(\lambda; j)} + \sum_{i=1}^{l(\lambda)} v_{\lambda + j\varepsilon_i} \right) \right),$$

then $V_1 \cong \boldsymbol{Q}$.

(2) *If we set*

$$V_r := V \Big/ \sum_{\lambda \in \mathscr{S}\mathscr{P}} \boldsymbol{Q}\left(\sum_{j \geq 1, \text{odd}} \left(\frac{1}{2} \sum_{i=1}^{(rj-1)/2} (-1)^i v_{(\lambda; rj-i, i)} + \frac{1}{2} v_{(\lambda; rj)} + \sum_{i=1}^{l(\lambda)} v_{\lambda + rj\varepsilon_i} \right) \right),$$

then $V_r \cong \Gamma_Q^{(r)}$.

PROOF. (1) There is a canonical linear surjection γ from Γ_Q to V_1 which maps $Q_\lambda(x)$ to v_λ for $\lambda \in \mathcal{SP}$. The kernel of γ coincides with the maximal ideal $\mathcal{I} = (p_1, p_3, p_5, \ldots)$ because of Theorem 1.1. Since the algebra $\Gamma_Q/\mathcal{I}$ is isomorphic to $\boldsymbol{Q}$, so is V_1.
(2) Consider a linear surjection $\gamma^{(r)}: \Gamma_Q \to V_r$ defined by $\gamma^{(r)}(Q_\lambda(x)) = v_\lambda$ for $\lambda \in \mathcal{SP}$. If we set $\mathcal{I}^{(r)} = (p_r, p_{3r}, p_{5r}, \ldots)$, then by Theorem 1.1, we see that $\gamma^{(r)}(\mathcal{I}^{(r)}) = 0$ and can define a linear surjection $\bar{\gamma}^{(r)}: \Gamma_Q/\mathcal{I}^{(r)} \to V_r$. On the other hand we can define a linear surjection η: $V_r \to \Gamma_Q^{(r)}$ by $\eta(v_\lambda) = Q_\lambda^{(r)}(x)$ for $\lambda \in \mathcal{SP}$. The composition $\eta \circ \bar{\gamma}^{(r)}: \Gamma_Q/\mathcal{I}^{(r)} \to \Gamma_Q^{(r)}$ gives a linear isomorphism. Hence η is a linear isomorphism as desired. ■

PROOF OF PROPOSITION 2.1. We give a proof only for the case $r = 3$ in order to avoid the too complicated notation. The general case can be shown by similar arguments [7].

First we shall see that the set $\{Q_\lambda^{(3)}(x); \lambda^0 = \varnothing\}$ spans $\Gamma_Q^{(3)}$. We introduce a filtration on $\Gamma_Q^{(3)}$ by $F_n\Gamma_Q^{(3)} = \sum_{\lambda; |\lambda^0| \le n} \boldsymbol{Q}Q^{(3)}_{(\lambda^c, \lambda^0, \lambda^1)}(x)$ and the associated graded module $\bar{F}(\Gamma_Q^{(3)}) := \bigoplus_{n \ge 0} F_n\Gamma_Q^{(3)}/F_{n-1}\Gamma_Q^{(3)}$, where $F_{-1}\Gamma_Q^{(3)} = \{0\}$. For any positive odd integer j the 3-reduced Q-functions $Q_\lambda^{(3)}(x)$ satisfy

$$\begin{aligned}
0 &= \frac{1}{2}\sum_{i=1}^{(3j-1)/2} Q^{(3)}_{(\lambda; 3j-i, i)}(x) + \frac{1}{2}Q^{(3)}_{(\lambda; 3j)}(x) + \sum_{i=1}^{l(\lambda)} Q^{(3)}_{\lambda + 3j\varepsilon_i}(x) \\
&= \frac{1}{2}\sum_{i=1}^{(j-1)/2} Q^{(3)}_{(\lambda^c, (\lambda^0; j-i, i), \lambda^1)}(x) + \frac{1}{2}Q^{(3)}_{(\lambda^c, (\lambda^0; j), \lambda^1)}(x) + \sum_{i=1}^{l(\lambda^0)} Q^{(3)}_{(\lambda^c, \lambda^0 + j\varepsilon_i, \lambda^1)}(x) \\
&\quad + \frac{1}{2}\sum_{i \not\equiv 0 \pmod 3} Q^{(3)}_{(\lambda^c, \lambda^0, (\lambda; 3j-i, i)^1)} + \sum_{i=1}^{l(\lambda^1)} Q^{(3)}_{(\lambda^c, \lambda^0, \lambda^1 + j\varepsilon_i)}(x).
\end{aligned}$$

