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A new family of filtration seven in the stable homotopy of spheres

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ABSTRACT. This paper proves the existence of a new family of nontrivial homotopy elements in the stable homotopy of spheres which is of degree $2(p-1)(p^n+3p^2+3p+3)-7$ and is represented by $b_{n-1}g_0\gamma_3$ in the $E_2^{7,*}$-term of the Adams spectral sequence, where $p \geqslant 7$ is a prime and $n \geqslant 4$. In the course of proof, a new family of homotopy elements in $\pi_* V(1)$ which is represented by $b_{n-1}g_0$ in the $E_2^{4,*}V(1)$-term of the Adams spectral sequence is detected.

1. Introduction

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z_p, Z_p) \Rightarrow \pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A. If a family of generators x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of homotopy elements f_i in π_*S and we say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in π_*S have been detected. For example, a family $\zeta_{n-1} \in \pi_{p^nq+q-3}S$ for $n \geqslant 2$ which has filtration 3 and is represented by $h_0b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p)$ has been detected in [2], where $q = 2(p-1)$. The main purpose of this paper is to detect a new family of homotopy elements in π_*S which has filtration 7 in the ASS.

From [3], $\mathrm{Ext}_A^{1,*}(Z_p, Z_p)$ has Z_p-base consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(Z_p, Z_p)$ for all $i \geqslant 0$ and $\mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ has Z_p-base consisting of α_2, a_0^2, a_0h_i $(i > 0)$, g_i $(i \geqslant 0)$, k_i $(i \geqslant 0)$, b_i $(i \geqslant 0)$, and h_ih_j $(j \geqslant i+2, i \geqslant 0)$ whose internal degree are $2q+1$, 2, p^iq+1, $p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq respectively. From [1] p.110 table 8.1, there is a generator $\gamma_3 \in \mathrm{Ext}_A^{3,(3p^2+2p+1)q}(Z_p, Z_p)$ whose name in [1] is $h_{0,1,2,3}$. Our main result is the following theorem.

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THEOREM I: *Let $p \geqslant 7$, $n \geqslant 4$, then the product*

$$b_{n-1}g_0\gamma_3 \neq 0 \in \mathrm{Ext}_A^{7,p^nq+3(p^2+p+1)q}(Z_p, Z_p)$$

and it converges in the ASS to a nontrivial element in $\pi_{p^nq+3(p^2+p+1)q-7}S$ of order p.

The above family of homotopy elements in π_*S is constructed based on a family of homotopy elements in $\pi_*V(1)$, the stable homotopy groups of Toda-Smith spectrum $V(1)$.

The spectrum $V(1)$ is closely related to S and is defined as follows. Let M be the Moore spectrum modulo a prime $p \geqslant 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S \tag{1.1}$$

Let $\alpha: \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M \tag{1.2}$$

where $q = 2(p-1)$. This spectrum which we briefly write as K is known to be the Toda-Smith spectrum $V(1)$. Theorem I will be proved based on the following result.

THEOREM II: *Let $p \geqslant 5$, $n \geqslant 2$, then*

$$b_{n-1}g_0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p),$$

the reduction of $b_{n-1}g_0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(Z_p, Z_p)$, converges in the ASS to a nontrivial homotopy element in $\pi_{p^nq+pq+2q-4}K$.

From [2], there is $\zeta_{n-1} \in \pi_{p^nq+q-3}S$ for $n \geqslant 2$ which is represented by $h_0b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p)$. By using ζ_{n-1} as a geometric input and some properties of K studied in [7], we will detect an element $\zeta''_{n-1} \in [\Sigma^{p^nq+q-4}K, K]$ satisfying

$$j'\zeta''_{n-1} = ijj'(\zeta_{n-1} \wedge 1_K) \quad \text{modulo higher filtration}$$

Moreover, we will show that $\zeta''\beta i'i \in \pi_{p^nq+pq+2q-4}K$ is a nontrivial element of filtration 4 represented by $b_{n-1}g_0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$, where $\beta \in [\Sigma^{(p+1)q}K, K]$ is the known v_2-periodicity element (cf. [7] p.426). This is a sketch of construction in the proof of Theorem *II* given in section 3.

Let $V(2)$ be the cofibre of $\beta: \Sigma^{(p+1)q}K \to K$ and $\gamma \in [\Sigma^{(p^2+p+1)q}V(2), V(2)]$ be the v_3-periodicity element for $p \geqslant 7$ (cf. [7] p.426). Reduct the element $\zeta''_{n-1}\beta i'i \in \pi_*K$ to $\pi_*V(2)$ and compose with $\gamma^3 \in [\Sigma^{3(p^2+p+1)q}V(2), V(2)]$, moreover, we pinch this resulting map to the top cell of $V(2)$, then we get an element in $\pi_{p^nq+3(p^2+p+1)q-7}S$ which will be shown to be represented by $b_{n-1}g_0\gamma_3 \in \mathrm{Ext}_A^{7,*}(Z_p, Z_p)$ in the ASS. This is a sketch of construction in the proof of Theorem I given in section 3. Note that the $b_{n-1}g_0\gamma_3$-element obtained in Theorem I is an indecomposable element in π_*S, i.e. it is not a composition of elements in π_*S of lower filtration, because b_{n-1} and $g_0 \in \mathrm{Ext}_A^{2,*}(Z_p, Z_p)$ are known to die in the ASS.

After giving some preliminaries on Ext groups of lower dimensional in section 2, the proof of the main theorems will be given in section 3.

2. Some preliminaries on Ext groups

In this section, we will prove some results on Ext groups of lower dimension which will be used in the proofs of the main theorems.

Proposition 2.1: *Let $p \geqslant 5$, $n \geqslant 2$, then*

(1) $\mathrm{Ext}_A^{4,t}(Z_p, Z_p) = 0$ *for* $t = p^nq$, $p^nq + q + r$ $(r = 0, 1)$, *and* $p^nq + 2q + r$ $(r = 0, 2)$ *and* $a_0^2 b_{n-1} \neq 0 \in \mathrm{Ext}_A^{4,p^nq+2}(Z_p, Z_p)$.

(2) $\mathrm{Ext}_A^{4,p^nq+2q+1}(Z_p, Z_p) = Z_p\{\alpha_2 b_{n-1}\}$ *with* $\alpha_2 \in \mathrm{Ext}_A^{2,2q+1}(Z_p, Z_p)$
$\mathrm{Ext}_A^{4,p^nq+pq+2q}(Z_p, Z_p) = Z_p\{b_{n-1}g_0\}$

(3) $\mathrm{Ext}_A^{s,t}(Z_p, Z_p) = 0$ *for* $s \leqslant 5$, $t \equiv -1, -2, -3 (\mathrm{mod}\, q)$ *except for* $p = 5$, $s = 5$ *and* $t \equiv 5$ (mod 8).

Proof. From [11], we have a quotient chain complex $(C^{*,*}, d)$ of the cobar complex of A whose cohomology is isomorphic to $\mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ (not an algebra isomorphism!) and as a vector space over Z_p

$$C^{*,*} = \mathrm{E}(h_{m,i} | m > 0, i \geqslant 0) \otimes \mathrm{P}(b_{m,i} | m > 0, i \geqslant 0) \otimes \mathrm{P}(a_n | n \geqslant 0)$$

where E is the exterior algebra and P the polynomial algebra, $h_{m,i}, b_{m,i}, a_n$ are represented in the cobar complex by

$$h_{m,i} = [\xi_m^{p^i}] \in C^{1,(p^{i+m-1}+p^{i+m-2}+\cdots+p^i)q}$$

$$b_{m,i} = [\xi_m^{(p-1)p^i} | \xi_m^{p^i}] \in C^{2,(p^{i+m}+p^{i+m-1}+\cdots+p^{i+1})q}$$

$$a_n = [\tau_n] \in C^{1,(p^{n-1}+p^{n-2}+\cdots+1)q+1}$$

To prove (1), observe the following internal degree mod p^nq for $1 \leqslant i < n$, $n \geqslant 2$,

$$\|h_{1,i}\| = p^iq \quad (\mathrm{mod}\, p^nq)$$

$$\|b_{1,i-1}\| = p^iq \quad (\mathrm{mod}\, p^nq)$$

$$\|h_{s,i-1}\| = (p^{s+i-2} + \cdots + p^{i-1})q \pmod{p^n q}, \quad i \leqslant s+i-2 < n$$

$$\|b_{s,i-1}\| = (p^{s+i-1} + \cdots + p^{i})q \pmod{p^n q}, \quad i \leqslant s+i-2 < n$$

$$\|a_{i+1}\| = (p^i + p^{i-1} + \cdots + 1)q + 1 \pmod{p^n q}$$

At degree $t = p^n q + mq + r$ with $m, r < p$, $C^{4,t}$ has no generator which has factors consisting of the above elements, because such generator will have internal degree $(c_{n-1}p^{n-1} + \cdots + c_1 p + c_0)q + d (\operatorname{mod} p^n q)$ with some $c_i \neq 0$ $(1 \leqslant i \leqslant n-1)$, where $0 \leqslant c_s < p$, $s = 0, \ldots, n-1$, $0 \leqslant d \leqslant 4$. Exclude the above factors and factors with internal degree $> p^n q$, we can easily show that

$$C^{4,p^n q} = 0, \quad C^{4,p^n q+q} = 0, \quad C^{4,p^n q+2q} = 0$$

$$C^{4,p^n q+q+1} = Z_p\{b_{1,n-1}h_{1,0}a_0\}, \quad C^{4,p^n q+2q+2} = Z_p\{h_{1,n}h_{1,0}a_1 a_0\}$$

But computing the coboundary in the cobar complex we have

$$d\sum_{k=1}^{p-1}\binom{p}{k}\Big/p[\xi_1^{kp^{n-1}}|\xi_1^{(p-k)p^{n-1}}|\tau_1] = \sum_{k=1}^{p-1}\binom{p}{k}\Big/p[\xi_1^{kp^{n-1}}|\xi_1^{(p-k)p^{n-1}}|\xi_1|\tau_0]$$

$$\begin{aligned} d([\xi_1^{p^n}|\tau_1|\tau_1] + [\xi_1^{p^n}|\xi_1\tau_1|\tau_0] - [\xi_1^{p^n}|\xi_1|\tau_0\tau_1]) = {} & -2[\xi_1^{p^n}|\xi_1|\tau_0|\tau_1] - [\xi_1^{p^n}|\xi_1|\xi_1\tau_0/\tau_0] \\ & - [\xi_1^{p^n}|\xi_1^2|\tau_0|\tau_0] \end{aligned}$$

Take the quotient of the above equality in $C^{*,*}$ we have

$$db_{1,n-1}a_1 = b_{1,n-1}h_{1,0}a_0, \quad dh_{1,n}a_1^2 = -2h_{1,n}h_{1,0}a_0 a_1$$

Similarly we have

$$C^{3,p^n q+2} = Z_p\{a_0^2 h_{1,n}\} \quad C^{4,p^n q+2} = Z_p\{a_0^2 b_{1,n-1}\}$$

It is obvious that $a_0^2 h_{1,n}$ is represented by $a_0^2 h_n$ in $\operatorname{Ext}_A^{3,*}(Z_p, Z_p)$, so $da_0^2 h_{1,n} = 0$, and (1) is proved.

By a similar method we can check that

$$C^{4,p^n q+2q+1} = Z_p\{b_{1,n-1}h_{1,0}a_1\}, \quad C^{4,p^n q+pq+2q} = Z_p\{b_{1,n-1}h_{1,0}h_{2,0}\}$$

$$C^{3,p^n q+2q+1} = Z_p\{h_{1,n}h_{1,0}a_1\}, \quad C^{3,p^n q+pq+2q} = Z_p\{h_{1,n}h_{1,0}h_{2,0}\}$$

However, the known generators $h_n\alpha_2$ and $h_n g_0$ in $\operatorname{Ext}_A^{3,*}(Z_p, Z_p)$ are represented respectively by $h_{1,n}h_{1,0}a_1$ and $h_{1,n}h_{1,0}h_{2,0}$ in $C^{3,*}$, thus $d(h_{1,n}h_{1,0}a_1) = 0$, $d(h_{1,n}h_{1.0}h_{2,0}) = 0$ and so (2) is proved.

