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An approach to generalized Radon-Nikodym sets and generalized Pettis sets

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Abstract. We give a simultaneous study of A-Radon-Nikodym sets and A-Pettis sets for nonempty bounded subsets A of Banach spaces with comparable characterizations of these two classes of sets by various properties, using certain weak*-measurable functions.

1. Introduction

Throughout this paper X denotes an arbitrary real Banach space; X^*, X^{**} and $B(X)$ are its topological dual space, bidual space and closed unit ball, respectively. For weak*-compact subsets of dual Banach spaces, we have two notions of Pettis sets and Radon-Nikodym sets (RN sets in brief), which are generalizations of weak*-compact convex sets with the weak Radon-Nikodym property and the Radon-Nikodym property, respectively. The RN sets were defined and studied in Reynow [17] and the Pettis sets in Talagrand [21]. Succeedingly, Fitzpatrick [4] defined the notion of separably related sets in X^* as a generalization of RN sets. On the other hand, Bator and Lewis [2] defined the K-weak precompactness as a generalization of weak precompactness. In this paper, in order to analyze these notions in a unified manner, we give attention to a continuity property of certain maps, and redefine two notions; They are generalizations of Pettis sets and norm-fragmented sets (cf. [7]).

Definition 1. Let A be a bounded subset of X and K a weak*-compact (not necessarily convex) subset of X^*. Then

(1) K is said to be an *A-Pettis set* if every weak*-compact subset D of K has the following property $(*)$.

$(*)$ For every $x^{**} \in \bar{A}^*$ (the weak*-closure of A in X^{**}) and every $\varepsilon > 0$, there exists a weak*-open subset U such that $U \cap D \neq \phi$ and $O(x^{**} | U \cap D)$ $(= \sup\{(u^*, x^{**}) : u^* \in U \cap D\} - \inf\{(v^*, x^{**}) : v^* \in U \cap D\}$, the oscillation of x^{**} on $U \cap D) < \varepsilon$.

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(2) K is said to be an *A-RN set* if every weak*-compact subset D of K has the following property $(**)$.

$(**)$ For every $\varepsilon > 0$, there exists a weak*-open subset U such that $U \cap D \neq \phi$ and $\mathrm{diam}_A(U \cap D)$ $(= \sup\{q_A(u^* - v^*) : u^*, v^* \in U \cap D\}$, the q_A-diameter of $U \cap D) < \varepsilon$.

Here q_A is the seminorm given by $q_A(x^*) = \sup_{x \in A} |(x, x^*)|$ for every $x^* \in X^*$.

A $B(X)$-Pettis (resp. $B(X)$-RN) set is nothing but a Pettis (resp. RN) set. We have called an A-RN set an A-fragmented set in [13].

Now, in a series of our papers [9], [10], [11] and [12] (resp. [13]), we have studied the notion of K-weakly precompact set A (resp. A-fragmented set K) which is equivalent to say that K is an A-Pettis set (resp. A-Radon-Nikodym set). In order to see various properties of these sets, we used there certain K-valued weak*-measurable functions constructed in the case where A is a non-K-weakly precompact set or K is a non-A-fragmented set.

In this paper, by following the same ideas of the use of K-valued weak*-measurable functions, we wish to study A-Pettis sets K and A-RN sets K in a unified manner, seeing them in parallel and comparison. Then we obtain some new characterizations of these sets, and clarify that there are many points in which these two notions for sets are similar and yet subtly different. A similar type research has been done by Riddle and Uhl [18] in the case where $A = B(X)$ and $K = B(X^*)$. Thus the results are also concerned with Pettis sets and RN sets in dual Banach spaces, and weakly precompact sets and GSP sets in Banach spaces. So this study has an aspect of the reconsideration of results in [9], [11], [12], [13] and so on.

The paper is organized as follows. In §2, we give notation, definitions and preliminary results. In §3, we recall the construction of certain K-valued weak*-measurable functions following [9] and [11]. In §4, we present basic functions to study A-Pettis sets K and A-RN sets K from our point of view, which contain informations to recognize not only the similarity but also the subtle difference between these two notions. As its consequence, in §5, we give a study of their similarity and difference in various aspects, such as geometric, operator theoretic, measure theoretic or convex analytic one. In §6, by an appropriate choice of the sets A and K, we present various characterizations of RN sets, GSP sets, Pettis sets and weakly precompact sets as a convenient summary.

2. Definitions and preliminary results

In the following, notation and terminology, unless otherwise stated, are as in [10] and [11]. The triple (I, Λ, λ) refers to the Lebesgue measure space on

I $(=[0,1])$, Λ^+ to the sets in Λ with positive measure, L_1 to $L_1(I,\Lambda,\lambda)$ and L_∞ to $L_\infty(I,\Lambda,\lambda)$. For each $E \in \Lambda^+$, denote $\Delta(E) = \{\chi_F/\lambda(F) : F \subset E, F \in \Lambda^+\}$. For each $g \in L_\infty$ and $E \in \Lambda^+$, ess-$O(g|E)$ denotes the essential oscillation of g (as a function) on E. We always understand that I is endowed with Λ and λ. If $f : I \to X^*$ is a bounded weak*-measurable function, we obtain a bounded linear operator $S_f : X \to L_\infty$ (resp. $T_f : X \to L_1$) given by $S_f(x) = x \circ f$ (resp. $T_f(x) = x \circ f$) for every $x \in X$, where $(x \circ f)(t) = (x, f(t))$ for every $t \in I$. The dual operator of T_f is denoted by T_f^* $(: L_\infty \to X^*)$. Define a vector measure α_f (associated with such a function f): $\Lambda \to X^*$ by $\alpha_f(E) = T_f^*(\chi_E)$ for every $E \in \Lambda$. Then

$$(x, \alpha_f(E)) = \int_E (x, f(t))d\lambda(t)$$

for every $x \in X$ and $E \in \Lambda$. When a vector measure $\alpha : \Lambda \to X^*$ satisfies that $\alpha(E) = T_f^*(\chi_E)$ for every $E \in \Lambda$, we say that it has the weak*-density f. Let $\{O(n,i) : n = 0,1,\dots; i = 0,\dots,2^n - 1\}$ be a system of open intervals in I given by $O(n,i) = (i/2^n, (i+1)/2^n)$ for every (n,i).

If $g : X \to R$ is a continuous convex function, we define $Dg(x,y)$ by

$$Dg(x,y) = \lim_{t\to 0}\{g(x+ty) - g(x)\}/t \qquad \text{for } x, y \in X$$

provided that this limit exists; g is Gateaux differentiable at $x \in X$ if $Dg(x,y)$ exists for every $y \in X$. Further, we recall some notions concerning differentiability.

Definition 2 (cf. [3] and [16]). Let $g : X \to R$ be a continuous convex function and A a bounded subset of X. Then

(1) The *subdifferential* of g, denoted by ∂g, is the set valued map from X to X^* defined by:

$$\partial g(x) = \{x^* \in X^* : g(y) - g(x) \geqq (y - x, x^*) \text{ for every } y \in X\}.$$

(2) The subdifferential map $\partial g : X \to \mathscr{P}(X^*)$ (power set of X^*) is said to be *A-continuous* at $x \in X$ if for every $\varepsilon > 0$ there exists a neighborhood V of x such that $\mathrm{diam}_A(\partial g(V)) \leqq \varepsilon$. Here $\partial g(V) = \bigcup\{\partial g(y) : y \in V\}$.