If we write $\bar{Q}_\lambda^{(3)}(x) = Q_\lambda^{(3)}(x) + F_{n-1}\Gamma_Q^{(3)}$ for $|\lambda^0| = n$, then we see that in $\bar{F}(\Gamma_Q^{(3)})$,

$$\sum_{i=1}^{l(\lambda^0)} \bar{Q}^{(3)}_{(\lambda^c, \lambda^0 + j\varepsilon_i, \lambda^1)}(x) + \frac{1}{2}\bar{Q}^{(3)}_{(\lambda^c, (\lambda^0; j), \lambda^1)}(x) + \frac{1}{2}\sum_{i=1}^{(j-1)/2} \bar{Q}^{(3)}_{(\lambda^c, (\lambda^0; j-i, i), \lambda^1)}(x) = 0.$$

By applying Lemma 2.2 (1) it turns out that $\sum_{\lambda^0 \in \mathcal{SP}} \boldsymbol{Q}\bar{Q}^{(3)}_{(\lambda^c, \lambda^0, \lambda^1)}(x) = \boldsymbol{Q}\bar{Q}^{(3)}_{(\lambda^c, \varnothing, \lambda^1)}(x)$. This proves that $\{Q_\lambda^{(3)}(x); \lambda^0 = \varnothing\}$ spans $\Gamma_Q^{(3)}$.

By looking at the one-to-one correspondence between the strict partitions and the tuples of r-bar cores and r-bar quotients, we see that the number of those strict partitions $\lambda \in \mathcal{SP}_n$ such that $\lambda^0 = \varnothing$ coincides with the dimension of the subspace of homogeneous polynomials of degree n in $\Gamma_Q^{(r)}$, where we count $\deg p_j = j$. This completes the proof. ■

We shall state the linear relations satisfied by the r-reduced Q-functions. To this end, we define two types of Littlewood-Richardson-like coefficients

$\tilde{c}^{\nu}_{\lambda_1\cdots\lambda_t}$ ($\lambda_1, \ldots, \lambda_t, \nu \in \mathscr{SP}$) and $h^{\nu}_{\lambda\mu}$ ($\lambda \in \mathscr{SP}, \mu, \nu \in \mathscr{P}$), respectively by

$$Q_{\lambda_1}(x) \cdots Q_{\lambda_t}(x) = \sum_{\nu \in \mathscr{SP}} \tilde{c}^{\nu}_{\lambda_1\cdots\lambda_t} Q_\nu(x),$$

$$P_\lambda(x) s_\mu(x) = \sum_{\nu \in \mathscr{P}} h^{\nu}_{\lambda\mu} s_\nu(x),$$

where we set $P_\lambda(x) = 2^{-l(\lambda)} Q_\lambda(x)$ for $\lambda \in \mathscr{SP}$. It can be shown that $\tilde{c}^{\nu}_{\lambda_1\cdots\lambda_t}$ and $h^{\nu}_{\lambda\mu}$ are non-negative integers [3].

THEOREM 2.3. *Let λ be a strict partition, λ^c and $\lambda^q = (\lambda^0, \lambda^1, \ldots, \lambda^t)$ be its r-bar core and r-bar quotient, respectively. Then*

$$Q^{(r)}_\lambda(x) = 2^{[(l(\lambda)+l(\lambda^0))/2]}(-1)^{|\lambda^0|}\delta_{\bar{r}}(\lambda) \sum_{\mu, \nu_1, \ldots, \nu_t} 2^{-[l(\mu)/2]}\delta_{\bar{r}}(\mu)\tilde{c}^{\lambda^0}_{\nu_1\cdots\nu_t} h^{\mu^1}_{\nu_1\lambda^1} \cdots h^{\mu^t}_{\nu_t\lambda^t} Q^{(r)}_\mu(x),$$

where summation runs over the strict partitons $\nu_1, \ldots, \nu_t$ and μ such that $|\mu| = |\lambda|$, $\mu^c = \lambda^c$ and $\mu^0 = \varnothing$.

PROOF. Again we give a proof only for the case $r = 3$. For the general case we should notice the following identity:

$$P_\nu(x^{(1)}, x^{(2)}, \ldots, x^{(t)}) = \sum_{\lambda_1, \ldots, \lambda_t} \tilde{c}^{\nu}_{\lambda_1\cdots\lambda_t} P_{\lambda_1}(x^{(1)}) P_{\lambda_2}(x^{(2)}) \cdots P_{\lambda_t}(x^{(t)}),$$

where the summation runs over the strict partitions $\lambda_1, \ldots, \lambda_t$. Here $(x^{(1)}, x^{(2)}, \ldots, x^{(t)})$ are t sets of variables and $P_\nu(x^{(1)}, \ldots, x^{(t)})$ denotes the P-function in the set of variables $(x^{(1)}_1, x^{(1)}_2, \ldots, x^{(2)}_1, x^{(2)}_2, \ldots, x^{(t)}_1, x^{(t)}_2, \ldots)$.