To prove (3), observe the following internal degree mod q,

$$\|h_{m,i}\| = 0 (\operatorname{mod} q), \quad \|b_{m,i}\| = 0 \pmod q, \quad \|a_n\| = 1 \pmod q$$

Similarly, look at the following exact sequence

$$\xrightarrow{p_*} \mathrm{Ext}_A^{r,p^nq+2}(Z_p,Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{r,p^nq+2}(\mathrm{H}^*M,Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{r,p^nq+1}(Z_p,Z_p) \xrightarrow{p_*}$$

induced by (1.1). The left group is zero for $r=2$ (cf. [3]) and has unique generator $a_0^2h_n$ for $r=3$ (cf. [1] table 8.1) which satisfies $i_*(a_0^2h_n)=i_*p_*(a_0h_n)=0$, then $\mathrm{im}\, i_*=0$. The right group has unique generator a_0h_n and a_0b_{n-1} for $r=2$ and 3 respectively which satisfies $p_*(a_0h_n)=a_0^2h_n\neq 0\in \mathrm{Ext}_A^{3,p^nq+2}(Z_p,Z_p)$ (cf. [1]) and $p_*(a_0b_{n-1})=a_0^2b_{n-1}\neq 0\in \mathrm{Ext}_A^{4,p^nq+2}(Z_p,Z_p)$ (cf. Prop. 2.1(1)), then $\mathrm{im} j_*=0$ and so the middle group is zero for $r=2,3$. It follows from the following exact sequence

$$0=\mathrm{Ext}_A^{r,p^nq+2}(\mathrm{H}^*M,Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{r,p^nq+1}(\mathrm{H}^*M,\mathrm{H}^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{r,p^nq+1}(\mathrm{H}^*M,Z_p)=0$$

induced by (1.1) that $\mathrm{Ext}_A^{r,p^nq+1}(\mathrm{H}^*M,\mathrm{H}^*M)=0$ for $r=2,3$.

(2) Consider the following exact sequence

$$\xrightarrow{p^*} \mathrm{Ext}_A^{r,p^nq+2}(Z_p,Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{r,p^nq+1}(Z_p,\mathrm{H}^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{r,p^nq+1}(Z_p,Z_p) \xrightarrow{p^*}$$

induced by (1.1). The left group is zero for $r=2$ (cf. [3]) and has unique generator $a_0^2h_n$ for $r=3$ (cf. [1] table 8.1) which satisfies $j^*(a_0^2h_n)=j^*p^*(a_0h_n)=0$, then $\mathrm{im} j^*=0$. The right group has unique generator a_0h_n and a_0b_{n-1} for $r=2$ and 3 respectively which satisfy $p^*(a_0h_n)=a_0^2h_n\neq 0$ and $p^*(a_0b_{n-1})=a_0^2b_{n-1}\neq 0$ then $\mathrm{im}\, i^*=0$ and so the middle group is zero for $r=2,3$.

Similarly, look at the following exact sequence

$$\xrightarrow{p^*} \mathrm{Ext}_A^{r,p^nq+1}(Z_p,Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{r,p^nq}(Z_p,\mathrm{H}^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{r,p^nq}(Z_p,Z_p) \xrightarrow{p^*}$$

induced by (1.1). The left group is zero for $r=1$ and has unique generator a_0h_n for $r=2$ which satisfies $j^*(a_0h_n)=j^*p^*(h_n)=0$, then $\mathrm{im} j^*=0$. The right group has unique generator h_n and b_{n-1} for $r=1$ and 2 respectively which satisfies $p^*(h_n)=a_0h_n\neq 0\in \mathrm{Ext}_A^{2,p^nq+1}(Z_p,Z_p)$ and $p^*(b_{n-1})=a_0b_{n-1}\neq 0\in \mathrm{Ext}_A^{3,p^nq+1}(Z_p,Z_p)$ then $\mathrm{im}\, i^*=0$ and the result follows. Q.E.D.

PROPOSITION 2.4: *Let $p\geqslant 5$, $n\geqslant 2$, then*

(1) $\mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M,\mathrm{H}^*M)=Z_p\{\tilde{b}_{n-1}\}$ *and* $\mathrm{Ext}_A^{3,p^nq+q+1}(\mathrm{H}^*M,\mathrm{H}^*M)=Z_p\{\alpha_*(\tilde{b}_{n-1})\}$, *where* $\alpha_*\colon \mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M,\mathrm{H}^*M)\to \mathrm{Ext}_A^{3,p^nq+q+1}(\mathrm{H}^*M,\mathrm{H}^*M)$ *is the boundary homomorphism induced by* $\alpha\colon \sum^q M\to M$.

(2) $\mathrm{Ext}_A^{1,p^nq}(\mathrm{H}^*M,\mathrm{H}^*M)=Z_p\{\tilde{h}_n\}$ *and* $\mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M,\mathrm{H}^*M)=Z_p\{\alpha_*(\tilde{h}_n)\}$.

PROOF. (1) Consider the following exact sequence

$$0=\mathrm{Ext}_A^{2,p^nq+1}(\mathrm{H}^*M,Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M,\mathrm{H}^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M,Z_p) \xrightarrow{p^*}$$

then when $t \equiv -1, -2, -3, C^{s,t}$ is spanned by generators with at least $q-1$, $q-2$, $q-3$ factors of a_i's, so $s \geqslant q-3 \geqslant 5$, (3) is proved. Q.E.D.

PROPOSITION 2.2: *Let $p \geqslant 7$, $n \geqslant 4$, then the product*

$$b_{n-1} g_0 \gamma_3 \neq 0 \in \mathrm{Ext}_A^{7, p^n q + 3(p^2+p+1)q}(Z_p, Z_p),$$

where $\gamma_3 = h_{0,1,2,3} \in \mathrm{Ext}_A^{3,(3p^2+2p+1)q}(Z_p, Z_p)$ *whose representative in* $C^{*,*}$ *is* $h_{1,2}h_{2,1}h_{3,0}$.

PROOF. The related internal degree $t = p^n q + 3(p^2+p+1)q$ is divisible by q, then $C^{s,t}(s = 6, 7)$ is spanned by generators which has no factor a_i's. Since $t \equiv 3q \pmod{pq}$, $\|h_{n,i}\| \equiv 0 \pmod{pq}$, $\|b_{n,i-1}\| \equiv 0 \pmod{pq}$ $(n, i > 0)$, then $C^{s,t}(s = 6, 7)$ must be spanned by generators which has factor $h_{i,0}h_{j,0}h_{k,0}$ $(i < j < k)$. Comparing $\|h_{i,0}h_{j,0}h_{k,0}\|$ modulo $p^n q$, we have $i = 1$, $j = 2$, $k = 3$. Exclude this factor $h_{i,0}h_{j,0}h_{k,0}$, the other factor is in $C^{s-3, p^n q + (2p^2+p)q}$ and we can easily check that $C^{3, p^n q + (2p^2+p)q} = Z_p\{h_{1,n}h_{1,2}h_{2,1}\}$, $C^{4, p^n q + (2p^2+p)q} = Z_p\{b_{1,n-1}h_{1,2}h_{2,1}\}$. So we have

$$C^{6, p^n q + 3(p^2+p+1)q} = Z_p\{h_{1,n}h_{1,2}h_{2,1}h_{1,0}h_{2,0}h_{3,0}\}$$

$$C^{7, p^n q + 3(p^2+p+1)q} = Z_p\{b_{1,n-1}h_{1,2}h_{2,1}h_{1,0}h_{2,0}h_{3,0}\}$$

Moreover, by computing modulo mixed words, we can check that $h_n g_0 \gamma_3 \in \mathrm{Ext}_A^{6, p^n q + 3(p^2+p+1)q}(Z_p, Z_p)$ and $b_{n-1} g_0 \gamma_3 \in \mathrm{Ext}_A^{7, p^n q + 3(p^2+p+1)q}(Z_p, Z_p)$ are represented respectively by $h_{1,n}h_{1,2}h_{2,1}h_{1,0}h_{2,0}h_{3,0}$ and $b_{1,n-1}h_{1,2}h_{2,1}h_{1,0}h_{2,0}h_{3,0}$. So $d(C^{6, p^n q + 3(p^2+p+1)q}) = 0$ and $b_{n-1} g_0 \gamma_3 \neq 0 \in \mathrm{Ext}_A^{7, p^n q + 3(p^2+p+1)q}(Z_p, Z_p)$. Q.E.D.

PROPOSITION 2.3: *Let $p \geqslant 5, n \geqslant 2$, then*

(1) $Ext_A^{r, p^n q+1}(\mathrm{H}^*M, Z_p) = 0, \mathrm{Ext}_A^{r, p^n q+2}(\mathrm{H}^*M, Z_p) = 0$ *and* $\mathrm{Ext}_A^{r, p^n q+1}(\mathrm{H}^*M, \mathrm{H}^*M) = 0$ *for* $r = 2, 3$.

(2) $\mathrm{Ext}_A^{r, p^n q+1}(Z_p, \mathrm{H}^*M) = 0$ *for* $r = 2, 3$ *and* $\mathrm{Ext}_A^{r, p^n q}(Z_p, \mathrm{H}^*M) = 0$ *for* $r = 1, 2$.

PROOF. (1) Consider the following exact sequence

$$\xrightarrow{p_*} \mathrm{Ext}_A^{r, p^n q+1}(Z_p, Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{r, p^n q+1}(\mathrm{H}^*M, Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{r, p^n q}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). The left group has unique generator $a_0 h_n$ and $a_0 b_{n-1}$ for $r = 2$ and 3 respectively (cf. [3] and [1] table 8.1) which satisfies $i_*(a_0 h_n) = i_* p_*(h_n) = 0$, $i_*(a_0 b_{n-1}) = i_* p_*(b_{n-1}) = 0$, then $\mathrm{im}\, i_* = 0$. The right group is zero for $r = 3$ (cf. [1] table 8.1) and has unique generator b_{n-1} for $r = 2$ which satisfies $p_*(b_{n-1}) = a_0 b_{n-1} \neq 0 \in \mathrm{Ext}_A^{3, p^n q+1}(Z_p, Z_p)$, then $\mathrm{im}\, j_* = 0$ and so the middle group is zero.

induced by (1.1). The left group is zero by Prop.2.3.(1), the right group has unique generator $i_*(b_{n-1})$ since $\mathrm{Ext}_A^{2,p^nq}(Z_p, Z_p) = Z_p\{b_{n-1}\}$ and $\mathrm{Ext}_A^{r,p^nq-1}(Z_p, Z_p) = 0$ for $r = 1,2$ (cf. [3]) and $p^* = 0$ since $\mathrm{Ext}_A^{3,p^nq+1}(\mathrm{H}^*M, Z_p) = 0$ by Prop.2.3 (1), then the middle group has unique generator $\tilde{b}_{n-1}$ such that $i^*(\tilde{b}_{n-1}) = i_*(b_{n-1})$.

Look at the following exact sequence

$$\xrightarrow{p_*} \mathrm{Ext}_A^{3,p^nq+q+1}(Z_p, Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{3,p^nq+q+1}(\mathrm{H}^*M, Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). The left group is zero and the right group has unique generator $h_0b_{n-1} = j_*\alpha_*i_*(b_{n-1})$ by [1] table 8.1, so the middle group has unique generator $\alpha_*i_*(b_{n-1})$. Consider the exact sequence

$$\mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*M, Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{3,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{3,p^nq+q+1}(\mathrm{H}^*M, Z_p) \xrightarrow{p^*=0}$$

induced by (1.1). As stated above, the right group has unique generator $\alpha_*i_*(b_{n-1})$. Moreover, $p^*\alpha_*i_*(b_{n-1}) = \alpha_*i_*p^*(b_{n-1}) = \alpha_*i_*(a_0b_{n-1}) = \alpha_*i_*p_*(b_{n-1}) = 0$, that is to say, the right $p^* = 0$. So, $\mathrm{Ext}_A^{3,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M)$ has unique generator $\alpha_*(\tilde{b}_{n-1})$ such that $i^*\alpha_*(\tilde{b}_{n-1}) = i^*\alpha^*(\tilde{b}_{n-1}) = \alpha_*i_*(b_{n-1})$ since $\mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*M, Z_p) = 0$ by the fact that $\mathrm{Ext}_A^{3,p^nq+q+r}(Z_p, Z_p) = 0$ for $r = 1,2$ (cf. [1]).

