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On maps into a co-H-space

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ABSTRACT. We prove that the domain of a map $X \to Y$ to a co-H-space inherits a co-H-structure provided some dimensionality and connectivity properties hold. Then we deduce that a space X admits a co-H-structure if and only if on all its skeletons there is such a structure as well. Moreover, if X is 1-connected then a co-H-structure on X is equivalent to such a structure on all its homology decomposition stages.

Introduction.

Recall [5, Chapter IX], where it was shown that a map $X \to Y$ of based spaces extends an H-space structure on X to one on Y provided some connectivity properties hold. In particular, the result implies that an H-structure on X is inherited on all its Postnikov stages.

This paper examines the dual problem. However, the arguments used for H-spaces do not seem to dualize. Therefore, we make use of the one-to-one correspondence (see [1] or [2, p. 209–212]) between homotopy classes of co-H-structures on a space X and those of coretractions $X \to \Sigma\Omega X$. A result due to Hilton (see [2, p. 185]) says that if a space X is dominated by a co-H-space Y then X also admits a co-H-structure. We prove (Theorem 1) that the domain of a map $X \to Y$ to a co-H-space inherits a co-H-structure provided some dimensionality and connectivity properties hold. Then we deduce (Corollary 1) that a space X admits a co-H-structure if and only if on all its skeletons there is such a structure as well. Moreover, if X is 1-connected then Corollary 3 states that a co-H-structure on X is equivalent to such a structure on all its homology decomposition stages and a dualization of Corollary 5.6 in [5, page 443] is obtained provided X is 2-connected.

1. Preliminaries.

We consider based spaces of the based homotopy type of a based CW-complex; the basepoints are assumed to be non-degenerate. All maps and

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homotopies are supposed to preserve basepoints and all spaces denoted by the symbols X, Y and their derivates (X_n, Y_n etc.) are implicitly assumed to be based CW-complexes. It follows from [4] the constructions we will perform do not lead outside the class of spaces considered. For a space X let $\Delta : X \to X \times X$ be the diagonal map and $j : X \vee X \to X \times X$ the inclusion map of the wedge into the product. A *co-H-structure* on X is a map $\sigma : X \to X \vee X$ such that the diagram

$$\begin{tikzcd} & X \vee X \arrow[d, "j"] \\ X \arrow[ur, "\sigma"] \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

commutes up to homotopy. A *co-H-space* is a pair (X, σ), where X is a space and σ a co-H-structure on X. An *inversion* for σ is a map $\eta : X \to X$ such that the composites

$$X \xrightarrow{\sigma} X \vee X \xrightarrow{\mathrm{id}_X \vee \eta} X \vee X \xrightarrow{\nabla} X \quad \text{and} \quad X \xrightarrow{\sigma} X \vee X \xrightarrow{\eta \vee \mathrm{id}_X} X \vee X \xrightarrow{\nabla} X$$

are both nullhomotopic maps; ∇ is the folding map. We point out that by [3] any co-H-structure on a 1-connected space admits an inversion.

With any space X we may associate $\Sigma\Omega X$, the (reduced) suspension of the loop space of X; the natural projection $p_X : \Sigma\Omega X \to X$ is given by $p_X([t, \omega]) = \omega(t)$ for $[t, \omega] \in \Sigma\Omega X$. A *coretraction* is a map $\phi : X \to \Sigma\Omega X$ such that $p_X\phi \simeq \mathrm{id}_X$, the identity map of X. For a space X the suspension co-H-structure $s_X : \Sigma X \to \Sigma X \vee \Sigma X$ is the pinch map and the coretraction $\Sigma e : \Sigma X \to \Sigma\Omega\Sigma X$ is given by $e(x)(t) = [t, x]$ for $x \in X$ and $t \in [0, 1]$. As it was pointed out in [1] and [2, p. 209–212] the map $(p_X \vee p_X)s_{\Omega X} : \Sigma\Omega X \to X \vee X$ induces a bijection between homotopy classes of coretractions and those of comultiplications on X.

Let $f : X \to Y$ be any map and consider the diagrams

$$\begin{tikzcd} \Sigma\Omega X \arrow[r, "\Sigma\Omega f"] & \Sigma\Omega Y \\ X \arrow[u, "\phi"] \arrow[r, "f"] & Y \arrow[u, "\psi"'] \end{tikzcd} \qquad \begin{tikzcd} X \vee X \arrow[r, "f \vee f"] & Y \vee Y \\ X \arrow[u, "\sigma"] \arrow[r, "f"] & Y \arrow[u, "\tau"'] \end{tikzcd}$$

in which ϕ, ψ are coretractions and σ, τ the corresponding comultiplications. Taking into account [1] the second square homotopy commutes if and only if the first does; we way then that f is a *map of co-H-spaces* (X, σ) and (Y, τ).