(3) ([1]). g is said to be *A-differentiable* at $x \in X$ if there exists an $x^* \in X^*$ such that

$$\lim_{t\to 0+}\left\{\sup_{y\in A}|(g(x+ty) - g(x))/t - (y, x^*)|\right\} = 0.$$

Needless to say, $B(X)$-differentiability at x is equivalent to Frechet differentiability at x.

(4) g is said to be *A-uniformly Gateaux differentiable* at $x \in X$ if $Dg(x, y)$ exists uniformly in $y \in A$.

Note that $\partial g(x)$ is a nonempty weak*-compact convex subset of X^* for every $x \in X$, and g is Gateaux differentiable at $x \in X$ if and only if g is $\{y, -y\}$-differentiable at $x \in X$ for every $y \in X$. In order to analyse A-RN sets and A-Pettis sets from our view-point, let us turn our attention to the special continuous convex functions defined as follows.

DEFINITION 3. Let H be a nonempty bounded subset of X^*. Then the continuous convex function c_H associated with H is defined by

$$c_H(x) = \sup_{x^* \in H} (x, x^*)$$

for each $x \in X$, and called the *support function* of H ([16]).

The convexity and the continuity of such functions c_H are clear, and the following is well-known.

PROPOSITION 1. *Let Y be a closed linear subspace of X and K a weak*-compact subset of X^*. Let $j : Y \to X$ be the inclusion operator and j^* its dual operator. Then, for every nonempty subset H of K, $c_H : Y \to R$ satisfies that $\partial c_H(y) \subset \overline{\mathrm{co}}^*(j^*(K))$ (the weak*-closed convex hull of $j^*(K)$) for every $y \in Y$. In particular, if $Y = X$, then $\partial c_H(x) \subset \overline{\mathrm{co}}^*(K)$ for every $x \in X$.*

Let us recall a fact useful to check the A-uniform Gateaux differentiability of continuous convex functions.

PROPOSITION 2. *Let $g : X \to R$ be a continuous convex function and A a bounded subset of X. Then*

(1) *g is A-uniformly Gateaux differentiable at $x \in X$ if and only if g satisfies that*

$$\inf_{t>0}\left\{\sup_{y \in A}(g(x + ty) + g(x - ty) - 2g(x))/t\right\} = 0.$$

(2) *g is $B(X)$-uniformly Gateaux differentiable at $x \in X$ if and only if g is Frechet differentiable at $x \in X$.*

In order to see geometric properties of A-RN sets and A-Pettis sets, we recall

DEFINITION 4 (cf. [4] and [2]). (1) Let K be a bounded subset of X^*. A *weak*-open slice* of K is a set of the form:

$$S(x, c, K) = \left\{ x^* \in K : (x, x^*) > \sup_{z^* \in K} (x, z^*) - c \right\}$$

where $x \in X$ and $c > 0$.

(2) Let A be a bounded subset of X and K a bounded subset of X^*. Then the set K is said to be *A-weak*-dentable* if K has weak*-open slices of arbitrarily small q_A-diameter. And the set K is said to be *weak*-$\bar{A}^*$-dentable* if for every $\varepsilon > 0$ and $x^{**} \in \bar{A}^*$, there exists a weak*-open slice S of K such that $O(x^{**}|S) < \varepsilon$.

Note that for any bounded subset D of X^*, $\mathrm{diam}_A(D) = \sup\{O(x|D) : x \in A\}$ and $O(x^{**}|D) \leqq \mathrm{diam}_A(D)$ for every $x^{**} \in \bar{A}^*$. So, A-RN sets are A-Pettis sets. $B(X)$-weak*-dentability (resp. weak*-$B(X^{**})$-dentability) is simply called the weak*-dentability (resp. weak*-scalar-dentability). To see the relation between geometric properties and operator theoretic properties of A-Pettis sets, we recall

Definition 5 ([11]). Let K be a bounded convex subset of X^* and A a bounded subset of X^{**}. Then the set K is said to be *A-weak*-strongly regular* if for every $\varepsilon > 0$ and every nonempty convex subset D of K, there exist positive numbers $\alpha_1, \dots, \alpha_n$ whose sum is one and weak*-open slices $S_1, \dots, S_n$ of D such that

$$\sup_{x^{**} \in A} O\left(x^{**} | \sum_{i=1}^{n} \alpha_i S_i \right) \left(= \sup_{x^{**} \in A} \sum_{i=1}^{n} \alpha_i O(x^{**}|S_i) \right) < \varepsilon.$$

If $A = B(X)$, this set is simply called *weak*-strongly regular*.

Before introducing the notions of the A-Radon-Nikodym property (A-RNP in brief) and the $\bar{A}^*$-weak Radon-Nikodym property ($\bar{A}^*$-WRNP in brief), we define A-strong measurability and $\bar{A}^*$-Pettis integrability for X^*-valued functions as follows.

Definition 6 (cf. [10] and [13]). (1) Let $f : I \to X^*$ be a function and A a bounded subset of X. Then f is said to be *A-strongly measurable* if f has the following two conditions.

(a) f is weak*-measurable.

(b) For every $\varepsilon > 0$ and $E \in \Lambda^+$, there exists a set $F \in \Lambda^+$ with $F \subset E$ such that $\mathrm{diam}_A(f(F)) < \varepsilon$.

(2) Let $f : I \to X^*$ be a weak*-measurable function and A a bounded subset of X. Then f is said to have the *A-strongly measurable decomposability* if there exists an A-strongly measurable function $g : I \to X^*$ such that $f - g$ is weak*-scalarly null (that is, $(x, f(t) - g(t)) = 0$ λ-a.e. on I for every $x \in X$).

(3) Let $f: I \to X^*$ be a bounded weak*-measurable function and A a bounded subset of X. Then f is said to be $\bar{A}^*$-*Pettis integrable* if f has the following two conditions.

(a) For every $x^{**} \in \bar{A}^*$, $x^{**} \circ f \in L_1$.

(b) For each $E \in \Lambda$, it holds that

$$(x^{**}, T_f^*(\chi_E)) = \int_E (x^{**}, f(t)) d\lambda(t)$$

for every $x^{**} \in \bar{A}^*$.

(4) Let $f: I \to X^*$ be a bounded weak*-measurable function and A a bounded subset of X. Then f is said to have the $\bar{A}^*$-*Pettis decomposability* (resp. $\bar{A}^*$-*measurable decomposability*) if there exists a $\bar{A}^*$-Pettis integrable (resp. $\bar{A}^*$-measurable) function $g: I \to X^*$ such that $f - g$ is weak*-scalarly null.

Setting $A = B(X)$ in Definition 6, we have the usual notion of strong measurability and Pettis integrability for X^*-valued bounded functions. Define the A-RNP and the $\bar{A}^*$-WRNP for weak*-compact convex subsets of X^* as follows.

DEFINITION 7. Let A be a bounded subset of X and K a weak*-compact convex subset of X^*. Then

(1) K is said to have the *A-RNP* if for any vector measure $\alpha: \Lambda \to X^*$ for which $\alpha(E) \in \lambda(E) \cdot K$ for every $E \in \Lambda$, there exists an A-strongly measurable function $f: I \to K$ such that $\alpha(E) = T_f^*(\chi_E)$ for every $E \in \Lambda$ (that is, any such vector measure α has an A-strongly measurable weak*-density f valued in K).