(2) Consider the following exact sequence

$$0 = \mathrm{Ext}_A^{1,p^nq+1}(Z_p, Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{1,p^nq+1}(\mathrm{H}^*M, Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{1,p^nq}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). The left group is zero and the right group has unique generator h_n (cf. [3]) which satisfies $p_*(h_n) = a_0h_n \neq 0 \in \mathrm{Ext}_A^{2,p^nq+1}(Z_p, Z_p)$, then $\mathrm{Ext}_A^{1,p^nq+1}(\mathrm{H}^*M, Z_p) = 0$. It follows from the exact sequence

$$0 = \mathrm{Ext}_A^{1,p^nq+1}(\mathrm{H}^*M, Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{1,p^nq}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{1,p^nq}(\mathrm{H}^*M, Z_p) \xrightarrow{p^*=0}$$

induced by (1.1) that $\mathrm{Ext}_A^{1,p^nq}(\mathrm{H}^*M, \mathrm{H}^*M)$ has unique generator $\tilde{h}_n$ so that $i^*(\tilde{h}_n) = i_*(h_n) \in \mathrm{Ext}_A^{1,p^nq}(\mathrm{H}^*M, Z_p)$, where the right $p^* = 0$ since $\mathrm{Ext}_A^{2,p^nq+1}(\mathrm{H}^*M, Z_p) = 0$ by Prop.2.3 (1).

Look at the following exact sequence

$$\mathrm{Ext}_A^{2,p^nq+q+1}(Z_p, Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{2,p^nq+q}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). The left group is zero and the right group has unique generator $h_0h_n = j_*\alpha_*i_*(h_n)$ (cf. [3]), then $\mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, Z_p) = Z_p\{\alpha_*i_*(h_n)\}$. It follows from the exact sequence

$$\mathrm{Ext}_A^{2,p^nq+q+2}(\mathrm{H}^*M, Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{i^*} \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, Z_p) \xrightarrow{p^*=0}$$

induced by (1.1) that $\mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M)$ has unique generator $\alpha_*(\tilde{h}_n)$ such that $i^*\alpha_*(\tilde{h}_n) = i^*\alpha^*(\tilde{h}_n) = \alpha_*i_*(h_n) \in \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, Z_p)$ since

$\mathrm{Ext}_A^{2,p^nq+q+2}(\mathrm{H}^*M, Z_p) = 0$ by the fact that $\mathrm{Ext}_A^{2,p^nq+q+r}(Z_p, Z_p) = 0$ for $r = 1, 2$ (cf. [3]). Q.E.D.

PROPOSITION 2.5: *Let* $p \geqslant 5, n \geqslant 2$, *then*

(1) $\mathrm{Ext}_A^{3,p^nq+q+3}(\mathrm{H}^*K, Z_p) = 0, \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*K) = 0$ *and* $\mathrm{Ext}_A^{3,p^nq-2}(Z_p, \mathrm{H}^*K) = Z_p\{(jj')^*(h_0b_{n-1})\}$.

(2) *For* $b_{n-1}g_0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(Z_p, Z_p), (i'i)_*(b_{n-1}g_0) \neq 0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$.

PROOF. (1) Consider the following exact sequence

$$\mathrm{Ext}_A^{3,p^nq+q+3}(\mathrm{H}^*M, Z_p) \xrightarrow{i'_*} \mathrm{Ext}_A^{3,p^nq+q+3}(\mathrm{H}^*K, Z_p) \xrightarrow{j'_*} \mathrm{Ext}_A^{3,p^nq+2}(\mathrm{H}^*M, Z_p)$$

induced by (1.2). The left group is zero since $\mathrm{Ext}_A^{3,p^nq+q+r}(Z_p, Z_p) = 0$ for $r = 2, 3$ (cf. [1]) and the right group is zero by Prop.2.3.(1), then the middle group is zero.

For the second result, we first show that $\mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*M) = Z_p\{(ij)^*(ij)_*\alpha_*(\tilde{b}_{n-1})\}$ and $j^*i_*(\alpha_2b_{n-1}) \neq 0 \in \mathrm{Ext}_A^{4,p^nq+2q}(\mathrm{H}^*M, \mathrm{H}^*M)$, where $\alpha_2b_{n-1} \in \mathrm{Ext}_A^{4,p^nq+2q+1}(Z_p, Z_p)$ is the generator in Prop.2.1.(2).

Look at the following exact sequence

$$\xrightarrow{p^*=0} \mathrm{Ext}_A^{3,p^nq+q}(\mathrm{H}^*M, Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{i_*} \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, Z_p)$$

induced by (1.2). The left group has unique generator $i_*(h_0b_{n-1}) = i_*j_*\alpha_*i_*(b_{n-1})$ since $\mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p) = Z_p\{h_0b_{n-1}\}$ and $\mathrm{Ext}_A^{r,p^nq+q-1}(Z_p, Z_p) = 0$ for $r = 2, 3$ (cf. [3] [1]) and the right group is zero since $\mathrm{Ext}_A^{3,p^nq+q+r}(Z_p, Z_p) = 0$ for $r = -1, -2$, then the middle group has unique generator $j^*i_*(h_0b_{n-1}) = (ij)^*(ij)_*\alpha_*(\tilde{b}_{n-1})$.

By the following exact sequence

$$0 = \mathrm{Ext}_A^{3,p^nq+2q}(Z_p, Z_p) \xrightarrow{p_*} \mathrm{Ext}_A^{4,p^nq+2q+1}(Z_p, Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{4,p^nq+2q+1}(\mathrm{H}^*M, Z_p)$$

induced by (1.1) we know that $i_*(\alpha_2b_{n-1}) \neq 0 \in \mathrm{Ext}_A^{4,p^nq+2q+1}(\mathrm{H}^*M, Z_p)$, where the left group is zero by [1] table 8.1. Moreover, by the following exact sequence

$$0 = \mathrm{Ext}_A^{3,p^nq+2q}(\mathrm{H}^*M, Z_p) \xrightarrow{p_*} \mathrm{Ext}_A^{4,p^nq+2q+1}(\mathrm{H}^*M, Z_p) \xrightarrow{j^*} \mathrm{Ext}_A^{4,p^nq+2q}(\mathrm{H}^*M, \mathrm{H}^*M)$$

induced by (1.1) we have $j^*i_*(\alpha_2b_{n-1}) \neq 0$, where the left group is zero since $\mathrm{Ext}_A^{3,p^nq+2q+r}(Z_p, Z_p) = 0$ for $r = -1, 0$ (cf. [1] table 8.1).

Now observe the following exact sequence

$$\mathrm{Ext}_A^{3,p^nq+2q}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{(j')^*} \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*K)$$

$$\xrightarrow{(i')^*} \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{(\alpha)^*}$$

induced by (1.2). Since $\mathrm{Ext}_A^{3,p^nq+2q+r}(Z_p, Z_p) = 0$ for $r = 0, -1$ and $\mathrm{Ext}_A^{3,p^nq+2q+1}(Z_p, Z_p) = Z_p\{\alpha_2 h_n\}$ by [1] table 8.1 and $\mathrm{Ext}_A^{2,p^nq+2q}(Z_p, Z_p) = 0$ (cf. [3]), then $\mathrm{Ext}_A^{3,p^nq+2q}(\mathrm{H}^*M, \mathrm{H}^*M)$ has unique generator $j^*i_*(\alpha_2 h_n)$ satisfying

$$(j')^*j^*i_*(\alpha_2 h_n) = (ijj')^*\alpha^*\alpha^*(ij)^*(\tilde{h}_n) = 0$$

since $\alpha^2 ijj' = 0 \in [\sum^{q-2} K, M]$. So the above $\mathrm{im}(j')^* = 0$. Moreover, the right group has unique generator $(ij)^*(ij)_*\alpha_*(\tilde{b}_{n-1})$ satisfying

$$\begin{aligned}\alpha^*(ij)^*(ij)_*\alpha_*(\tilde{b}_{n-1}) = (ij)^*(ij)_*\alpha_*\alpha_*(\tilde{b}_{n-1}) &= j^*i_*(\alpha_2 b_{n-1}) \\ &\neq 0 \in \mathrm{Ext}_A^{4,p^nq+2q}(\mathrm{H}^*M, \mathrm{H}^*M),\end{aligned}$$

then the above α^* is monic and so $\mathrm{im}(i')^* = 0$ and we have $\mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*K) = 0$.

At last, observe the following exact sequence

$$\begin{aligned}\mathrm{Ext}_A^{2,p^nq-2}(Z_p, \mathrm{H}^*M) \xrightarrow{\alpha^*} \mathrm{Ext}_A^{3,p^nq+q-1}(Z_p, \mathrm{H}^*M) \xrightarrow{(j')^*} \mathrm{Ext}_A^{3,p^nq-2}(Z_p, \mathrm{H}^*K) \\ \xrightarrow{(i')^*} \mathrm{Ext}_A^{3,p^nq-2}(Z_p, \mathrm{H}^*M)\end{aligned}$$

induced by (1.2). Since $\mathrm{Ext}_A^{r,p^nq-t}(Z_p, Z_p) = 0$ for $t = 1, 2$ and $r = 2, 3$, then $\mathrm{Ext}_A^{r,p^nq-2}(Z_p, \mathrm{H}^*M) = 0$ for $r = 2, 3$ and so the above $(j')^*$ is an isomorphism. Moreover, $\mathrm{Ext}_A^{3,p^nq+q-1}(Z_p, Z_p) = 0$ and $\mathrm{Ext}_A^{2,p^nq+q}(Z_p, Z_p) = Z_p\{h_0 b_{n-1}\}$ by [1] table 8.1, then $\mathrm{Ext}_A^{3,p^nq+q-1}(Z_p, \mathrm{H}^*M) = Z_p\{j^*(h_0 b_{n-1})\}$ and so $\mathrm{Ext}_A^{3,p^nq-2}(Z_p, \mathrm{H}^*K)$ has unique generator $(jj')^*(h_0 b_{n-1})$.

(2) This follows from Prop.2.1.(3). Q.E.D.

Let K' be the cofibre of $\alpha i \colon \Sigma^q S \to M$ given by the cofibration

(2.6) $$\Sigma^q S \xrightarrow{\alpha i} M \xrightarrow{v} K' \xrightarrow{y} \Sigma^{q+1} S$$

then K' is also the cofibre of $jj' \colon \Sigma^{-1}K \to \Sigma^{q+1}S$. This can be seen by the following commutative diagram of 3×3 lemma in stable homotopy category

(2.7)
$$\begin{tikzcd}
\Sigma^q S \arrow[rr] \arrow[dr, "i"] & & M \arrow[rr, "i'"] \arrow[dr, "v"] & & K \\
& \Sigma^q M \arrow[ur, "\alpha"] \arrow[dr, "j"] & & K' \arrow[ur, "x"] \arrow[dr, "y"] & \\
\Sigma^{-1} K \arrow[rr] \arrow[ur, "j'"] & & \Sigma^{q+1} S \arrow[rr, "p"] \arrow[ur, "z"] & & \Sigma^{q+1} S
\end{tikzcd}$$

That is, we have a cofibration

(2.8) $$\Sigma^{-1}K \xrightarrow{jj'} \Sigma^{q+1}S \xrightarrow{z} K' \xrightarrow{x} K$$

From (2.7), $yz = p$, then the composition $zy = 1_{K'} \wedge p\colon K' \xrightarrow{y} \Sigma^{q+1}S \xrightarrow{z} K'$. To check this, observe that $yzy = py = y(1_{K'} \wedge p)$ and so $zy = 1_{K'} \wedge p + vh$ for some $h \in [K', M]$ and this group is zero by the following exact sequence

$$\xleftarrow{(\alpha i)^*} [M, M] \xleftarrow{v^*} [K', M] \xleftarrow{y^*} [\Sigma^{q+1}S, M] = 0$$

induced by (2.6), where $(\alpha i)^*$ is monic.