We now show how maps from some spaces into a co-H-space reflect co-H-structures on their domains. For a map $f : X \to Y$ we write $\operatorname{conn} f = n$, if the induced map of homotopy groups $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and an epimorphism for $k = n$. In particular, for the map $f : X \to *$ to the single point space we put $\operatorname{conn} X = \operatorname{conn} f$. Write $X \flat Y$ and $X \wedge Y$ for the flat and smash product of spaces X and Y, respectively; now there is a homotopy equivalence $X \flat Y \simeq \Sigma\Omega X \wedge \Omega Y$ (see e.g., [2, page 216]).

LEMMA 1. *For a map* $f : X \to Y$, $\operatorname{conn}(f \wedge f) = \operatorname{conn} f + \min\{\operatorname{conn} X, \operatorname{conn} Y\} + 1$ *and* $\operatorname{conn}(f \flat f) = \operatorname{conn} f + \min\{\operatorname{conn} X, \operatorname{conn} Y\}$.

PROOF. Observe that $f \wedge f = (f \wedge \mathrm{id}_X) \circ (\mathrm{id}_Y \wedge f)$ and $f \flat f = (f \flat \mathrm{id}_X) \circ (\mathrm{id}_Y \flat f)$. From the above $f \flat \mathrm{id}_X \simeq \Omega f \wedge \mathrm{id}_{\Sigma\Omega X}$ and $\mathrm{id}_Y \flat f \simeq \mathrm{id}_{\Sigma\Omega Y} \wedge \Omega f$. Hence $\operatorname{conn}(f \wedge \mathrm{id}_X) = \operatorname{conn} f + \operatorname{conn} X + 1$, $\operatorname{conn}(\mathrm{id}_Y \wedge f) = \operatorname{conn} f + \operatorname{conn} Y + 1$ and $\operatorname{conn}(f \flat \mathrm{id}_X) = \operatorname{conn} X + \operatorname{conn} f$, $\operatorname{conn} \mathrm{id}_Y \flat f = \operatorname{conn} Y + \operatorname{conn} f$. The result follows. $\square$

This lemma will be also useful in the next section.

LEMMA 2. *Let* $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots$ *be a countable direct system of co-H-spaces and co-H-maps. Then its homotopy colimit* $\operatorname{holim}_{\to} X_n$ *admits a co-H-structure such that the canonical imbeddings* $X_n \to \operatorname{holim}_{\to} X_n$ *are co-H-maps for all* $n \geq 0$.

PROOF. By the "small object" argument we get that the canonical map

$$\operatorname{holim}_{\to} \Sigma\Omega X_n \longrightarrow \Sigma \operatorname{holim}_{\to} \Omega X_n = \Sigma\Omega \operatorname{holim}_{\to} X_n$$

is a homotopy equivalence. Then the coretractions $\phi_n : X_n \to \Sigma\Omega X_n$ corresponding to co-H-structures on X_n for $n \geq 0$ yield the required coretraction $\phi = \operatorname{holim}_{\to} \phi_n : \operatorname{holim}_{\to} X_n \to \Sigma\Omega \operatorname{holim}_{\to} X_n$. $\square$

2. Main results.

The behaviour of co-H-structures with respect to maps is taken into account in this section.

THEOREM 1. *Let* $f : X \to Y$ *be a map with* Y *a co-H-space. If* $\dim X \leq \operatorname{conn} f + \min\{\operatorname{conn} X, \operatorname{conn} Y\}$ *then there exists a (which is unique up to homotopy if strict inequality holds) co-H-structure on* X *such that* $f : X \to Y$ *is a co-H-map.*

Proof. Consider the commutative (up to homotopy) diagram

$$\begin{tikzcd} P \arrow[r, "\pi'"] \arrow[d, "\pi"'] & \Sigma\Omega Y \arrow[r, "(p_Y \vee p_Y)s_Y"] \arrow[d, "pY"] & Y \vee Y \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \arrow[r, "\Delta"] & Y \times Y, \end{tikzcd}$$

where the first square is the homotopy pullback and the second one is also a homotopy pullback by [1]. Therefore, the homotopy fibres of the maps $p_Y : \Sigma\Omega Y \to Y$ and $\pi : P \to X$ have the homotopy type of the space $Y \flat Y$. Let now $\phi : Y \to \Sigma\Omega Y$ be the coretraction corresponding to the co-H-structure on Y. Then the pairs of maps $\Sigma\Omega f : \Sigma\Omega X \to \Sigma\Omega Y$, $p_X : \Sigma\Omega X \to X$ and $\phi f : X \to \Sigma\Omega Y$, id_X yield maps $\alpha : \Sigma\Omega X \to P$ and $\gamma : X \to P$, respectively with the appropriate composite property.