(2) K is said to have the $\bar{A}^*$-*WRNP* if for any vector measure $\alpha: \Lambda \to X^*$ for which $\alpha(E) \in \lambda(E) \cdot K$ for every $E \in \Lambda$, there exists a $\bar{A}^*$-Pettis integrable function $f: I \to K$ such that $\alpha(E) = T_f^*(\chi_E)$ for every $E \in \Lambda$ (that is, any such vector measure α has a $\bar{A}^*$-Pettis integrable weak*-density f valued in K).

Concerning notions stated in Definitions 5, 6 and 7, we have the following result. For the proof, see [9], [10], [13] and [14].

PROPOSITION 3. (1) *Let K be a weak*-compact convex subset of X^*. Then the $B(X)$-RNP (resp. $B(X^{**})$-WRNP) for K coincides the (usual) RNP (resp. WRNP) for K.*

(2) *Let $f: I \to X^*$ be a bounded weak*-measurable function and A a bounded subset of X. Suppose that the set $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is A-weak*-dentable (resp. weak*-$\bar{A}^*$-dentable) for every $E \in \Lambda^+$. Then f has the A-strongly measurable decomposability (resp. $\bar{A}^*$-Pettis decomposability).*

(3) *Let K be a weak*-compact convex subset of X^* and A a bounded subset of X. Then, for every weak*-measurable function $f: I \to K$ and every*

$E \in \Lambda^+$, the set $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is A-weak-dentable (resp. weak*-$\bar{A}^*$-dentable) if and only if K has the A-RNP (resp. $\bar{A}^*$-WRNP).*

In order to characterize A-RN sets and A-Pettis sets in terms of operators $S_f : X \to L_\infty$, we need the following two notions concerning subsets of L_∞. These notions act similarly in each case.

Definition 8 ([20] and [6]). Let M be a subset of L_∞.

(1) The set M is said to be *equimeasurable* if for every $\varepsilon > 0$ there exists a set $E \in \Lambda$ with $\lambda(E) > 1 - \varepsilon$ such that $\{f\chi_E : f \in M\}$ is a relatively norm compact subset of L_∞.

(2) The set M is said to be a *set of small oscillation with respect to* λ if for every $\varepsilon > 0$ there exists a positive measurable partition $(= \{E_1, \dots, E_n\})$ of I such that

$$\sum_{i=1}^{n} \lambda(E_i)\ \text{ess-}O(f|E_i) < \varepsilon$$

for every $f \in M$.

Concerning bounded linear operators $S_f : X \to L_\infty$ (resp. $T_f : X \to L_1$) associated with A-strongly (resp. $\bar{A}^*$-) measurable functions f, we note the following result, part of which has appeared in [9], [11] and [13].

Proposition 4. *Let A be a bounded subset of X.*

(1) *Suppose that $f : I \to X^*$ is a bounded weak*-measurable function having the A-strongly measurable decomposability. Then $S_f(A)$ is equimeasurable in L_∞, and*

$$\inf_{n \geqq 1} \int_I q_A(f_n(t) - f_{n+1}(t))d\lambda(t) = 0.$$

(2) *Suppose that $f : I \to X^*$ is a bounded weak*-measurable function having the $\bar{A}^*$-measurable decomposability. Then $T_f(A)$ is relatively norm compact in L_1, and*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} |(x, T_f^*(r_n))| \right\} = 0$$

(*Here r_n denotes the n-th Rademacher function on I*).

Proof. (1) For the proof of equimeasurability of the set $S_f(A)$, refer to that of Proposition 3 in [13].

In order to prove the latter part, we first show that

$$\inf_{n \geqq 1} \int_I q_A(f_n(t) - f_{n+1}(t))d\lambda(t) = 0$$

for the special case that $S_f(A)$ is relatively norm compact in L_∞. The proof is easily given as follows by a standard and routine argument. Let $\varepsilon > 0$. Since $S_f(A)$ is relatively norm compact in L_∞, there exists $x_1, \dots, x_p \in A$ such that

$$S_f(A) \subset \bigcup_{i=1}^{p} \{g \in L_\infty : \|g - x_i \circ f\| < \varepsilon/3\}.$$

Denoting the usual conditional expectation operator with respect to Λ_n by E_n $(= E(\cdot|\Lambda_n))$, we have that $\|x \circ f_n\| = \|E_n(x \circ f)\| \leqq \|x \circ f\|$ for every $x \in X$ and every $n \geqq 1$ and $\|E_n(x \circ f) - x \circ f\|_1 \to 0$ $(n \to \infty)$ for each $x \in X$ by the well-known martingale theorem in L_1. So, there exists a natural number m such that if $n > m$, then

$$\int_I \left\{ \sup_{1 \leqq i \leqq p} |(x_i, f_n(t) - f_{n+1}(t))| \right\} d\lambda(t) < \varepsilon/3.$$

Let $x \in A$ and let x_j be such that $\|x \circ f - x_j \circ f\| < \varepsilon/3$. Then, if $n > m$, we have that for every $x \in A$ and every $t \in I$,

$$\begin{aligned}
&|(x, f_n(t) - f_{n+1}(t))| \\
&\quad \leqq |(x - x_j, f_n(t) - f_{n+1}(t))| + \sup_{1 \leqq i \leqq p} |(x_i, f_n(t) - f_{n+1}(t))| \\
&\quad \leqq \|E_n[(x - x_j) \circ f]\| + \|E_{n+1}[(x - x_j) \circ f]\| + \sup_{1 \leqq i \leqq p} |(x_i, f_n(t) - f_{n+1}(t))| \\
&\quad < 2\varepsilon/3 + \sup_{1 \leqq i \leqq p} |(x_i, f_n(t) - f_{n+1}(t))|.
\end{aligned}$$

Thus we get that $\inf_{n \geqq 1} \int_I q_A(f_n(t) - f_{n+1}(t))d\lambda(t) = 0$.

Next let us consider the general case that $S_f(A)$ is equimeasurable in L_∞. Let $\varepsilon > 0$. Then there exists $E \in \Lambda$ such that $\lambda(I \backslash E) < \varepsilon$ and $\{x \circ f\chi_E : x \in A\}$ is relatively norm compact in L_∞. Let $g = f\chi_E$. Then, by the special case above, we have that

$$\inf_{n \geqq 1} \int_I q_A(g_n(t) - g_{n+1}(t))d\lambda(t) = 0.$$

Then the routine calculation easily deduces that

$$\inf_{n \geqq 1} \int_I q_A(f_n(t) - f_{n+1}(t))d\lambda(t) = 0.$$

Thus the proof of the statement (1) is completed.

(2) The former part of the statement (2) follows immediately from a deep

result due to Fremlin (Theorem 2F in [5]) and the latter part has been shown in the proof of Theorem in [9].

Consequently, the proof of Proposition 4 is completed.

Finally, for a given nonempty bounded subset A of X, we define slightly new types of tree structures in X^* associated with A as follows.

Definition 9. A system $\{x^*(n,i) : n = 0,1,\dots; i = 0,\dots,2^n - 1\}$ in X^* is called a *tree* if $x^*(n,i) = \{x^*(n+1,2i) + x^*(n+1,2i+1)\}/2$ for $n = 0,1,\dots$ and $i = 0,\dots,2^n - 1$.