The cofibre of $1_{K'} \wedge p\colon K' \to K'$ is $K' \wedge M$ and it is also the cofibre of $\alpha ijj'\colon \Sigma^{-1}K \to \Sigma M$, this can be seen by the following commutative diagram of 3×3 lemma in stable homotopy category

(2.9)
$$\begin{tikzcd}
K' \arrow[rr, "1_{K'}\wedge p"] \arrow[dr, "y"] & & K' \arrow[rr, "x"] \arrow[dr, "1_{K'}\wedge i"] & & K \\
& \Sigma^{q+1}S \arrow[ur, "z"] \arrow[dr, "\alpha i"] & & K' \wedge M \arrow[ur, "p"] \arrow[dr, "1_{K'}\wedge j"] & \\
\Sigma^{-1}K \arrow[rr] \arrow[ur, "jj'"] & & \Sigma M \arrow[rr, "v"] \arrow[ur, "\psi"] & & \Sigma K'
\end{tikzcd}$$

That is, we have a cofibration

(2.10) $$\Sigma^{-1}K \xrightarrow{\alpha ijj'} \Sigma M \xrightarrow{\psi} K' \wedge M \xrightarrow{p} K$$

Let L be the cofibre of $\alpha_1 = j\alpha i\colon \sum^{q-1} S \to S$ given by the cofibration

(2.11) $$\Sigma^{q-1}S \xrightarrow{\alpha_1} S \xrightarrow{i''} L \xrightarrow{j''} \textstyle\sum^{q} S.$$

From [7] p.434, there is $\bar{\Delta} \in [\sum^{-q-1} L \wedge K, K]$ such that

(2.12) $$\bar{\Delta}(i'' \wedge 1_K) = i'j' \in [\textstyle\sum^{-q-1} K, K], \quad j'\bar{\Delta} = ijj'(j'' \wedge 1_K).$$

PROPOSITION 2.13: *Let* $p \geqslant 5$, $n \geqslant 2$, *then* $\mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*K', \mathrm{H}^*K) = 0$, $\mathrm{Ext}_A^{2,p^nq+q+2}(\mathrm{H}^*K', \mathrm{H}^*M) = 0$, $\mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*K', \mathrm{H}^*M) = 0$ *and* $\mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*K', Z_p)$ *has unique generator* $(a_0b_{n-1})_{K'}$ *satisfying* $y_*(a_0b_{n-1})_{K'} = a_0b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+1}(Z_p, Z_p)$.

PROOF. We first prove that $\mathrm{Ext}_A^{2,p^nq-q-1}(Z_p, \mathrm{H}^*K) = 0$ and $\mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M, \mathrm{H}^*K) = 0$. To check this, consider the following exact sequences

$$\mathrm{Ext}_A^{2,p^nq}(Z_p,\mathrm{H}^*M) \xrightarrow{(j')^*} \mathrm{Ext}_A^{2,p^nq-q-1}(Z_p,\mathrm{H}^*K) \xrightarrow{(i')^*} \mathrm{Ext}_A^{2,p^nq-q-1}(Z_p,\mathrm{H}^*M)$$

$$\mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M,\mathrm{H}^*M) \xrightarrow{(j')^*} \mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M,\mathrm{H}^*K) \xrightarrow{(i')^*} \mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M,\mathrm{H}^*M) \xrightarrow{(\alpha)^*}$$

induced by (1.2). The upper left group is zero by Prop.2.3 (2) and the upper right group is also zero since $\mathrm{Ext}_A^{2,p^nq-q-r}(Z_p,Z_p)=0$ for $r=0,1$ (cf. [3]), then the upper middle group is zero. The lower left group has unique generator $\alpha^*(\tilde{h}_n)$ by Prop.2.4.(2) so that $\mathrm{im}(j')^*=0$. The lower right group has unique generator $\tilde{b}_{n-1}$ by Prop.2.4.(1) satisfying $\alpha^*(\tilde{b}_{n-1})\neq 0\in \mathrm{Ext}_A^{3,p^nq+q+1}(\mathrm{H}^*M,\mathrm{H}^*M)$, then $\mathrm{im}(i')^*=0$ and so the lower middle group is zero. Hence, by the following exact sequence

$$0=\mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*M,\mathrm{H}^*K) \xrightarrow{v_*} \mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*K',\mathrm{H}^*K) \xrightarrow{y_*} \mathrm{Ext}_A^{2,p^nq-q-1}(Z_p,\mathrm{H}^*K)=0$$

induced by (2.6) we know that $\mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*K',\mathrm{H}^*K)=0$.

Consider the following exact sequence

$$\mathrm{Ext}_A^{r,p^nq+q+2}(\mathrm{H}^*M,\mathrm{H}^*M) \xrightarrow{v_*} \mathrm{Ext}_A^{r,p^nq+q+2}(\mathrm{H}^*K',\mathrm{H}^*M) \xrightarrow{y_*} \mathrm{Ext}_A^{r,p^nq+1}(Z_p,\mathrm{H}^*M)$$

induced by (2.6). The left group is zero since $\mathrm{Ext}_A^{r,p^nq+q+t}(Z_p,Z_p)=0$ for $t=1,2,3$ and $r=2,3$ (cf. [3] and [1] table 8.1) and the right group is zero by Prop.2.3.(2), so $\mathrm{Ext}_A^{r,p^nq+q+2}(\mathrm{H}^*K',\mathrm{H}^*M)=0$ for $r=2,3$.

At last, look at the following exact sequence

$$\mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*M,Z_p) \xrightarrow{v_*} \mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*K',Z_p) \xrightarrow{y_*} \mathrm{Ext}_A^{3,p^nq+1}(Z_p,Z_p) \xrightarrow{(\alpha i)_*}$$

induced by (2.6). The left group is zero since $\mathrm{Ext}_A^{3,p^nq+q+t}(Z_p,Z_p)=0$ for $t=1,2$ and the right group has unique generator a_0b_{n-1} (cf. [1]) satisfying $(\alpha i)_*(a_0b_{n-1})=0$, then $\mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*K',Z_p)$ has unique generator $(a_0b_{n-1})_{K'}$ as desired. Q.E.D.

The main theorem II will be proved by some technique processing in the Adams resolution of certain spectra and need some more knowledge on Ext groups. Let

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\bar{a}_2} & \Sigma^{-2}E_2 & \xrightarrow{\bar{a}_1} & \Sigma^{-1}E_1 & \xrightarrow{\bar{a}_0} & S \\
& \downarrow{\scriptstyle \bar{b}_2} & & \downarrow{\scriptstyle \bar{b}_1} & & \downarrow{\scriptstyle \bar{b}_0} \\
& \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0=KZ_P
\end{array}$$

be the minimal Adams resolution of S satisfying

(1) $E_s \xrightarrow{\bar{b}_s} KG_s \xrightarrow{\bar{c}_s} E_{s+1} \xrightarrow{\bar{a}_s} \Sigma E_s$ are cofibrations for all $s \geqslant 0$ which induce short exact sequences $0 \to \mathrm{H}^*E_{s+1} \xrightarrow{\bar{c}_s^*} \mathrm{H}^*KG_s \xrightarrow{\bar{b}_s^*} \mathrm{H}^*E_s \to 0$ in Z_p-cohomology.

(2) KG_s is a wedge sum of Eilenberg-Maclane spectra of type KZ_p.

(3) $\pi_t KG_s$ are the $\mathrm{E}_1^{s,t}$-terms, $(\bar{b}_s\bar{c}_{s-1})_*: \pi_t KG_{s-1} \to \pi_t KG_s$ are the $d_1^{s,t}$-differentials of the ASS and $\pi_t KG_s \cong \mathrm{Ext}_A^{s,t}(Z_p, Z_p)$. Then

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\bar{a}_2 \wedge 1_w} & \Sigma^{-2}E_2 \wedge w & \xrightarrow{\bar{a}_1 \wedge 1_w} & \Sigma^{-1}E_1 \wedge w & \xrightarrow{\bar{a}_0 \wedge 1_w} & W \\
& \downarrow{\scriptstyle \bar{b}_2 \wedge 1_w} & & \downarrow{\scriptstyle \bar{b}_1 \wedge 1_w} & & \downarrow{\scriptstyle \bar{b}_0 \wedge 1_w} \\
& \Sigma^{-2}KG_2 \wedge W & & \Sigma^{-1}KG_1 \wedge W & & KG_0 \wedge W
\end{array}$$

is an Adams resolution of arbitrary finite spectrum W.

From [7] p.430, there is $\alpha'' \in [\sum^{q-2} K, K]$ satisfying

$$\alpha'' i' = i'ij\alpha ij, \quad j'\alpha'' = ij\alpha ijj'. \tag{2.14}$$

It follows that $jj'\alpha'' = 0$ and so by (2.8), $\alpha'' = x\bar{\alpha}''$ with $\bar{\alpha}'' \in [\sum^{q-2} K, K']$ and $j'x\bar{\alpha}'' = iy\bar{\alpha}'' = ij\alpha ijj'$. So, $y\bar{\alpha}'' = j\alpha ijj' \in [K, \sum^3 S]$. Moreover, $x\bar{\alpha}'' i' = \alpha'' i' = i'ij\alpha ij = xvij\alpha ij$, then $\bar{\alpha}'' i' = vij\alpha ij$ since $[M, \sum^3 S] = 0$. That is, we have

$$y\bar{\alpha}'' = j\alpha ijj', \quad \bar{\alpha}'' i' = vij\alpha ij. \tag{2.15}$$

Let X be the cofibre of $\alpha'': \sum^{q-2} K \to K$ given by the cofibration

$$\Sigma^{q-2} K \xrightarrow{\alpha''} K \xrightarrow{w} X \xrightarrow{u} \Sigma^{q-1} K \tag{2.16}$$

PROPOSITION 2.17: *Let $p \geqslant 5, n \geqslant 2$, then*

(1) *There is* $(b_{n-1})' \in \mathrm{Ext}_A^{2,p^n q}(\mathrm{H}^*K, \mathrm{H}^*K)$ *and* $(b_{n-1})'_{K'} \in \mathrm{Ext}_A^{2,p^n q}(\mathrm{H}^*K', \mathrm{H}^*K')$ *such that* $(jj')_*(b_{n-1})' = (jj')^*(b_{n-1}) \in \mathrm{Ext}_A^{2,p^n q-q-2}(Z_p, \mathrm{H}^*K)$ *and* $y_*(b_{n-1})'_{K'} = y^*(b_{n-1}) \in \mathrm{Ext}_A^{2,p^n q-q-1}(Z_p, \mathrm{H}^*K'), (vi)^*(b_{n-1})'_{K'} = (vi)_*(b_{n-1}) \in \mathrm{Ext}_A^{2,p^n q}(\mathrm{H}^*K', Z_p)$.

(2) $\mathrm{Ext}_A^{3,p^n q+q-1}(\mathrm{H}^*K', \mathrm{H}^*K)$ *has unique generator* $(h_0 b_{n-1})''_{K'}$ *satisfying* $(i')^*(h_0 b_{n-1})''_{K'} = v_* i_* j^*(h_0 b_{n-1})$ *and* $(h_0 b_{n-1})''_{K'} = \bar{\alpha}''_*(b_{n-1})' = (\bar{\alpha}'')^*(b_{n-1})'_{K'}$.

(3) $\mathrm{Ext}_A^{3,p^n q+q-1}(\mathrm{H}^*K, \mathrm{H}^*K)$ *has unique generator* $(h_0 b_{n-1})'' = \alpha''_*(b_{n-1})' = (\alpha'')^*(b_{n-1})'$ *satisfying* $j'_*(h_0 b_{n-1})'' = i_*(jj')^*(h_0 b_{n-1}) \in \mathrm{Ext}_A^{3,p^n q-2}(\mathrm{H}^*M, \mathrm{H}^*K)$.

PROOF. (1) We use an argument in the Adams resolution. Let $b_{n-1} \in \pi_{p^n q} KG_2 \cong \mathrm{Ext}_A^{2,p^n q}(Z_p, Z_p)$, then $b_{n-1}jj' = (1_{KG_2} \wedge jj')(b_{n-1} \wedge 1_K)$ and $(b_{n-1} \wedge 1_K) \in [\sum^{p^n q} K, KG_2 \wedge K]$ is a d_1-cycle which represents an element in $\mathrm{Ext}_A^{2,p^n q}(\mathrm{H}^*K, \mathrm{H}^*K)$ and we write it as $(b_{n-1})'$. The equation $b_{n-1}jj' = (1_{KG_2} \wedge jj')(b_{n-1} \wedge 1_K)$ implies $(jj')^*(b_{n-1}) = (jj')_*(b_{n-1})' \in \mathrm{Ext}_A^{2,p^n q-q-2}(Z_p, \mathrm{H}^*K)$. Moreover, $b_{n-1}y = (1_{KG_2} \wedge y)(b_{n-1} \wedge 1_{K'}) \in [\sum^{p^n q-q-1} K', KG_2]$ and $(b_{n-1} \wedge 1_{K'}) \in [\sum^{p^n q} K', KG_2 \wedge K']$ is a d_1-cycle which represents an element in

$\mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*K', \mathrm{H}^*K')$ and we write it as $(b_{n-1})'_{K'}$. Then, the equation $b_{n-1}y = (1_{KG_2} \wedge y)(b_{n-1} \wedge 1_{K'})$ and $(1_{KG_2} \wedge vi)b_{n-1} = (b_{n-1} \wedge 1_{K'})vi$ implies $y^*(b_{n-1}) = y_*(b_{n-1})'_{K'} \in \mathrm{Ext}_A^{2,p^nq-q-1}(Z_p, \mathrm{H}^*K')$ and $(vi)^*(b_{n-1})'_{K'} = (vi)_*(b_{n-1}) \in \mathrm{Ext}_A^{2,p^nq}(\mathrm{H}^*K', Z_p)$ respectively.