Analyze now the map of fibrations

$$\begin{tikzcd} X \flat X \arrow[r] \arrow[d, "f \flat f"'] & \Sigma\Omega X \arrow[r, "p_X"] \arrow[d, "\alpha"] & X \arrow[d, equal] \\ Y \flat Y \arrow[r] & P \arrow[r, "\pi"] & X. \end{tikzcd}$$

From the 5-lemma we get that $\operatorname{conn}\alpha = \operatorname{conn} f \flat f$. But by Lemma 1 $\operatorname{conn} f \flat f = \operatorname{conn} f + \min\{\operatorname{conn} X, \operatorname{conn} Y\}$ and the obstruction theory yields a (unique up to homotopy if there is strict inequality) map $\psi : X \to \Sigma\Omega X$ such that $\alpha\psi \simeq \gamma$. Then $\pi\alpha\psi \simeq p_X\psi \simeq \pi\gamma \simeq \mathrm{id}_X$ and the map ψ determines a co-H-structure on X. Moreover, $(\Sigma\Omega f)\psi \simeq \pi'\alpha\psi \simeq \pi'\gamma \simeq \phi f$ and $f : X \to Y$ is a co-H-map. $\square$

Lemma 3. *Let $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ and $g : X_1 \to X_2$ be maps with $f_2 g \simeq f_1$ and Y a co-H-space. If $\dim X_1 \le \min\{\operatorname{conn} f_2 - 1, \operatorname{conn} f_1 + \min\{\operatorname{conn} X_1, \operatorname{conn} Y\}\}$ and $\dim X_2 \le \operatorname{conn} f_2 + \min\{\operatorname{conn} X_2, \operatorname{conn} Y\}$ then a co-H-structure on Y yields co-H-structures on X_1 and X_2 such that $g : X_1 \to X_2$ is a co-H-map.*

Proof. Let $\phi : Y \to \Sigma\Omega Y$ be the coretraction determined by the co-H-structure on Y and $\phi_1 : X_1 \to \Sigma\Omega X_1$, $\phi_2 : X_2 \to \Sigma\Omega X_2$ the coretractions determined by the co-H-structures on X_1 and X_2 described in Theorem 1. To show the commutativity (up to homotopy) of the diagram

$$\begin{tikzcd} X_1 \arrow[r, "\phi_1"] \arrow[d, "g"'] & \Sigma\Omega X_1 \arrow[d, "\Sigma\Omega g"] \\ X_2 \arrow[r, "\phi_2"] & \Sigma\Omega X_2 \end{tikzcd}$$

observe that $(\Sigma\Omega f_2)\phi_2 g \simeq \phi f_2 g \simeq \phi f_1$ and $(\Sigma\Omega f_2)(\Sigma\Omega g)\phi_1 \simeq (\Sigma\Omega f_2 g)\phi_1 \simeq (\Sigma\Omega f_1)\phi_1 \simeq \phi f_1$. But $\operatorname{conn}\Sigma\Omega f_2 = \operatorname{conn} f_2$ and $\dim X_1 \leq \operatorname{conn} f_2 - 1$, hence obstruction theory yields a homotopy $(\Sigma\Omega g)\phi_1 \simeq \phi_2 g$. □

Our final results can be stated now. For a space X let $X^{(n)}$ be its n-skeleton, $n \geq 0$. Then Theorem 1, Lemmas 2 and 3 yield

COROLLARY 1. *A connected space X admits a co-H-structure if and only if on each skeleton $X^{(n)}$ there exists such a co-H-structure that the canonical imbedding $X^{(n)} \to X^{(n+1)}$ is a co-H-map. Furthermore, in this case, the imbedding $X^{(n)} \to X$ is a co-H-map for suitable co-H-structures on X and $X^{(n)}$.*

For a co-H-map $f : X \to Y$ let C_f be its mapping cone and $q : Y \to C_f$ the canonical imbedding. By [3] we may give C_f a co-H-structure in such a way that q is a co-H-map. We now show that the converse of this fact also holds, provided some conditions are satisfied. Recall first that for a co-H-space X with a co-H-structure σ and an inversion map, and any space Y there is a (naturally) split short exact sequence

$$1 \longrightarrow [X, Y\flat Y] \longrightarrow [X, Y \vee Y] \underset{\gamma_Y}{\overset{j_*}{\rightleftarrows}} [X, Y \times Y] \longrightarrow 1,$$

where $j_*\gamma_Y = \mathrm{id}$ with $\gamma_Y([\alpha_1, \alpha_2)]) = [(\alpha_1 \vee \alpha_2)\sigma]$ for $(\alpha_1, \alpha_2) : X \to Y \times Y$. Denote the induced operation on $[X, Y \vee Y]$ additively and let $\beta_Y = \mathrm{id} - \gamma_Y j_*$. Then we get $j_*\beta_Y = 0$.