Let $\delta > 0$. Then a tree $\{x^*(n,i) : n = 0,1,\dots; i = 0,\dots,2^n - 1\}$ in X^* is called an *A-δ-tree* if $q_A(x^*(n+1,2i) - x^*(n+1,2i+1)) > 2\delta$ for $n = 0,1,\dots$ and $i = 0,\dots,2^n - 1$. A tree $\{x^*(n,i) : n = 0,1,\dots; i = 0,\dots,2^n - 1\}$ is called an *A-separated δ-tree* if there exists a sequence $\{x_n\}_{n \geqq 1}$ in A such that for $n = 0,1,\dots$ and $i = 0,\dots,2^n - 1$

$$(x_{n+1}, x^*(n+1,2i) - x^*(n+1,2i+1)) > 2\delta.$$

In this case, we say that the tree *is separated by* $\{x_n\}_{n \geqq 1}$. A $B(X)$-δ-tree (resp. $B(X)$-separated δ-tree) is simply called a *δ-tree* (resp. *separated δ-tree*).

Note that a tree $\{x^*(n,i) : n = 0,1,\dots; i = 0,\dots,2^n - 1\}$ is an A-δ-tree if and only if there exists a (double) sequence $\{x(n,i) : n = 0,1,\dots; i = 0,\dots, 2^n - 1\}$ in A such that for $n = 0,1,\dots$ and $i = 0,\dots,2^n - 1$,

$$(x(n,i), x^*(n+1,2i) - x^*(n+1,2i+1)) > 2\delta.$$

Further we note that for every bounded weak*-measurable function $f : I \to X^*$, we get a system $\{2^n \alpha_f(I(n,i)) : n = 0,1,\dots; i = 0,\dots,2^n - 1\}$ which is a tree in X^*. In the following, this is called a tree associated with f. Clearly, all the elements of this tree are contained in $T_f^*(\Delta(I))$.

3. A first step for constructing certain weak*-measurable functions

For the sake of necessity and importance, we must recall the construction of certain weak*-measurable functions according to [9] (or [11]).

Let D be a weak*-compact subset of X^*. Suppose that there exists a system $\{V(n,i) : n = 0,1,\dots; i = 0,\dots,2^n - 1\}$ of nonempty weak*-closed subsets of D such that $V(n+1,2i) \cup V(n+1,2i+1) \subset V(n,i)$ and $V(n+1,2i) \cap V(n+1,2i+1) = \phi$ for $n = 0,1,\dots$ and $i = 0,\dots,2^n - 1$. Then, if we put $A_n = \bigcup\{V(n,2i+1) : i = 0,\dots,2^{n-1} - 1\}$ and $B_n = \bigcup\{V(n,2i) : i = 0,\dots, 2^{n-1} - 1\}$ for every $n \geqq 1$, $(A_n, B_n)_{n \geqq 1}$ is an independent sequence (cf. [19]) of pairs of weak*-closed subsets of D. Then $\Gamma = \bigcap_{n \geqq 1}(A_n \cup B_n)$ is a nonempty weak*-compact subset of D, since $(A_n, B_n)_{n \geqq 1}$ is independent. Now, define $\psi : \Gamma \to \mathscr{P}(N)$ (Cantor space, with its usual compact metric topology) by $\psi(x^*) = \{j : A_j \ni x^*\} \in \mathscr{P}(N)$. Then ψ is a continuous surjection and so we have a Radon probability measure γ on Γ such that $\psi(\gamma) = \nu$ (the normalized Haar measure if we identify $\mathscr{P}(N)$ with $\{0,1\}^N$) and $\{u \circ \psi : u \in L_1(\mathscr{P}(N), \Sigma_\nu, \nu)\} = L_1(\Gamma, \Sigma_\gamma, \gamma)$, where Σ_ν (resp. Σ_γ) is the family of all ν (resp. γ)-measurable subsets of $\mathscr{P}(N)$ (resp. Γ). Consider a function $\tau : \mathscr{P}(N) \to I$ defined by $\tau(J) = \sum_{j \in J} 1/2^j$ for every $J \in \mathscr{P}(N)$. Then τ is a continuous surjection such that $\tau(\nu) = \lambda$ and $\{v \circ \tau : v \in L_1\} = L_1(\mathscr{P}(N), \Sigma_\nu, \nu)$. Then, making use of the lifting theory, we have a weak*-measurable function $k : I \to \Gamma(\subset D)$ such that

(a) $\rho(f \circ k)(t) = f(k(t))$ for every $f \in C(\Gamma)$ and every $t \in I$,

(b) $\displaystyle\int_E f(k(t))\,d\lambda(t) = \int_{\psi^{-1}(\tau^{-1}(E))} f(x^*)\,d\gamma(x^*)$

for every $E \in \Lambda$ and every $f \in C(\Gamma)$. Here ρ is a lifting of L_∞. Further $\tau(\psi(\gamma)) = \lambda$, $\bigcup\{\psi^{-1}(\tau^{-1}(O(n,2i))) : 0 \leqq i \leqq 2^{n-1}-1\} \equiv \Gamma \cap B_n$, $\bigcup\{\psi^{-1}(\tau^{-1}(O(n,2i+1))) : 0 \leqq i \leqq 2^{n-1}-1\} \equiv \Gamma \cap A_n$ (with respect to γ) for $n = 1,2,\ldots,$ and it also holds that $\psi^{-1}(\tau^{-1}(O(n,2i))) \subset V(n,2i)$ and $\psi^{-1}(\tau^{-1}(O(n,2i+1))) \subset V(n,2i+1)$ for $n = 1,2,\ldots$ and $i = 0,\ldots,2^{n-1}-1$. Hence we easily get that this function $k : I \to D$ satisfies the followings:

$$\begin{aligned}\int_{O(n,2i)} f(k(t))\,d\lambda(t) &= \int_{\psi^{-1}(\tau^{-1}(O(n,2i)))} f(x^*)\,d\gamma(x^*)\\ &= \int_{\Gamma \cap V(n,2i)} f(x^*)\,d\gamma(x^*)\end{aligned}$$

and

$$\begin{aligned}\int_{O(n,2i+1)} f(k(t))\,d\lambda(t) &= \int_{\psi^{-1}(\tau^{-1}(O(n,2i+1)))} f(x^*)\,d\gamma(x^*)\\ &= \int_{\Gamma \cap V(n,2i+1)} f(x^*)\,d\gamma(x^*)\end{aligned}$$

for $f \in C(\Gamma)$, $n = 1,2,\ldots$ and $i = 0,\ldots,2^{n-1}-1$. This function k plays a very important role in the development of our argument in this paper, especially, in §4.

4. Basic functions associated with non-A-RN sets or non-A-Pettis sets

The main result of this section is Proposition 5 which makes it possible to analyze A-RN sets and A-Pettis sets simultaneously. As far as we know, we have never seen Proposition 5 in the literature. The main statements are (Q) of (i) and (S) of (ii). As a result, we are able to see their similarity and subtle difference in many sides. Comparing the statements (R_0) and (R_1) in (ii), (R_1) has more information than (R_0). But, (R_0) usually provides us with imformation enough to analyze the property of A-Pettis sets.

Proposition 5. *Let D be a weak*-compact subset of X^* and A a bounded subset of X.*

(i) *Suppose that there exists an $\varepsilon > 0$ such that $\operatorname{diam}_A(U \cap D) > \varepsilon$ whenever U is a weak*-open subset with $U \cap D \neq \phi$ (that is, D does not have the property $(**)$ in Definition 1). Then the following statements hold.*

(P) *There exist a system $\{x(n,i) : n = 0, 1, \dots; i = 0, \dots, 2^n - 1\}$ in A and a system $\{V(n,i) : n = 0, 1, \dots; i = 0, \dots, 2^n - 1\}$ of nonempty weak*-closed subsets of D such that*

(1) $V(n+1, 2i) \cup V(n+1, 2i+1) \subset V(n,i)$,

(2) $x^* \in V(n+1, 2i)$ *and* $y^* \in V(n+1, 2i+1)$ *imply* $(x(n,i), x^* - y^*) \geqq \varepsilon$ *for* $n = 0, 1, \dots$ *and* $i = 0, \dots, 2^n - 1$.