(2) Consider the following exact sequence

$$\mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*K) \xrightarrow{v_*} \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*K', \mathrm{H}^*K) \xrightarrow{y_*} \mathrm{Ext}_A^{3,p^nq-2}(Z_p, \mathrm{H}^*K) \xrightarrow{(\alpha i)_*}$$

induced by (2.6). The left group is zero and the right group has unique generator $(jj')^*(h_0b_{n-1})$ by Prop.2.5. Moreover, $(\alpha i)_*(jj')^*(h_0b_{n-1}) = (jj')^*(\alpha i)_*(j\alpha i)_*(b_{n-1}) = (\alpha i)_*(j\alpha i)_*(jj')_*(b_{n-1})' = 0 \in \mathrm{Ext}_A^{4,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*K)$ since $\alpha ij\alpha ijj' = 0 \in [\sum^{q-3} K, M]$. (Note: $\alpha ij\alpha ij = (1/2)ij\alpha^2 ij = ij\alpha ij\alpha$). Then, the middle group has unique generator $(h_0b_{n-1})''_{K'}$ such that $y_*(h_0b_{n-1})''_{K'} = (jj'')^*(h_0b_{n-1}) = (jj')^*\alpha_1^*(b_{n-1}) = (\alpha_1 jj')^*(b_{n-1}) = (y\bar{\alpha}'')^*(b_{n-1}) = (\bar{\alpha}'')^*y_*(b_{n-1})'_{K'}$ and so we have $(h_0b_{n-1})''_{K'} = (\bar{\alpha}'')^*(b_{n-1})'_{K'}$. On the other hand, $y_*(h_0b_{n-1})''_{K'} = (jj')^*(h_0b_{n-1}) = (jj')^*(\alpha_1)_*(b_{n-1}) = (\alpha_1)_*(jj')_*(b_{n-1})' = y_*\bar{\alpha}''_*(b_{n-1})'$, then $(h_0b_{n-1})''_{K'} = \bar{\alpha}''_*(b_{n-1})'$ as desired.

Moreover, $(i')^*(h_0b_{n-1})''_{K'} = (i')^*(\bar{\alpha}'')^*(b_{n-1})'_{K'} = (\bar{\alpha}''i')^*(b_{n-1})'_{K'} = (vij\alpha ij)^*(b_{n-1})'_{K'} = (j\alpha ij)^*(vi)_*(b_{n-1}) = vi_*j^*(h_0b_{n-1})$.

(3) We first prove that $\mathrm{Ext}_A^{r,p^nq-3}(Z_p, \mathrm{H}^*K) = 0$ for $r = 2, 3$, this can be seen by the following exact sequence

$$0 = \mathrm{Ext}_A^{r,p^nq+q-2}(Z_p, \mathrm{H}^*M) \xrightarrow{(j')^*} \mathrm{Ext}_A^{r,p^nq-3}(Z_p, \mathrm{H}^*K) \xrightarrow{(i')^*} \mathrm{Ext}_A^{r,p^nq-3}(Z_p, \mathrm{H}^*M) = 0$$

induced by (1.2), where both side of groups are zero since $\mathrm{Ext}_A^{r,p^nq-i}(Z_p, Z_p) = 0$ for $t = 2, 3, r = 2, 3$ and $\mathrm{Ext}_A^{r,p^nq+q-t}(Z_p, Z_p) = 0$ for $t = 1, 2, r = 2, 3$ (cf. [3] and [1] table 8.1). So, from the following exact sequence

$$0 = \mathrm{Ext}_A^{2,p^nq-3}(Z_p, \mathrm{H}^*K) \xrightarrow{z_*} \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*K', \mathrm{H}^*K) \xrightarrow{x_*} \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*K, \mathrm{H}^*K)$$
$$\xrightarrow{(jj')_*} \mathrm{Ext}_A^{3,p^nq-3}(Z_p, \mathrm{H}^*K) = 0$$

induced by (2.8) we know that $\mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*K, \mathrm{H}^*K)$ has unique generator $(h_0b_{n-1})'' = x_*(h_0b_{n-1})''_{K'} = x_*\bar{\alpha}''_*(b_{n-1})' = \alpha''_*(b_{n-1})'$. Moreover, $j'_*(h_0b_{n-1})'' = j'_*\alpha''_*(b_{n-1})' = (ij\alpha ijj')_*(b_{n-1})' = i_*(jj')^*(h_0b_{n-1})$, $(h_0b_{n-1})'' = x_*(h_0b_{n-1})''_{K'} = x_*(\bar{\alpha}'')^*(b_{n-1})'_{K'} = (\bar{\alpha}'')^*x_*(b_{n-1})'_{K'} = (\bar{\alpha}'')^*x^*(b_{n-1})' = (\alpha'')^*(b_{n-1})'$. Q.E.D.

3. Proof of the main theorems

We will first prove theorem II. Before proving it, we need to prove some lemmas.

LEMMA 3.1: *Let $p \geqslant 5$ and $\alpha_*(\tilde{h}_n) \in \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M)$ be the generator stated in* Prop.2.4.(2), *then $\psi_*\alpha_*(\tilde{h}_n) \neq 0 \in \mathrm{Ext}_A^{2,p^nq+q+2}(\mathrm{H}^*K' \wedge M, \mathrm{H}^*M)$*

and it is a permanent cycle in the ASS, where $\psi: \sum M \to K' \wedge M$ *is the map in* (2.10).

PROOF. We need to prove $v_*\alpha_*(\tilde{h}_n) = (1_{K'} \wedge j)_* \psi_* \alpha_*(\tilde{h}_n) \neq 0 \in \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*K', \mathrm{H}^*M)$ and this can be seen from the following exact sequence

$$0 = \mathrm{Ext}_A^{1,p^nq}(Z_p, \mathrm{H}^*M) \xrightarrow{(\alpha i)_*} \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{v_*} \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*K', \mathrm{H}^*M)$$

induced by (2.6), where the left group is zero by Prop.2.3.(2).

For the second result, we do work in the Adams resolution. Let $\widetilde{h_0 h_n} \in [\sum^{p^nq+q+1} M, KG_2 \wedge M]$ be the d_1-cycle which represents $\alpha_*(\tilde{h}_n) \in \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M)$, then it suffices to prove that $(\tilde{c}_2 \wedge 1_{K' \wedge M}) \cdot (1_{KG_2} \wedge \psi)(\widetilde{h_0 h_n}) = 0$.

Since the d_1-cycle $(1_{KG_2} \wedge v)(\widetilde{h_0 h_n})i \in \pi_{p^nq+q+1} KG_2 \wedge K'$ represents $v_*(i^*\alpha_*(\tilde{h}_n)) = v_*\alpha_* i_*(h_n) = 0$, then $(1_{KG_2} \wedge v)(\widetilde{h_0 h_n})i$ is a d_1-boundary and so we have

$$(3.2) \qquad (\tilde{c}_2 \wedge 1_{K'})(1_{KG_2} \wedge v)(\widetilde{h_0 h_n})ij = 0.$$

From (2.9), there is a factorization $v = (1_{K'} \wedge j)\psi: \sum M \xrightarrow{\psi} K' \wedge M \xrightarrow{1_{K'} \wedge j} \sum K'$, then from (3.2) we have $(\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0 h_n})ij = (1_{E_3} \wedge 1_{K'} \wedge i)f$ for some $f \in [\sum^{p^nq+q+1} M, E_3 \wedge K']$. Hence, $(\bar{a}_2 \wedge 1_{K' \wedge M}) \cdot (1_{E_3} \wedge 1_{K'} \wedge i)f = 0$ and $(\bar{a}_2 \wedge 1_{K'})f = (1_{E_2} \wedge 1_{K'} \wedge p)f_2 = f_2(1_M \wedge p) = 0$ with $f_2 \in [\sum^{p^nq+q} M, E_2 \wedge K']$. Thus $f = (\bar{c}_2 \wedge 1_{K'})g$ with $g \in [\sum^{p^nq+q+1} M, KG_2 \wedge K']$ and we have

$$(3.3) \qquad (\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0 h_n})ij = (\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge 1_{K'} \wedge i)g.$$

It follows that $(\bar{b}_3 \bar{c}_2 \wedge 1_{K'})g = 0$ since $1_{KG_3} \wedge 1_{K'} \wedge p = 0$, i.e., g is a d_1-cycle which represents an element in $\mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*K', \mathrm{H}^*M)$. We claim that this group has unique generator $v_*\alpha_*(\tilde{h}_n)$, this can be seen from the following exact sequence

$$\xrightarrow{(\alpha i)_*} \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, \mathrm{H}^*M) \xrightarrow{v_*} \mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*K', \mathrm{H}^*M) \xrightarrow{y_*} \mathrm{Ext}_A^{2,p^nq}(Z_p, \mathrm{H}^*M)$$

induced by (2.6), where the right group is zero by Prop.2.3.(2) and the left group has unique generator $\alpha_*(\tilde{h}_n)$ by Prop.2.4.(2), moreover, $\mathrm{Ext}_A^{1,p^nq}(Z_p, \mathrm{H}^*M) = 0$ (cf. Prop.2.3.(2)) so that $\mathrm{im}(\alpha i)_* = 0$. Then, $g = c(1_{KG_2} \wedge v)(\widetilde{h_0 h_n})$ modulo d_1-boundary with $c \in Z_p$ and so (3.3) becomes

$$(3.4) \quad (\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0 h_n})ij = (\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge (1_{K'} \wedge i)v)(\widetilde{h_0 h_n}),$$

where we omit the scalar $c \in Z_p$ which is inessential in the argument below.

Now we will use some technique on derivations of maps between M-module spectra. The spectra $E_3 \wedge K' \wedge M, KG_2 \wedge K' \wedge M$ and $KG_2 \wedge M$ are M-module spectra with M-module structure determined by the right M, then the derivation $d(\bar{c}_2 \wedge 1_{K' \wedge M}) = \bar{c}_2 \wedge d(1_{K' \wedge M}) = 0$ (cf. [10] p.210 theorem 2.2) and $d(1_{KG_2} \wedge \psi) = 1_{KG_2} \wedge d(\psi) = 0$ since $d(\psi) \in [\sum^2 M, K' \wedge M] = 0$ by the following exact sequence

$$0 = [\Sigma^2 M, \Sigma M] \xrightarrow{\psi_*} [\Sigma^2 M, K' \wedge M] \xrightarrow{\rho_*} [\Sigma^2 M, K] = 0$$

induced by (2.10). So the derivation of the left hand side of (3.4) equals $-(\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0 h_n})$ since $d(ij) = -1_M$ and $d(\widetilde{h_0 h_n}) \in [\sum^{p^n q+q+2} M, KG_2 \wedge M] = 0$ since $\pi_{p^n q+q+r} KG_2 \wedge M = 0$ for $r = 2, 3$ by the fact that $\pi_{p^n q+q+r} KG_2 = \mathrm{Ext}_A^{2, p^n q+q+r}(Z_p, Z_p) = 0$ for $r = 1, 2, 3$.