THEOREM 2. *Let $f : X \to Y$ be a map with X and Y 1-connected co-H-spaces. If the mapping cone C_f is a co-H-space, $q : Y \to C_f$ a co-H-map and $\dim X < \operatorname{conn} X + \min\{\operatorname{conn} Y, \operatorname{conn} C_f\}$ then $f : X \to Y$ is a co-H-map.*

PROOF. Observe that by [3] a co-H-structure on a 1-connected space X admits an inversion. Let $\sigma : Y \to Y \vee Y$ and $\sigma' : C_f \to C_f \vee C_f$ be co-H-structures on Y and C_f, respectively and consider the commutative diagram

$$\begin{array}{ccccc}
[X, Y] & \xrightarrow{\sigma_*} & [X, Y \vee Y] & \xrightarrow{\beta_Y} & [X, Y\flat Y] \\
{\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle (q \vee q)_*} & & \downarrow{\scriptstyle (q\flat q)_*} \\
[X, C_f] & \xrightarrow{\sigma'_*} & [X, C_f \vee C_f] & \xrightarrow{\beta_{C_f}} & [X, C_f\flat C_f].
\end{array}$$

Then $(q\flat q)_*\beta_Y\sigma_*([f]) = \beta_{C_f}\sigma'_*q_*([f]) = \beta_{C_f}\sigma'_*([qf]) = 0$. But $\operatorname{conn} q = \operatorname{conn} X$ and by Lemma 1 $\operatorname{conn}(q\flat q) = \operatorname{conn} q + \min\{\operatorname{conn} Y, \operatorname{conn} C_f\}$, hence $\dim X < \operatorname{conn}(q\flat q)$. By obstruction theory the map $(q\flat q)_*$ is an isomorphism, so we get that $\beta_Y\sigma_*[f]) = 0$. From the definition of β_Y we derive that

$\sigma f \simeq (f \vee f)\tau$, where $\tau : X \to X \vee X$ is a co-H-structure on Y and finally f is a co-H-map. $\square$

In particular, let $M(A,n)$ be the Moore space of type (A,n) for $n \geq 2$. Then $\dim M(A,n) \leq n+1$ and $\operatorname{conn} M(A,n) = n-1$. Thus we get the following

COROLLARY 2. *Let $M(A,n)$ be the Moore space of type (A,n) for $n \geq 2$, X a 2-connected co-H-space and $f : M(A,n) \to X$ a map. If the mapping cone C_f of f is a co-H-space, $q : X \to C_f$ a co-H-map then $f : M(A,n) \to X$ is a co-H-map.*

Let now X be a 1-connected space and

$$\begin{tikzcd}
X_2 \arrow[r, "i_2"] \arrow[drr, "f_2"'] & X_3 \arrow[r, "i_3"] \arrow[dr, "f_3"] & \cdots \arrow[r, "i_{n-1}"] & X_n \arrow[r, "i_n"] \arrow[dl, "f_n"] & \cdots \\
& & X & &
\end{tikzcd}$$

its homology decomposition due to Hilton [2, Chapter 8]. Roughly speaking the space X with homology groups $H_2(X)$, $H_3(X), \ldots$ is built-up by a process of successively attaching cones $C(M(H_n(X), n-1))$ to X_{n-1} by the homologically trivial maps $M(H_n(X), n-1) \to X_{n-1}$ determined by the k'-classes $k'_{n-1} \in [M(H_n(X), n-1), X_n]$, where $M(H_n(X), n-1)$ is the Moore space of type $(H_n(X), n-1)$. Thus, by Lemmas 2 and 3, Theorem 1 and Corollary 2 it may be stated that a dualization of Corollary 5.6 in [5, page 443] there exists.

COROLLARY 3. (1) *A 1-connected X space admits a co-H-structure if and only if on each nth stage X_n there exists such a co-H-structure that $i_n : X_n \to X_{n+1}$ is a co-H-map. Furthermore, in this case, $f_n : X_n \to X$ is a co-H-map for suitable co-H-structures on X and X_n.*

(2) *If X is a 2-connected co-H-space with homology groups $H_3(X)$, $H_4(X), \ldots$ then the maps $M(H_n(X), n-1) \to X_{n-1}$ determined by the k'-classes are co-H-maps for all $n \geq 3$.*

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