(Q) *In view of* (P), *we have a weak*-measurable function $g : I \to D$ satisfying the following properties*:

(1) *The set $\overline{\mathrm{co}}^*(T_g^*(\Delta(E)))$ is not A-weak*-dentable for every $E \in \Lambda^+$.*

(2) *The tree associated with g is an A-δ-tree for some postive number δ.*

(3) $\inf_{n \geqq 1} \int_I q_A(g_n(t) - g_{n+1}(t))\,d\lambda(t) > 0$.

(4) *The set $S_g(A)$ is not equimeasurable in L_∞.*

(5) *The function g does not have the A-strongly measurable decomposability.*

(6) *The vector measure α_g has no A-strongly measurable weak*-density.*

(7) *There exists a sequence $\{y_n\}_{n \geqq 1}$ in A such that ∂c_G is nowhere Ψ-continuous in Y, where $G = g(I)$, $\Psi = \{y_n : n \geqq 1\}$ and Y denotes the closed linear span of Ψ.*

(8) *There exists a sequence $\{y_n\}_{n \geqq 1}$ in A such that c_G is nowhere Ψ-uniformly Gateaux differentiable in Y.*

(9) *The function c_G is nowhere A-differentiable in X if $A = -A$.*

(ii) *Suppose that there exist an element $a^{**} \in \bar{A}^*$ and an $\varepsilon > 0$ such that $O(a^{**} | U \cap D) > \varepsilon$ whenever U is a weak*-open subset with $U \cap D \neq \phi$ (that is, D does not have the property $(*)$ in Definition 1). Then the following statements hold.*

(R_0) *There exist a sequence $\{x_n\}_{n \geqq 1}$ in A and a system $\{W(n,i) : n = 0, 1, \dots; i = 0, \dots, 2^n - 1\}$ of nonempty weak*-closed subsets of D such that*

(1) $W(n+1, 2i) \cup W(n+1, 2i+1) \subset W(n,i)$,

(2) $x^* \in W(n+1, 2i)$ *and* $y^* \in W(n+1, 2i+1)$ *imply* $(x_{n+1}, x^* - y^*) \geqq \varepsilon$ *for* $n = 0, 1, \dots$ *and* $i = 0, \dots, 2^n - 1$.

(R_1) (A stronger form of (R_0)) *There exist a weak*-compact subset L of D, a real number r and a sequence $\{a_n\}_{n\geqq 1}$ in A such that putting $G_n = \{x^* \in L : (a_n, x^*) \leqq r\}$ and $H_n = \{x^* \in L : (a_n, x^*) \geqq r + \varepsilon\}$, then $(G_n, H_n)_{n\geqq 1}$ is an independent sequence of pairs of weak*-closed subsets of L.*

(S) *In view of (R_0), we have a weak*-measurable function $h : I \to D$ satisfying the following properties*:

(1) *The set $\overline{\mathrm{co}}^*(T_h^*(\Delta(E)))$ is not A-weak*-strongly regular for every $E \in \Lambda^+$.*

(2) *The tree associated with h is an A-separated δ-tree (separated by $\{x_n\}_{n\geqq 1}$) for some positive number δ.*

(3) $\left(\inf_{n\geqq 1}\left\{\sup_{x\in A}\|x\circ h_n - x\circ h_{n+1}\|_1\right\} \geqq\right) \inf_{n\geqq 1}\left\{\sup_{x\in A}|(x, T_h^*(r_n))|\right\} > 0.$

(4) *The set $S_h(A)$ is not a set of small oscillation with respect to λ.*

(5) *The function h does not have the $\bar{A}^*$-Pettis decomposability.*

(6) *The vector measure α_h has no $\bar{A}^*$-Pettis integrable weak*-density.*

(7) *For every subsequence $\{x_{n(k)}\}_{k\geqq 1}$ of $\{x_n\}_{n\geqq 1}$, ∂c_H is nowhere Φ-continuous in Z, where $H = h(I)$, $\Phi = \{x_{n(k)} : k \geqq 1\}$ and Z denotes the closed linear span of $\{x_n : n \geqq 1\}$.*

(8) *For every subsequence $\{x_{n(k)}\}_{k\geqq 1}$ of $\{x_n\}_{n\geqq 1}$, c_H is nowhere Φ-uniformly Gateaux differentiable in Z.*

PROOF. (I) For the proof of (P) of (i), refer to that of Proposition 4 in [13].

(II) Let us prove the statement (R_0) of (ii). This also can be proved by an argument analogous to (I). In virtue of the assumption, there exist an element $a^{**} \in \bar{A}^*$ and a positive number ε such that $O(a^{**}|U \cap D) > \varepsilon$ whenever U is a nonempty weak*-open subset with $U \cap D \neq \phi$. Let $U(0,0) = X$. Suppose that for some positive integer $k, \{U(n,i) : n = 0, 1, \ldots, k; i = 0, \ldots, 2^k - 1\}$ and $\{x_n\}_{1\leqq n\leqq k}$ have already been defined so that properties (a), (b) and (c) hold.

(a) $U(n,i) \cap D \neq \phi$ for $n = 0, 1, \ldots, k$ and $i = 0, \ldots, 2^k - 1$,

(b) $(U(n+1, 2i) \cap D) \cup (U(n+1, 2i+1) \cap D) \subset U(n,i) \cap D$ for $n = 0, 1, \ldots, k-1$ and $i = 0, \ldots, 2^{k-1} - 1$,

(c) $x^* \in U(n+1, 2i) \cap D$ and $y^* \in U(n+1, 2i+1) \cap D$ imply $(x_{n+1}, x^* - y^*) \geqq \varepsilon$ for $n = 0, 1, \ldots k-1$ and $i = 0, \ldots, 2^{k-1} - 1$.

Then, by assumption, we have $O(a^{**}|U(k,i) \cap D) > \varepsilon$ for $i = 0, \ldots, 2^k - 1$, and hence, for every such i there exist elements $x^*(k+1, 2i)$ and $x^*(k+1, 2i+1)$ of $U(k,i) \cap D$ such that $(x^*(k+1, 2i) - x^*(k+1, 2i+1), a^{**}) > \varepsilon$. Since A is weak*-dense in $\bar{A}^*$, we can choose an element $x_{k+1} \in A$ such that for every i with $0 \leqq i \leqq 2^k - 1$, $(x_{k+1}, x^*(k+1, 2i) - x^*(k+1, 2i+1)) > \varepsilon$. Take a positive number δ such that $(x_{k+1}, x^*(k+1, 2i) - x^*(k+1, 2i+1)) > \varepsilon + \delta$ for every i with $0 \leqq i \leqq 2^k - 1$, and let $U(k+1, 2i) = \{z^* \in U(k, i) : (x_{k+1}, z^*) > (x_{k+1}, x^*(k+1, 2i)) - \delta/2\}$ and $U(k+1, 2i+1) = \{z^* \in U(k, i) : (x_{k+1}, z^*) < (x_{k+1}, x^*(k+1, 2i+1)) + \delta/2\}$ for every i with $0 \leqq i \leqq 2^k - 1$. Then they are nonempty weak*-open subsets with $U(k+1, i) \cap D \neq \phi$ for every i with $0 \leqq i \leqq 2^{k+1} - 1$. Furthermore, we easily get that $x^* \in U(k+1, 2i) \cap D$ and $y^* \in U(k+1, 2i+1) \cap D$ imply $(x_{k+1}, x^* - y^*) \geqq \varepsilon$ for every i with $0 \leqq i \leqq 2^k - 1$. Hence, letting $W(n, i) = w^* - \mathrm{cl}(U(n, i) \cap D)$, we have desired systems $\{x_n\}_{n \geqq 1}$ and $\{W(n, i) : n = 0, 1, \dots ; i = 0, \dots, 2^n - 1\}$.