Moreover, we consider the derivation of the right hand side of (3.4). Note that KG_s is an M-module spectrum with M-module action $m_{G_s}: M \wedge KG_s \to KG_s, \bar{m}_{G_s}: \sum KG_s \to M \wedge KG_s$. Then $KG_s \wedge K'$ is also an M-module spectrum with M-module action

$$m_{G_s} \wedge 1_{K'}: M \wedge KG_s \wedge K' \to KG_s \wedge K'$$

$$\bar{m}_{G_s} \wedge 1_{K'}: \Sigma KG_s \wedge K' \to M \wedge KG_s \wedge K'$$

So by applying d to (3.4) we have

$$\begin{aligned}(3.5) \qquad &(\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0 h_n}) \\ &= (\bar{c}_2 \wedge 1_{K' \wedge M}) d(1_{KG_2} \wedge 1_{K'} \wedge i)(1_{KG_2} \wedge v)(\widetilde{h_0 h_n}) \\ &\quad + (\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge 1_{K'} \wedge i) d(1_{KG_2} \wedge v)(\widetilde{h_0 h_n})\end{aligned}$$

Since $(\widetilde{h_0 h_n})$ is a d_1-cycle, i.e., $(\bar{b}_3 \bar{c}_2 \wedge 1_M)(\widetilde{h_0 h_n}) = 0$, then $d(1_{KG_2} \wedge v)(\widetilde{h_0 h_n}) \in [\sum^{p^n q+q+2} M, KG_2 \wedge K']$ is also a d_1-cycle. To check this, we need to prove the commutativity $(\bar{b}_3 \bar{c}_2 \wedge 1_{K'}) d(1_{KG_2} \wedge v) = d(1_{KG_3} \wedge v)(\bar{b}_3 \bar{c}_2 \wedge 1_M)$. Note that $d(1_{KG_2} \wedge v) = (m_{G_2} \wedge 1_{K'})(1_M \wedge 1_{KG_2} \wedge v) \cdot (T \wedge 1_M)(1_{KG_2} \wedge \bar{m})$, where $\bar{m}: \sum M \to M \wedge M$ is the M-module action of M. Then it suffices to prove the following diagram commutes up to homotopy

$$\begin{array}{ccc} M \wedge KG_2 & \xrightarrow{m_{G_2}} & KG_2 \\ \Big\downarrow{\scriptstyle 1_M \wedge \bar{b}_3 \bar{c}_2} & & \Big\downarrow{\scriptstyle \bar{b}_3 \bar{c}_2} \\ M \wedge KG_3 & \xrightarrow{m_{G_3}} & KG_3 \end{array}$$

Consider the induced homomorphisms in Z_p-cohomology. Since $m_{G_s}(i \wedge 1_{KG_s}) = 1_{KG_s}$, then $m_{G_s}^*(a) = 1 \otimes a$ for any $a \in \mathrm{H}^* KG_s$. So we have

$m_{G_3}^*(\bar{b}_3\bar{c}_2)^*(a) = 1 \otimes (\bar{b}_3\bar{c}_2)^*(a) = (1_M \wedge \bar{b}_3\bar{c}_2)^*(1 \otimes a) = (1_M \wedge \bar{b}_3\bar{c}_2)^* m_{G_2}^*(a)$ for any $a \in \mathrm{H}^*KG_2$. This proves the above commutativity and so $d(1_{KG_2} \wedge v)(\widetilde{h_0h_n}) \in [\sum^{p^nq+q+2} M, KG_2 \wedge K']$ is a d_1-cycle which represents an element in $\mathrm{Ext}_A^{2,p^nq+q+2}(\mathrm{H}^*\underline{K'}, \mathrm{H}^*M)$. However, this group is zero by Prop.2.13., then $d(1_{KG_2} \wedge v)(\widetilde{h_0h_n})$ is a d_1-boundary and so (3.5) becomes

$$(3.6) \qquad (\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0h_n}) = (\bar{c}_2 \wedge 1_{K' \wedge M}) d(1_{KG_2} \wedge 1_{K'} \wedge i)(1_{KG_2} \wedge v)(\widetilde{h_0h_n})$$

Recall from (3.2) we have $(1_{KG_2} \wedge v)(\widetilde{h_0h_n})ij = (\bar{b}_2\bar{c}_1 \wedge 1_{K'})g_2$ for some $g_2 \in \left[\sum^{p^nq+q} M, KG_1 \wedge K'\right]$. Moreover, for the same reason as stated above, we have the commutativity $d(1_{KG_2} \wedge 1_{K'} \wedge i)(\bar{b}_2\bar{c}_1 \wedge 1_{K'}) = (\bar{b}_2\bar{c}_1 \wedge 1_{K' \wedge M}) d(1_{KG_1} \wedge 1_{K'} \wedge i)$. (Note: Here, we need only to check $\bar{m}_{G_2}(\bar{b}_2\bar{c}_1) = (1_M \wedge \bar{b}_2c_1)\bar{m}_{G_1}$ and this can be proved by the induced homomorphism in Z_p-cohomology by using the fact that $(j \wedge 1_{KG_s})\bar{m}_{G_s} = 1_{KG_s}$). Then $(\bar{c}_2 \wedge 1_{K' \wedge M}) d(1_{KG_2} \wedge 1_{K'} \wedge i)(1_{KG_2} \wedge v)(\widetilde{h_0h_n})ij = 0$ and by (3.6) we have

$$(\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0h_n})ij = 0$$

By applying d to this equation, we have $(\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0h_n}) = 0$ since $d(ij) = -1_M$ and $d(\widetilde{h_0h_n}) = 0, d(1_{KG_2} \wedge \psi) = 0, d(\bar{c}_2 \wedge 1_{K' \wedge M}) = 0$. This finishes the proof of Lemma.3.1. Q.E.D.

LEMMA 3.7. *Let* $p \geqslant 5$ *and* $(a_0b_{n-1})_{K'} \in \mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*K', Z_p)$ *be the generator stated in* Prop. 2.13, *then this* $(a_0b_{n-1})_{K'}$ *is a permanent cycle in the ASS.*

PROOF. From [3] or [8] p.11 Theorem 1.2.14, we have $d_2(h_n) = a_0b_{n-1}$, where $d_2\colon E_2^{1,p^nq} \to E_2^{3,p^nq+1}$ is the differential of the ASS. Then, for $h_n \in \pi_{p^nq}KG_1 \cong \mathrm{Ext}^{1,p^nq}(Z_p, Z_p)$, we have $\bar{c}_1h_n = \bar{a}_2f'$ with $f' \in \pi_{p^nq+1}E_3$ and $\bar{b}_3f' = a_0b_{n-1} \in \pi_{p^nq+1}KG_3 \cong \mathrm{Ext}^{3,p^nq+1}(Z_p, Z_p)$.

It follows that $(\bar{a}_2 \wedge 1_M)(1_{E_3} \wedge \alpha i)f' = 0$ and $(1_{E_3} \wedge \alpha i)f' = (\bar{c}_2 \wedge 1_M)g''$ with d_1-cycle $g'' \in \pi_{p^nq+q+1}KG_2 \wedge M$ which represents an element in $\mathrm{Ext}_A^{2,p^nq+q+1}(\mathrm{H}^*M, Z_p) = \underline{Z_p}\{i^*\alpha_*(\tilde{h}_n)\}$ by the proof of Prop. 2.4., then $(1_{E_3} \wedge \alpha i)f' = (\bar{c}_2 \wedge 1_M)(\widetilde{h_0h_n})i$ and so by Lemma 3.1, we have

$$(1_{E_3} \wedge \psi\alpha i)f' = (\bar{c}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge \psi)(\widetilde{h_0h_n})i = 0$$

So by (2.10), $(1_{E_3} \wedge \alpha i)f' = (1_{E_3} \wedge \alpha ijj')f''$ with $f'' \in \pi_{p^nq+q+3}E_3 \wedge K$ and by (2.6)

$$f' = (1_{E_3} \wedge jj')f'' + (1_{E_3} \wedge y)f_2''$$

with $f_2'' \in \pi_{p^nq+q+2}E_3 \wedge K'$. It follows that

$$(1_{KG_3} \wedge jj')(\bar{b}_3 \wedge 1_K)f'' + (1_{KG_3} \wedge y)(\bar{b}_3 \wedge 1_{K'})f_2'' = \bar{b}_3 f' = a_0 b_{n-1}$$

Since $(\bar{b}_3 \wedge 1_K)f'' \in \pi_{p^nq+q+3}KG_3 \wedge K$ represents $[(\bar{b}_3 \wedge 1_K)f''] \in \mathrm{Ext}_A^{3,p^nq+q+3}(\mathrm{H}^*K, Z_p) = 0$ by Prop. 2.5., then $(\bar{b}_3 \wedge 1_{K'})f_2'' \in \pi_{p^nq+q+2}KG_3 \wedge K'$ must represents the generator $(a_0b_{n-1})_{K'} \in \mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*K', Z_p)$. This finishes the proof of the lemma. Q.E.D.

Proof of Theorem II: The relation $\alpha'' = x\bar{\alpha}''$ in (2.14) yields an element $\sigma \in \left[X, \sum^{q+2} S\right]$ fitting into the commutative diagram

$$\begin{array}{ccccccccc}
\sum^{q-2} K & \xrightarrow{\alpha''} & K & \xrightarrow{w} & X & \xrightarrow{u} & \sum^{q-1} K & \xrightarrow{\alpha''} & \sum K \\
\downarrow{\scriptstyle \bar{\alpha}} & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \bar{\alpha}''} & & \| \\
K' & \xrightarrow{x} & K & \xrightarrow{jj'} & \sum^{q+2} S & \xrightarrow{z} & \sum K' & \xrightarrow{x} & \sum K
\end{array},$$

in which both sequences are cofibrations. Thus we have relations $jj' = \sigma w$ and $\bar{\alpha}''u = z\sigma$.

For the generator $(h_0b_{n-1})_{K'}'' \in \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*K', \mathrm{H}^*K)$, $u^*(h_0b_{n-1})_{K'}'' = u^*(\bar{\alpha}'')^*((b_{n-1})_{K'}') = \sigma^*z^*((b_{n-1})_{K'}')$ and $z^*((b_{n-1})_{K'}') = (a_0b_{n-1})_{K'} \in \mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*K', Z_p)$ is a permanent cycle by Lemma 3.7, then $u^*(h_0b_{n-1})_{K'}''$ is also a permanent cycle in the ASS. (Note: $y_*z^*(b_{n-1})_{K'}' = z^*y_*(b_{n-1})_{K'}' = z^*y^*(b_{n-1}) = p^*(b_{n-1}) = a_0b_{n-1} = y_*(a_0b_{n-1})_{K'}$, then we have the above equation $z^*(b_{n-1})_{K'}' = (a_0b_{n-1})_{K'}$ since $\mathrm{Ext}_A^{3,p^nq+q+2}(\mathrm{H}^*M, Z_p) = 0$).

Therefore, $(c_3 \wedge 1)(h_0b_{n-1})_{K'}''u = 0$, which yields a commutative diagram

$$\begin{array}{ccccccccc}
\sum^{p^nq} K & \xrightarrow{w} & \sum^{p^nq} X & \xrightarrow{u} & \sum^{p^nq+q-1} K & \xrightarrow{\alpha''} & \sum^{p^nq+1} K & \xrightarrow{w} & \sum^{p^nq+1} X \\
\downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \bar{f}_2} & & \downarrow{\scriptstyle (h_0b_{n-1})_K''} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle \bar{f}_2} \\
\sum^{-1} E_4 \wedge K' & \xrightarrow{\bar{a}_3 \wedge 1} & E_3 \wedge K' & \xrightarrow{\bar{b}_3 \wedge 1} & KG_3 \wedge K' & \xrightarrow{\bar{c}_3 \wedge 1} & E_4 \wedge K' & \xrightarrow{\bar{a}_3 \wedge 1} & \sum E_3 \wedge K',
\end{array}$$

by which we obtain maps $\bar{f}: \sum^{p^nq+1} K \to E_4 \wedge K'$ and $\bar{f}_2: \sum^{p^nq} X \to E_3 \wedge K'$. By Lemma 3.7, let $\tilde{f}: \sum^{p^nq+q+2} S \to E_3 \wedge K'$ denote a homotopy element represented by $(a_0b_{n-1})_{K'}$, that is, $(\bar{b}_3 \wedge 1)\tilde{f} = (a_0b_{n-1})_{K'}$. Therefore, we have $(\bar{b}_3 \wedge 1_{K'})\bar{f}_2 = (h_0b_{n-1})_{K'}''u = (a_0b_{n-1})_{K'}\sigma = (\bar{b}_3 \wedge 1_{K'})\tilde{f}\sigma \bmod d_1$-boundary.