Set

$$F_\lambda(x) = \operatorname{sgn}(\lambda) 2^{[(l(\lambda)+l(\lambda^0))/2]}(-1)^{|\lambda^0|}\delta_{\bar{3}}(\lambda) \sum_\mu 2^{-[l(\mu)/2]}\delta_{\bar{3}}(\mu) h^{\mu^1}_{\lambda^0\lambda^1} Q^{(3)}_\mu(x).$$

We shall prove that

$$\sum_{i=1}^{(3j-1)/2} (-1)^i F_{(\lambda; 3j-i, i)}(x) + \frac{1}{2} F_{(\lambda; 3j)}(x) + \sum_{i=1}^{l(\lambda)} F_{\lambda + 3j\varepsilon_i}(x) = 0$$

for any positive odd integer j. For our purpose it is enough to show that the coefficient of $Q^{(3)}_\mu(x)$ vanishes for each $\mu \in \mathscr{SP}$. Since Corollary (3.8) in [6] leads to

$$\delta_{\bar{3}}(\tilde{\lambda}, \lambda) = \begin{cases} \operatorname{sgn}(\tilde{\lambda}) \operatorname{sgn}(\tilde{\lambda}^k) & \text{for } \tilde{\lambda} = \lambda + 3j\varepsilon_i \quad (k = 0, 1, \quad i = 1, \ldots, l(\lambda)) \\ \operatorname{sgn}(\tilde{\lambda}) \operatorname{sgn}(\tilde{\lambda}^0) & \text{for } \tilde{\lambda} = (\lambda; 3j) \text{ and } (\lambda; 3j - 3j', 3j') \left(j' \le \dfrac{j-1}{2}\right) \\ (-1)^i \operatorname{sgn}(\tilde{\lambda}) \operatorname{sgn}(\tilde{\lambda}^1) & \text{for } \tilde{\lambda} = (\lambda; 3j - i, i) \left(i \le \dfrac{3j-1}{2}, i \not\equiv 0 \ (\mathrm{mod}\ 3)\right), \end{cases}$$

our equation can be rewritten as follows:

$$\frac{1}{2}\sum_{i<j/2} 2^2(-1)^i \operatorname{sgn}((\lambda^0; j-i, i))(-1)^j h^{\mu^1}_{(\lambda^0; j-i, i)\lambda^1}$$

$$+\frac{1}{2}\sum_{i\not\equiv 0 \pmod 3} 2\operatorname{sgn}((\lambda; 3j-i, i)^1) h^{\mu^1}_{\lambda^0(\lambda; 3j-i, i)^1} + \frac{1}{2}\operatorname{sgn}((\lambda^0; j))2(-1)^j h^{\mu^1}_{(\lambda^0; j)\lambda^1}$$

$$+\sum_i \operatorname{sgn}((\lambda+3j\varepsilon_i)^0)(-1)^j h^{\mu^1}_{(\lambda+3j\varepsilon_i)^0\lambda^1} + \sum_i \operatorname{sgn}((\lambda+3j\varepsilon_i)^1) h^{\mu^1}_{\lambda^0(\lambda+3j\varepsilon_i)^1}$$

$$= 0.$$

Multiplying both sides by $s_{\mu^1}(x)$ and taking summation over μ^1, we see that the equation reduces to

$$(-1)^j\left\{\frac{1}{2}\sum_{i<j/2}(-1)^i Q_{(\lambda^0; j-i, i)}(x)s_{\lambda^1}(x) + \frac{1}{2}Q_{(\lambda^0; j)}(x)s_{\lambda^1}(x) + \sum_i Q_{\lambda^0+j\varepsilon_i}(x)s_{\lambda^1}(x)\right\}$$

$$+\left\{\sum_i Q_{\lambda^0}(x)s_{\lambda^1+j\varepsilon_i}(x) + \sum_{\substack{i<j/2\\ i\not\equiv 0 \pmod 3}} Q_{\lambda^0}(x)s_{(\lambda, 3j-i, i)^1}(x)\right\} = 0.$$

Note that there is a one-to-one correspondence between the $3j$-bars in λ and the j-bars in $\lambda^q = (\lambda^0, \lambda^1)$ (a j-bar in λ^q means a j-bar in λ^0 or a j-hook in λ^1). Hence we have

$$-(p_j Q_{\lambda^0}(x))s_{\lambda^1}(x) + Q_{\lambda^0}(x)(p_j s_{\lambda^1}(x)) = 0.$$

By Lemma 2.2 (3) there exists a linear isomorphism sending $F_\lambda(x)$ to $Q^{(3)}_\lambda(x)$ and it turns out to be the identity since $Q^{(3)}_\lambda(x)$ equals $F_\lambda(x)$ for the basis given in Proposition 2.1. ■

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