Concerning the statement (R_1) of (ii), a more strict argument (that is, Odell and Rosenthal's one in Lemmas 2 and 3 of [15]) yields this assertion.

(III) (Construction of functions) In order to prove the statement (Q) of (i) (resp. (S) of (ii)), take $\Gamma_1 = \bigcap_{n \geqq 1} (A_n \cup B_n)$ (resp. $\Gamma_2 = \bigcap_{n \geqq 1} (C_n \cup D_n)$), where $A_n = \bigcup \{V(n, 2i+1) : i = 0, \dots, 2^{n-1} - 1\}$ (resp. $C_n = \bigcup \{W(n, 2i+1) : i = 0, \dots, 2^{n-1} - 1\}$) and $B_n = \bigcup \{V(n, 2i) : i = 0, \dots, 2^{n-1} - 1\}$ (resp. $D_n = \bigcup \{W(n, 2i) : i = 0, \dots, 2^{n-1} - 1\}$). Then, by the result in §3, we have a weak*-measurable function g (resp. h): $I \to D$ such that

(a) $\rho(f \circ g)(t) = f(g(t))$ (resp. $\rho(f \circ h)(t) = f(h(t))$) for every $f \in C(\Gamma_1)$ (resp. $C(\Gamma_2)$) and every $t \in I$,

(b) $\int_E f(g(t)) d\lambda(t) = \int_{\psi_1^{-1}(\tau^{-1}(E))} f(x^*) d\gamma_1(x^*)$ (resp. $\int_E f(h(t)) d\lambda(t) = \int_{\psi_2^{-1}(\tau^{-1}(E))} f(x^*) d\gamma_2(x^*)$)
for every $E \in \Lambda$ and every $f \in C(\Gamma_1)$ (resp. $C(\Gamma_2)$). Here ψ_1 (resp. ψ_2) is the function defined by $\psi_1(x^*)$ (resp. $\psi_2(x^*)) = \{j : A_j \ni x^*\}$ (resp. $\{j : C_j \ni x^*\}) \in \mathscr{P}(N)$ for each $x^* \in \Gamma_1$ (resp. Γ_2) and γ_1 (resp. γ_2) is the Radon probability measure on Γ_1 (resp. Γ_2) such that $\psi_1(\gamma_1)$ (resp. $\psi_2(\gamma_2)) = \nu$ as stated in §3.

(IV) We intend to show that this function g (resp. h) has all properties $(1) \sim (9)$ in (Q) of (i) (resp. $(1) \sim (8)$ in (S) of (ii)). To this end, we note the following fact used repeatedly to show such properties of g and h.

Lemma 1 (Lemma 2 in [8]). *Let $E_1, \dots, E_m$ be arbitrary members of Λ^+. Then there exist a natural number p and a finite collection $\{i_1, \dots, i_m\}$ of non-negative integers such that*

(1) $0 \leqq 2 \cdot i_1, \dots, 2 \cdot i_m < 2^p - 1$,

(2) *Both $E_k \cap O(p, 2 \cdot i_k)$ and $E_k \cap O(p, 2 \cdot i_k + 1)$ are in Λ^+ for $k = 1, \dots, m$.*

In the following, let $a(n, i)$ (resp. $c(n, i)) = \inf\{(x(n, i), x^*) : x^* \in V(n+1, 2i)\}$ (resp. $\inf\{(x_{n+1}, x^*) : x^* \in W(n+1, 2i)\}$) and $b(n, i)$ (resp. $d(n, i)$) $= \sup\{(x(n, i), x^*) : x^* \in V(n+1, 2i+1)\}$ (resp. $\sup\{(x_{n+1}, x^*) : x^* \in W(n+1, 2i+1)\}$) for every (n, i). Then it holds that $a(n, i) - b(n, i)$ (resp. $c(n, i) - d(n, i)) \geqq \varepsilon$ for all (n, i).

(1) In order to prove the property (1) of g, take $E \in \Lambda^+$, and let $B = \rho(E)$. Consider the set $M_g = \overline{\mathrm{co}}^*(T_g^*(\Delta(B)))$ $(= \overline{\mathrm{co}}^*(T_g^*(\Delta(E))))$, and take any weak*-open slice $S(x, c, M_g)$. Then we have by the same argument as in the proof of the implication (c) $\Rightarrow$ (d) of Theorem in [9] that

$$S(x, c, M_g) = \left\{x^* \in M_g : (x, x^*) > \sup_{t \in B}(x, g(t)) - c\right\}.$$

Let $B_0 = \left\{s \in I : (x, g(s)) > \sup_{t \in B}(x, g(t)) - c\right\}$. Then $B_0 \in \Lambda^+$, and $g(B_0) \subset M_g$, which is easily proved by the separation theorem. Hence, in virtue of Lemma 1, there exist a natural number p and a non-negative integer k such that $0 \leqq 2k < 2^p - 1$ and both $B_0 \cap O(p, 2k)$ and $B_0 \cap O(p, 2k+1)$ are in Λ^+. Let $\{x(n, i) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ be the system in A obtained in (P) of (i). Then it follows from the remark after Definition 4 that $\mathrm{diam}_A(S(x, c, M_g)) \geqq O(x(p-1, k) \mid S(x, c, M_g))$. Further, setting $F = B_0 \cap O(p, 2k)$ and $G = B_0 \cap O(p, 2k+1)$, we easily know that both $T_g^*(\chi_F)/\lambda(F)$ and $T_g^*(\chi_G)/\lambda(G)$ are in $S(x, c, M_g)$. Hence we have that

$$\begin{aligned}
\mathrm{diam}_A(S(x, c, M_g)) &\geqq O(x(p-1, k) \mid S(x, c, M_g)) \\
&\geqq (x(p-1, k), T_g^*(\chi_F)/\lambda(F)) - (x(p-1, k), T_g^*(\chi_G)/\lambda(G)) \\
&= \left\{\int_F (x(p-1, k), g(t)) d\lambda(t)\right\}/\lambda(F) - \left\{\int_G (x(p-1, k), g(t)) d\lambda(t)\right\}/\lambda(G) \\
&= \left\{\int_{\psi_1^{-1}(\tau^{-1}(F))} (x(p-1, k), x^*) d\gamma_1(x^*)\right\}/\lambda(F) \\
&\quad - \left\{\int_{\psi_1^{-1}(\tau^{-1}(G))} (x(p-1, k), x^*) d\gamma_1(x^*)\right\}/\lambda(G) \\
&\geqq a(p-1, k) - b(p-1, k) \geqq \varepsilon.
\end{aligned}$$

by virtue of the fact stated in §3. Hence the set $\overline{\mathrm{co}}^*(T_g^*(\Delta(E))$ is not A-weak*-dentable.