Thus, we have $(\bar{b}_3 \wedge 1_{K'})\bar{f}_2 = (\bar{b}_3 \wedge 1_{K'})\tilde{f}\sigma + (\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_3$ with $\bar{g}_3 \in [\sum^{p^nq} X, KG_2 \wedge K']$ and $\bar{f}_2 = \tilde{f}\sigma + (\bar{c}_2 \wedge 1_{K'})\bar{g}_3 + (\bar{a}_3 \wedge 1_{K'})\bar{f}_3$, and so

$$\begin{aligned}
(\bar{a}_3 \wedge 1_{K'})\bar{f} &= \bar{f}_2 w = \tilde{f}\sigma w + (\bar{a}_3 \wedge 1_{K'})\bar{f}_3 w + (\bar{c}_2 \wedge 1_{K'})\bar{g}_3 w \\
&= \tilde{f}jj' + (\bar{a}_3 \wedge 1_{K'})\bar{f}_3 w + (\bar{c}_2 \wedge 1_{K'})\bar{g}_3 w
\end{aligned}$$

By (2.12), $jj'\bar{\Delta} = jijj'(j'' \wedge 1_K) = 0$ for $\bar{\Delta} \in [\sum^{-q-1} L \wedge K, K]$ and we obtain

$$(3.8) \qquad (\bar{a}_3 \wedge 1_{K'})\bar{f}\bar{\Delta} = \bar{f}_2 w = (\bar{a}_3 \wedge 1_{K'})\bar{f}_3 w\bar{\Delta} + (\bar{c}_2 \wedge 1_{K'})\bar{g}_3 w\Delta$$

It follows that $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_3 w\bar{\Delta} = 0$, i.e., $\bar{g}_3 w\bar{\Delta} \in [\sum^{p^n q-q-1} L \wedge K, KG_2 \wedge K']$ is a d_1-cycle which represents $[\bar{g}_3 w\bar{\Delta}] \in \mathrm{Ext}_A^{2,p^n q-q-1}(\mathrm{H}^*K', \mathrm{H}^*L \wedge K)$. This group is nonzero, but we claim that $[\bar{g}_3 w\bar{\Delta}] = 0$, which can be proved as follows.

Consider the derivation $d(\bar{g}_3 w\bar{\Delta}) \in [\sum^{p^n q-q} L \wedge K, KG_2 \wedge K']$. We have $d(\bar{g}_3 w\bar{\Delta}) = (m_{G_2} \wedge 1_{K'})(1_M \wedge \bar{g}_3 w\bar{\Delta})\bar{m}_{L\wedge K}$, where $(m_{G_2} \wedge 1_{K'})\colon M \wedge KG_2 \wedge K' \to KG_2 \wedge K', \bar{m}_{L\wedge K}\colon \sum L \wedge K \to M \wedge L \wedge K$ are the M-module actions of $KG_2 \wedge K'$ and $L \wedge K$ respectively. In the proof of Lemma 3.1, we have the commutativity $(\bar{b}_3\bar{c}_2)m_{G_2} = m_{G_3}(1_M \wedge \bar{b}_3\bar{c}_2)$, then $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})d(\bar{g}_3 w\bar{\Delta}) = (m_{G_3} \wedge 1_{K'}) \cdot (1_M \wedge \bar{b}_3\bar{c}_2 \wedge 1_{K'})(1_M \wedge \bar{g}_3 w\bar{\Delta})\bar{m}_{L\wedge K} = 0$, i.e., $d(\bar{g}_3 w\bar{\Delta}) \in [\sum^{p^n q-q} L \wedge K, KG_2 \wedge K']$ is a d_1-cycle.

Moreover, by the derivation formula, $d(\bar{g}_3 w\bar{\Delta}) = d(\bar{g}_3 w)\bar{\Delta} + \bar{g}_3 w(j'' \wedge 1_K)$ is a d_1-cycle (Note: $d(\bar{\Delta}) = j'' \wedge 1_K$ by [7] p.434), then $d(\bar{g}_3 w)i'j' = d(\bar{g}_3 w)\bar{\Delta}(i'' \wedge 1_K)$ (cf. (2.12)) is also a d_1-cycle.

Let K_2 be the cofibre of $\alpha^2\colon \sum^{2q} M \to M$ given by the cofibration

$$(3.9) \qquad \sum^{2q} M \xrightarrow{\alpha^2} M \xrightarrow{i_2'} K_2 \xrightarrow{j_2'} \sum^{2q+1} M.$$

Then we also have a cofibration (cf. [7] p.422)

$$(3.10) \qquad \sum^{-1} K \xrightarrow{i'j'} \sum^{q} K \xrightarrow{\bar{\psi}} K_2 \xrightarrow{\bar{\rho}} K.$$

satisfying $\bar{\psi}i' = i_2'\alpha$, $j'\bar{\rho} = \alpha j_2'$.

Following from $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})d(\bar{g}_3 w)i'j' = 0$, we have $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})d(\bar{g}_3 w) = \bar{g}_4\bar{\psi}$ with $\bar{g}_4 \in [\sum^{p^n q-q+1} K_2, KG_3 \wedge K']$. But $\bar{g}_4\bar{\psi}i' = \bar{g}_4 i_2'\alpha = 0$, then $\bar{g}_4\bar{\psi} = \bar{g}_5 j'$ with $\bar{g}_5 \in [\sum^{p^n q+q+2} M, KG_3 \wedge K']$. So we have

$$(3.11) \qquad (\bar{b}_3\bar{c}_2 \wedge 1_{K'})d(\bar{g}_3 w) = \bar{g}_5 j'$$

and $(\bar{b}_4\bar{c}_3 \wedge 1_{K'})\bar{g}_5 = \bar{g}_6\alpha = 0$ (with $\bar{g}_6 \in [\sum^{p^n q+2} M, KG_4 \wedge K']$), i.e., $\bar{g}_5$ is a d_1-cycle which represents $[\bar{g}_5] \in \mathrm{Ext}^{3,p^n q+q+2}(\mathrm{H}^*K', \mathrm{H}^*M) = 0$ by Prop. 2.13. Then $\bar{g}_5 = (\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_7$ for some $\bar{g}_7 \in [\sum^{p^n q+q+2} M, KG_2 \wedge K']$ and (3.11) becomes $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})d(\bar{g}_3 w) = (\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_7 j'$ and $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})d(\bar{g}_3 w)\bar{\Delta} = (\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_7 j'\bar{\Delta} = (\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_7 ijj'(j'' \wedge 1_K)$ (cf. 2.12)).

Following from the conclusion that $d(\bar{g}_3 w\bar{\Delta}) = d(\bar{g}_3 w)\bar{\Delta} + \bar{g}_3 w(j'' \wedge 1_K)$ is a d_1-cycle, we have $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})d(\bar{g}_3 w)\bar{\Delta} + (\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_3 w(j'' \wedge 1_K) = 0$ and so $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})\bar{g}_7 ijj'(j'' \wedge 1_K) + g_3 w(j'' \wedge 1_K)) = 0$ and $(\bar{b}_3\bar{c}_2 \wedge 1_{K'})(\bar{g}_7 ijj' + g_3 w) = \bar{g}_8(\alpha_1 \wedge 1_K) = 0$ (with $\bar{g}_8 \in [\sum^{p^n q-q+1} K, KG_3 \wedge K'])$. It means that $\bar{g}_7 ijj' + \bar{g}_3 w \in [\sum^{p^n q} K, KG_2 \wedge K']$ is a d_1-cycle which represents an element in

$\mathrm{Ext}^{2,p^nq}(\mathrm{H}^*K', \mathrm{H}^*K) = 0$ by Prop. 2.13. Hence, $\bar{g}_7 ijj' + \bar{g}_3 w$ is a d_1-boundary and so $\bar{g}_3 w\bar{\Delta}$ is also a d_1-boundary since $jj'\bar{\Delta} = 0$. This shows the claim.

So, (3.8) becomes $(\bar{a}_3 \wedge 1_{K'})\bar{f}\bar{\Delta} = (\bar{a}_3 \wedge 1_{K'})\bar{f}_3 w\bar{\Delta}$ and so $\bar{f}\bar{\Delta} = f_3 w\bar{\Delta} + (\bar{c}_3 \wedge 1_{K'})\bar{g}_9$ for some $\bar{g}_9 \in [\sum^{p^nq-q} L \wedge K, KG_3 \wedge K']$. Since $\alpha'': \sum^{q-2} K \to K$ induces zero homomorphism in Z_p-cohomology, then $\bar{f}\bar{\Delta}(1_L \wedge \alpha'') = f_3 w\bar{\Delta}(1_L \wedge \alpha'')$. Now $\bar{\Delta}(1_L \wedge \alpha'') = \alpha''\bar{\Delta}$, then we have

$$(3.12) \qquad (\bar{c}_3 \wedge 1_{K'})(h_0 b_{n-1})''_{K'}\bar{\Delta} = \bar{f}\alpha''\bar{\Delta} = \bar{f}\bar{\Delta}(1_L \wedge \alpha'') = \bar{f}_3 w\bar{\Delta}(1_L \wedge \alpha'') = 0$$

(Note: we prove $\bar{\Delta}(1_L \wedge \alpha'') = \alpha''\bar{\Delta}$ as follows. Since $(\bar{\Delta}(1_L \wedge \alpha'') - \alpha''\bar{\Delta}) \cdot (i'' \wedge 1_K) = (\bar{\Delta}(i'' \wedge 1_K)\alpha'' - (\alpha''\bar{\Delta})(i'' \wedge 1_K) = ij\alpha'' - \alpha'' ij = 0$ by (3.9), then $(\bar{\Delta}(1_L \wedge \alpha'') - \alpha''\bar{\Delta}) \in (j'' \wedge 1_K)^*[\sum^{q-3} K, K] = 0$ since $[\sum^{q-3} K, K] = 0$ by [7] p. 431.)

It follows from (3.12) that there is $\bar{f}_4 \in [\sum^{p^nq-2} L \wedge K, E_3 \wedge K]$ such that $(\bar{b}_3 \wedge 1_{K'})\bar{f}_4 = (h_0 b_{n-1})''_{K'}\bar{\Delta}$ and we have

$$\begin{aligned}(3.13) \qquad (\bar{b}_3 \wedge 1_{K'})\bar{f}_4(i'' \wedge 1_K) &= (h_0 b_{n-1})''_{K'}\bar{\Delta}(i'' \wedge 1_K) = (h_0 b_{n-1})''_{K'} i'j' \\ &= (1_{KG_3} \wedge v)(h_0 b_{n-1} \wedge 1_M)ijj' \text{ modulo } d_1\text{-boundary} \\ &\qquad \text{(cf. (2.12) and Prop. 2.17 (2)).}\end{aligned}$$

From [2], there is $\zeta_{n-1} \in \pi_{p^nq+q-3}S$ which is represented by $h_0 b_{n-1} \in \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p)$ in the ASS. Then ζ_{n-1} can be lifted to $\zeta_{n-1,3} \in \pi_{p^nq+q}E_3$ such that $\bar{a}_0\bar{a}_1\bar{a}_2\zeta_{n-1,3} = \zeta_{n-1}$ and $\bar{b}_3\zeta_{n-1,3} = h_0 b_{n-1} \in \pi_{p^nq+q}KG_3 \cong \mathrm{Ext}_A^{3,p^nq+q}(Z_p, Z_p)$.

So, by (3.13) we have

$$\begin{aligned}(\bar{b}_3 \wedge 1_K)(1_{E_3} \wedge x)\bar{f}_4(i'' \wedge 1_K) &= (1_K \wedge i')(h_0 b_{n-1} \wedge 1_M)ijj' + (\bar{b}_3\bar{c}_2 \wedge 1_K)\tilde{g} \\ &= (\bar{b}_3 \wedge 1_K)(1_{E_3} \wedge i')(\zeta_{n-1,3} \wedge 1_M)ijj' + (\bar{b}_3\bar{c}_2 \wedge 1_K)\tilde{g}\end{aligned}$$

with $\tilde{g} \in [\sum^{p^nq-2} K, KG_2 \wedge K]$ and

$$(1_{E_3} \wedge x)\bar{f}_4(i'' \wedge 1_K) = (1_{E_3} \wedge i')(\zeta_{n-1,3} \wedge 1_M)ijj' + (\bar{c}_2 \wedge 1_K)\tilde{g} + (\bar{a}_3 \wedge 1_K)\bar{f}_5$$

for some $\bar{f}_5 \in [\sum^{p^nq-1} K, E_4 \wedge K]$ and we have

$$(3.14) \qquad (1_{E_3} \wedge i'i)\zeta_{n-1,3} jj'(\alpha_1 \wedge 1_K) + (\bar{a}_3 \wedge 1_K)\bar{f}_5(\alpha_1 \wedge 1_K) = 0$$

From [7] p.433, there is a homotopy equivalence $K \wedge K = K \vee \sum L \wedge K \vee \sum^{q+2} K$ and there are multiplication $\mu: K \wedge K \to K$ and injection $v: \sum^{q+2} K \to K \wedge K$ satisfying $\mu(i'i \wedge 1_K) = 1_K, (jj' \wedge 1_K)v = 1_K$. Now write $\alpha' = \alpha_1 \wedge 1_K$ and following (3.14) we have

$$(1_{E_3} \wedge \alpha' i'i)\zeta_{n-1,3} jj' = (\bar{a}_3 \wedge 1_K)(1_{E_4} \wedge \alpha')\bar{f}_5 \quad \text{(up to sign)}$$

and

$$(1_{E_3} \wedge \mu)(\alpha' i' i \wedge 1_K)(\zeta_{n-1,3} \wedge 1_K)(jj' \wedge 1_K)\nu = (\bar{a}_3 \wedge 1_K)(1_{E_4} \wedge \mu(\alpha' \wedge 1_K))(\bar{f}_5 \wedge 1_K)\nu$$

and

$$(1_{E_3} \wedge \alpha')(\zeta_{n-1,3} \wedge 1_K) = (\bar{a}_3 \wedge 1_K)(1_{E_4} \wedge \alpha')\bar{f}_6$$

where we write $\bar{f}_6 = (1_{E_4} \wedge \mu)(\bar{f}_5 \wedge 1_K)\nu$ and use $\mu(\alpha' \wedge 1_K) = \alpha'\mu$. Since $j'\alpha' = \alpha ijj'$, then $(1_{E_3} \wedge \alpha ijj')(\zeta_{n-1,3} \wedge 1_K) = (1_{E_3} \wedge \alpha ijj')(\bar{a}_3 \wedge 1_K)\bar{f}_6$ and so

$$(3.15) \qquad (1_{E_3} \wedge ijj')(\zeta_{n-1,3} \wedge 1_K) = (1_{E_3} \wedge ijj')(\bar{a}_3 \wedge 1_K)\bar{f}_6 + (1_{E_3} \wedge j')\zeta''_{n-1,3}$$

with $\zeta''_{n-1,3} \in [\sum^{p^nq+q-1} K, E_3 \wedge K]$. From (3.15),

$$\begin{aligned}(1_{KG_3} \wedge j')(\bar{b}_3 \wedge 1_K)\zeta''_{n-1,3} &= (1_{KG_3} \wedge ijj')(\bar{b}_3 \wedge 1_K)(\zeta_{n-1,3} \wedge 1_K) \\ &= (1_{KG_3} \wedge ijj')(h_0 b_{n-1} \wedge 1_K) \\ &= (1_{KG_3} \wedge j')(h_0 b_{n-1})'' + (\bar{b}_3 \bar{c}_2 \wedge 1_M)(1_{KG_2} \wedge j')\tilde{g}_2\end{aligned}$$

with $(1_{KG_2} \wedge j')\tilde{g}_2 \in [\sum^{p^nq-2} K, KG_2 \wedge M]$ by Prop.2.17(3). Then $(\bar{b}_3 \wedge 1_K)\zeta''_{n-1,3} = (h_0 b_{n-1})'' + (\bar{b}_3\bar{c}_2 \wedge 1_K)\tilde{g}_2 + (1_{KG_3} \wedge i')\tilde{g}_3$ and $\tilde{g}_3 \in [\sum^{p^nq+q-1} K, KG_3 \wedge M]$ is a d_1-boundary since $\mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*M, \mathrm{H}^*K) = 0$ by Prop.2.5.(1). That is, we have $(\bar{b}_3 \wedge 1_K)\zeta''_{n-1,3} = (h_0 b_{n-1})''$ modulo d_1-boundary.

Let $\zeta''_{n-1} = (\bar{a}_0\bar{a}_1\bar{a}_2 \wedge 1_K)\zeta''_{n-1,3} \in [\sum^{p^nq+q-4} K, K]$ and consider the map $\zeta''_{n-1}\beta i' i \in \pi_{p^nq+pq+2q-4}K$, where $\beta \in [\sum^{(p+1)q} K, K]$ is the known v_2-map (cf.[7] p.426) which has filtration 1 in the ASS. Since ζ''_{n-1} is represented by $(h_0 b_{n-1})'' \in \mathrm{Ext}_A^{3,p^nq+q-1}(\mathrm{H}^*K, \mathrm{H}^*K)$, then $\zeta''_{n-1}\beta i' i$ is represented by $(\beta i' i)^*(h_0 b_{n-1})'' = (\beta i' i)^*\alpha''_*(b_{n-1})' = \alpha''_*(\beta i' i)_* b_{n-1} \in \mathrm{Ext}_A^{4,p^npq+2q}(\mathrm{H}^*K, Z_p)$.

Now from [10] p.219 Theorem 3.2 and [9] p.60 Theorem 5.2 we know that the map $\alpha''\beta i' i \in \pi_{pq+2q-2}K$ is nontrivial and is represented by $(i' i)_*(g_0) \in \mathrm{Ext}_A^{2,pq+2q}(\mathrm{H}^*K, Z_p)$ up to nonzero scalar. Then

$$1 \in \mathrm{Ext}_A^{0,0}(Z_p, Z_p) \xrightarrow{(\beta i' i)_*} \mathrm{Ext}_A^{1,pq+q+1}(\mathrm{H}^*K, Z_p) \xrightarrow{(\alpha'')_*} \mathrm{Ext}_A^{2,pq+2q}(\mathrm{H}^*K, Z_p)$$

we have $\alpha''_*(\beta i' i)_*(1) = (i' i)_*(g_0) \in \mathrm{Ext}_A^{2,pq+2q}(\mathrm{H}^*K, Z_p)$ up to nonzero scalar and so $\zeta''_{n-1}\beta i' i$ is represented by $\alpha''_*(\beta i' i)_*(b_{n-1}) = (i' i)_*(g_0 b_{n-1}) \neq 0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$ by Prop.2.5. Moreover, $(i' i)_*(g_0 b_{n-1}) \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$ can not be hit by differential since $\mathrm{Ext}_A^{2,p^nq+pq+2q-1}(\mathrm{H}^*K, Z_p) = 0, \mathrm{Ext}_A^{1,p^nq+pq+2q-2}(\mathrm{H}^*K, Z_p) = 0$ by several steps of exact sequences induced by (1.2) (1.1) and using Prop.2.1.(3). Hence, $\zeta''_{n-1}\beta i' i \in \pi_{p^nq+pq+2q-4}K$ is a nontrivial map which is represented by $(i' i)_*(b_{n-1} g_0) \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$ (up to nonzero scalar) in the ASS.

Q.E.D.

PROOF of THEOREM I: Let $V(2)$ be the cofibre of $\beta\colon \Sigma^{(p+1)q}K \to K$ given by the cofibration

$$\Sigma^{(p+1)q}K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1}K,$$

from Theorem *II*, there is $\zeta''_{n-1}\beta i'i \in \pi_{p^nq+pq+2q-4}K$ which is represented by $b_{n-1}g_0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$. Let $\gamma\colon \Sigma^{(p^2+p+1)q}V(2) \to V(2)$ be the v_3-map and consider the following composition $(t = p^nq + pq + 2q - 4)$

$$\tilde{f}\colon \Sigma^t \xrightarrow{\bar{i}\zeta''_{n-1}\beta i'i} V(2) \xrightarrow{\gamma^3} \Sigma^{-3(p^2+p+1)q}V(2) \xrightarrow{jj'\bar{j}} \Sigma^{-3(p^2+p+1)q+(p+2)q+3}S$$

Since $\zeta''_{n-1}\beta i'i$ is represented by $b_{n-1}g_0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$ which is the image of $b_{n-1}g_0 \in \mathrm{Ext}_A^{4,p^nq+pq+2q}(Z_p, Z_p)$ under the homomorphism $(i'i)_*\colon \mathrm{Ext}_A^{4,p^nq+pq+2q}(Z_p, Z_p) \to \mathrm{Ext}_A^{4,p^nq+pq+2q}(\mathrm{H}^*K, Z_p)$, then the above $\tilde{f}$ is represented by

$$c = jj'\bar{j}_*(\gamma_*)^3(\bar{i}i'i)_*(b_{n-1}g_0) \in \mathrm{Ext}_A^{7,p^nq+3(p^2+p+1)q}(Z_p, Z_p)$$

The proof of the following lemma will be postponed to the last of the paper.

LEMMA 3.16: *Let $p \geqslant 7$, then $\gamma_3 = jj'\bar{j}\gamma^3\bar{i}i'i \in \pi_{3p^2q+2pq+q-3}S$ is represented (up to nonzero scalar) by the unique generator $\gamma_3 = h_{0,1,2,3}$ in $Ext_A^{3,3p^2+2pq+q}(Z_p, Z_p)$ in the ASS.*

From Lemma 3.16 and the knowledge of Yoneda products we know that the composition

$$\mathrm{Ext}_A^{0,0}(Z_pZ_p) \xrightarrow{(\bar{i}i'i)_*} \mathrm{Ext}_A^{0,0}(\mathrm{H}^*V(2), Z_p)$$
$$\xrightarrow{(\gamma_*)^3} \mathrm{Ext}_A^{3,3(p^2+p+1)q+3}(\mathrm{H}^*V(2), Z_p) \xrightarrow{(jj'\bar{j})_*} \mathrm{Ext}_A^{3,3p^2+2pq+q}(Z_p, Z_p)$$

is a multiplication (up to nonzero scalar) by $\gamma_3 = h_{0,1,2,3} \in \mathrm{Ext}_A^{3,3p^2+2pq+q}(Z_p, Z_p)$. Hence, $\tilde{f} \in \pi_*S$ is represented (up to nonzero scalar) by $c = b_{n-1}g_0\gamma_3 \neq 0 \in \mathrm{Ext}_A^{7,p^nq+3(p^2+p+1)q}(Z_p, Z_p)$ in the ASS. Moreover, from Prop.2.1 (3), $\mathrm{Ext}_A^{7-r,p^nq+3(p^2+p+1)q-r+1}(Z_p, Z_p) = 0$ for $r \geqslant 2$, then $b_{n-1}g_0\gamma_3$ can not be hit by differentials in the ASS and so the corresponding homotopy element $\tilde{f} \in \pi_*S$ is nontrivial and of order p. This finishes the proof of Theorem I. Q.E.D.

PROOF of LEMMA 3.16: From [5] theorem 2.12, $\gamma_3 = jj'\bar{j}\gamma^3\bar{i}i'i \in \pi_{3p^2q+2pq+q-3}S$ is represented by $\delta_0\delta_1\delta_2(v_3^3) \in \mathrm{Ext}_{BP_*BP}^{3,*}(BP_*, BP_*)$ in the Adams-Novikov spectral sequence, where $v_3^3 \in \mathrm{Ext}_{BP_*BP}^{0,*}(BP_*, BP_*V(2))$ and

$\delta_k\colon \mathrm{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*V(k)) \to \mathrm{Ext}_{BP_*BP}^{*+1,*}(BP_*, BP_*V(k-1))$ is the boundary homomorphism associated with the exact sequence $0 \to BP_*V(k-1) \xrightarrow{v_k} BP_*V(k-1) \to BP_*V(k) \to 0$. By a computation due to K. Shimomura, in the cobar complex $\Omega^3 BP_*$ we have

$$\begin{aligned}\delta_0\delta_1\delta_2(v_3^3) = {} & 6t_3 \otimes t_2^p \otimes t_1^{p^2} \\ & + t_1 \otimes (3t_2^p \otimes t_1^{2p^2} + t_1^p t_2^p \otimes t_1^{2p^2} + t_1^{2p} \otimes t_1^{3p^2}) \\ & + t_2 \otimes [6t_1^{p^2} t_2^p \otimes t_1^{p^2} + 3(t_2^p + t_1^{p^2+p}) \otimes t_1^{2p^2} + 2t_1^p \otimes t_1^{3p^2}] \\ & + 3t_3 \otimes t_1^p \otimes t_1^{2p^2} \\ & + (\text{other terms with } v_n).\end{aligned}$$

Consider the Thom reduction map $\Phi\colon \mathrm{Ext}_{BP_*BP}^{3,*}(BP_*, BP_*) \to \mathrm{Ext}_A^{3,*}(Z_p, Z_p)$. Since $\Phi(t_n) = \bar{\xi}_n$ and $\Phi(v_n) = 0$, where $\bar{\xi}_n$ means the conjugate of $\xi_n \in A_*$ (A_* is the dual of A), thus modulo mixed words (cf.[11]) we have

$$\Phi(\delta_0\delta_1\delta_2 v_3^3) \equiv \Psi(6t_3 \otimes t_2^p \otimes t_1^{p^2}) \equiv 6\bar{\xi}_3|\bar{\xi}_2^p|\bar{\xi}_1^{p^2} \equiv 6h_{1,2}h_{2,1}h_{3,0} \in C^{*,*}$$

This proves the lemma. Q.E.D.

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