In order to prove the property (1) of h, take $E \in \Lambda^+$, and $B = \rho(E)$. Consider the set $M_h = \overline{\mathrm{co}}^*(T_h^*(\Delta(B)))$ $(= \overline{\mathrm{co}}^*(T_h^*(\Delta(E))))$, and take any positive numbers $\alpha_1, \ldots, \alpha_p$ whose sum is one and any weak*-open slices $S_1, \ldots, S_p$ of M_h. Let $S_n = S(z_n, c_n, M_h)$ for each n with $1 \leqq n \leqq p$. Then, by the same argument as above, we have a finite collection $\{B_1, \ldots, B_p\}$ of Λ^+ such that $h(B_n) \subset M_h$ for each n with $1 \leqq n \leqq p$. In virtue of Lemma 1, there exist a natural number q and a finite collection $\{i_1, \ldots, i_p\}$ of non-negative integers such that $0 \leqq 2 \cdot i_n < 2^q - 1$, $B_n \cap O(q, 2 \cdot i_n)$ $(= F_n) \in \Lambda^+$

and $B_n \cap O(q, 2 \cdot i_n + 1)$ $(= G_n) \in \Lambda^+$ for $n = 1, \ldots, p$. Then we know that for every n with $1 \leqq n \leqq p$, both $T_h^*(\chi_{F_n})/\lambda(F_n)$ and $T_h^*(\chi_{G_n})/\lambda(G_n)$ are in $S(z_n, c_n, M_h)$ $(= S_n)$. Hence we have that

$$\begin{aligned}\sup_{x \in A} O\left(x \middle| \sum_{n=1}^{p} \alpha_n S_n\right) &\geqq O\left(x_q \middle| \sum_{n=1}^{p} \alpha_n S_n\right)\\ &= \sum_{n=1}^{p} \alpha_n \cdot O(x_q | S_n) \geqq \sum_{n=1}^{p} \alpha_n \cdot (x_q, T_h^*(\chi_{F_n})/\lambda(F_n) - T_h^*(\chi_{G_n})/\lambda(G_n)).\end{aligned}$$

Further, it holds that for every n with $1 \leqq n \leqq p$

$$\begin{aligned}&(x_q, T_h^*(\chi_{F_n})/\lambda(F_n) - T_h^*(\chi_{G_n})/\lambda(G_n))\\ &\quad = \left\{\int_{F_n} (x_q, h(t)) d\lambda(t)\right\}/\lambda(F_n) - \left\{\int_{G_n} (x_q, h(t)) d\lambda(t)\right\}/\lambda(G_n)\\ &\quad = \left\{\int_{\psi_2^{-1}(\tau^{-1}(F_n))} (x_q, x^*) d\gamma_2(x^*)\right\}/\lambda(F_n)\\ &\qquad - \left\{\int_{\psi_2^{-1}(\tau^{-1}(G_n))} (x_q, x^*) d\gamma_2(x^*)\right\}/\lambda(G_n)\\ &\quad \geqq c(q-1, i_n) - d(q-1, i_n) \geqq \varepsilon.\end{aligned}$$

Thus we get that

$$\sup_{x \in A} O\left(x \middle| \sum_{n=1}^{p} \alpha_n S_n\right) \geqq \sum_{n=1}^{p} \alpha_n \cdot \varepsilon = \varepsilon.$$

Hence we have that $\overline{\mathrm{co}}^*(T_g^*(\Delta(E)))$ is not A-weak*-strongly regular.

(2) Since it holds that

$$\begin{aligned}&\int_I q_A(g_n(t) - g_{n+1}(t)) d\lambda(t) \qquad (\text{resp. } (x, T_h^*(r_n)))\\ &\qquad = \sum_{i=0}^{2^{n-1}-1} q_A(\alpha_g(O(n, 2i)) - \alpha_g(O(n, 2i+1)))\\ &\qquad\qquad \left(\text{resp. } \sum_{i=0}^{2^{n-1}-1} (x, \alpha_h(O(n, 2i)) - \alpha_h(O(n, 2i+1)))\right),\end{aligned}$$

(which can be easily shown by a straightfoward calculation), the same argument as in Remark 1 of [13] easily deduces the properties (2) and (3) of g and h.

(3) In order to prove the property (4) of g, assume that $S_g(A)$ is equimeasurable. Then, by virtue of the statement (1) in Proposition 4, this contradicts the property (3) of g. Thus the property (4) of g holds. Next, in order to prove the property (5) (resp. (6)), assume that g has the A-strongly measurable decomposability (resp. α_g has an A-strongly measurable weak*-density). Then we know in each case that $S_g(A)$ is equimeasurable, which is a contradiction. Thus the properties (5) and (6) of g hold.

In order to prove the property (4) of h, assume that $S_h(A)$ is a set of small oscillation with respect to λ. Then we easily get the relative norm compactness of $T_h(A)$ in L_1 by making use of the well-known fact: a bounded subset H of L_1 is relatively norm compact if and only if for every $\varepsilon > 0$ there is a finite measurable partition Π of I such that $\|f - E(f|\Sigma)\|_1 < \varepsilon$ for each $f \in H$ (Here Σ is the σ-algebra generated by Π). But, this contradicts the property (3) of h, in virtue of the statement (2) in Proposition 4. Further, properties (5) and (6) of h follow immediately by an argument similar to the case of g.

(4) Let us prove the properties (7), (8) and (9) of g. Let $\{y_n\}_{n\geqq 1}$ be a sequence defined by $y_n = x(m,i)$ for $n = 2^m + i$ with $m = 0, 1, \dots$ and $i = 0, \dots, 2^m - 1$. Let $\partial c_G : Y \to \mathscr{P}(Y^*)$ and take any point y of Y. Consider a family of weak*-open slices of M_g $(= \overline{\mathrm{co}}^*(j^*(T_g^*(\Delta(I)))))$: $\{S(y, \varepsilon/3n, M_g) : n \geqq 1\}$. Then we have that for every n

$$
\begin{aligned}
S(y,\varepsilon/3n, M_g) &= \left\{ y^* \in M_g : (y, y^*) > \sup_{z^* \in M_g} (y, z^*) - \varepsilon/3n \right\} \\
&= \left\{ y^* \in M_g : (y, y^*) > \operatorname*{ess\text{-}sup}_{t \in I} (j(y), g(t)) - \varepsilon/3n \right\} \\
&= \{ y^* \in M_g : (y, y^*) > c_G(y) - \varepsilon/3n \}.
\end{aligned}
$$

So, letting $E_n = \{t \in I : (j(y), g(t)) > c_G(y) - \varepsilon/3n\}$, we know that $E_n \in \Lambda^+$ and $j^*(g(E_n)) \subset S(y, \varepsilon/3n, M_g)$ for every n. Hence, in virtue of Lemma 1 (and its proof in [8]), there exist a strictly increasing sequence $\{p_n\}_{n\geqq 1}$ of natural numbers and a sequence $\{i_n\}_{n\geqq 1}$ of non-negative integers such that $0 \leqq 2 \cdot i_n < 2^{p_n} - 1$, $E_n \cap O(p_n, 2 \cdot i_n) \in \Lambda^+$ and $E_n \cap O(p_n, 2 \cdot i_n + 1) \in \Lambda^+$ for every $n \geqq 1$. Let $F_n = E_n \cap O(p_n, 2 \cdot i_n)$ and $G_n = E_n \cap O(p_n, 2 \cdot i_n + 1)$, and define $u_n^* = j^*(T_g^*(\chi_{F_n}/\lambda(F_n)))$ and $v_n^* = j^*(T_g^*(\chi_{G_n}/\lambda(G_n)))$ for every $n \geqq 1$. Then we have by the same argument as in [13] that for every n

(a) $(y, u_n^*) > c_G(y) - \varepsilon/3n$ and $(y, v_n^*) > c_G(y) - \varepsilon/3n$,

(b) $(z_n, u_n^* - v_n^*) \geqq \varepsilon$ (Here, $z_n = x(p_n - 1, i_n)$, and so, $\{z_n\}_{n\geqq 1}$ is a subsequence of $\{y_n\}_{n\geqq 1}$),

(c) $c_G(y + z_n/n) \geqq (y + z_n/n, u_n^*)$ and $c_G(y - z_n/n) \geqq (y - z_n/n, v_n^*)$.

Now, making use of these properties, let us show that ∂c_G is not Ψ-continuous at y. To this end, take $a_n^* \in \partial c_G(y + z_n/n)$ and $b_n^* \in \partial c_G(y - z_n/n)$ for every $n \geqq 1$. Then, by the property of subdifferential, (a), (b) and (c) we get that for every n

$$\begin{aligned}(z_n/n, a_n^* - b_n^*) &\geqq c_G(y + z_n/n) + c_G(y - z_n/n) - 2 \cdot c_G(y) \\ &> (y + z_n/n, u_n^*) + (y - z_n/n, v_n^*) - \{(y, u_n^* + v_n^*) + 2\varepsilon/3n\} \\ &= (z_n, u_n^* - v_n^*)/n - 2\varepsilon/3n \geqq \varepsilon/3n \text{ (that is, } (z_n, a_n^* - b_n^*) \geqq \varepsilon/3).\end{aligned}$$

whence, ∂c_G is not Ψ-continuous at y. Further, since it follows from this inequality that $\{c_G(y + z_n/n) + c_G(y - z_n/n) - 2 \cdot c_G(y)\}/(1/n) > \varepsilon/3$ for every n, c_G is not Ψ-uniformly Gateaux differentiable at y by virtue of the statement (1) of Proposition 2. Finally, to show the property (9) of g, assume that there exists a point $x \in X$ such that c_G is A-differentiable at x, and let x^* be its A-differential. Well, since $A = -A$, by a slight modification of the argument above, we have sequences $\{w_n\}_{n \geqq 1}$ in A and $\{u_n^*\}_{n \geqq 1}$ in X^* such that for every $n \geqq 1$, $(x, u_n^*) > c_G(x) - \varepsilon/3n$, $(w_n, u_n^* - x^*) \geqq \varepsilon/2$ and $c_G(x + w_n/n) \geqq (x + w_n/n, u_n^*)$. Then we have that for every $n \geqq 1$,

$$\begin{aligned}&c_G(x + w_n/n) - c_G(x) - (w_n/n, x^*) \\ &\quad \geqq (x + w_n/n, u_n^*) - \{(x, u_n^*) + \varepsilon/3n\} - (w_n/n, x^*) \\ &\quad = (w_n/n, u_n^* - x^*) - \varepsilon/3n > \varepsilon/6n.\end{aligned}$$

But, this is contradictory to that x^* is an A-differential of c_G at x. Hence we complete the proof of properties (7), (8) and (9) of g.

(5) In order to prove the properties (7) and (8) of h, take any subsequence $\{x_{n(k)}\}_{k \geqq 1}$ of $\{x_n\}_{n \geqq 1}$ and any point z of Z, and set $y_k = x_{n(k)}$ for every k. Then, by the same argument as in the proof of Theorem in [12] and (4), taking an adequate subsequence $\{y_{k(i)}\}_{i \geqq 1}$ of $\{y_k\}_{k \geqq 1}$, we get that $(y_{k(i)}, a_i^* - b_i^*) \geqq \varepsilon/3$ for $a_i^* \in \partial c_H(z + y_{k(i)}/i)$ and $b_i^* \in \partial c_H(z - y_{k(i)}/i)$, whence ∂c_H is not Φ-continuous at z. Further, since it holds that $\{c_H(z + y_{k(i)}/i) + c_H(z - y_{k(i)}/i) - 2 \cdot c_H(z)\}/(1/i) > \varepsilon/3$ for every i, c_H is not Φ-uniformly Gateaux differentiable at z. Thus the proof of this part is completed.

Consequently, the proof of Proposition 5 is completed.

Remark 1. We easily know that the property (1) of g also can be stated as follows: $\overline{\text{co}}^*(T_g^*(\Delta(E)))$ has the following property (P*) for every $E \in \Lambda^+$.

(P*) There exists an $\eta > 0$ such that for any positive number $\alpha_1, \dots, \alpha_n$ whose sum is one and any weak*-open slices $S_1, \dots, S_n$ of $\overline{\text{co}}^*(T_g^*(\Delta(E)))$, it holds that

$$\sum_{j=1}^{n} \alpha_j \cdot \text{diam}_A(S_j) \geqq \eta.$$

Moreover we have that

$$\sup_{x \in A} O\left(x \Big| \sum_{j=1}^{n} \alpha_j S_j\right) = \sup_{x \in A} \sum_{j=1}^{n} \alpha_j \cdot O(x|S_j) \leqq \sum_{j=1}^{n} \alpha_j \cdot \left\{\sup_{x \in A} O(x|S_j)\right\}$$
$$= \sum_{j=1}^{n} \alpha_j \cdot \mathrm{diam}_A(S_j).$$

Thus the property (1) of h can be regarded as a stronger one corresponding to that of g.

REMARK 2. Note that the properties (5) and (6) concerning the function h constructed above are proved by invoking a deep result due to Fremlin noted above. However, in stead of the function h, making use of a weak*-measurable function $k : I \to D$ whose existence can be guaranteed by the statement (R_1) in (ii), we can directly prove without using Fremlin's result that k does not have the $\bar{A}^*$-measurable decomposability (for further details of this result, refer to [9]). That is, without invoking Fremlin's result, we are able to know the existence of weak*-measurable functions k having such properties (5) and (6) of h.

5. The similarity and difference between A-RN sets and A-Pettis sets in various aspects

In this section, a parallel study of A-RN sets and A-Pettis sets is presented as the following series of propositions. Their similarity and difference in various aspects are fully indicated from our view-point. In each proposition, the statement (2) is a companion of the statement (1). Their proofs are given later in Theorems 1 and 2 with the help of results in preceding sections.

(I) Dentability and strong regularity.

PROPOSITION 6. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then*

(1) *K is an A-RN set if and only if for every weak*-measurable function $f : I \to K$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is A-weak*-dentable.*

(2) *K is an A-Pettis set if and only if for every weak*-measurable function $f : I \to K$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is weak*-$\bar{A}^*$-dentable.*

PROPOSITION 7. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then*

(1) *K is an A-RN set if and only if for every weak*-measurable function $f : I \to K$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ has the following property* (P_1).

(P_1) *For every positive number* η, *there exist positive numbers* $\alpha_1, \dots, \alpha_n$ *whose sum is one and weak*-open slices* $S_1, \dots, S_n$ *of* $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ *such that*

$$\sum_{j=1}^{n} \alpha_j \cdot \mathrm{diam}_A(S_j) < \eta.$$

(2) *K is an A-Pettis set if and only if for every weak*-measurable function* $f : I \to K$ *and every* $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ *has the following property* (P_2).

(P_2) *For every positive number* η, *there exist positive numbers* $\alpha_1, \dots, \alpha_n$ *whose sum is one and weak*-open slices* $S_1, \dots, S_n$ *of* $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ *such that*

$$\mathrm{diam}_A\left(\sum_{j=1}^{n} \alpha_j S_j\right) < \eta.$$

Proposition 8. *Let A be a bounded subset of X and K a weak*-compact subset of* X^*. *Then*

(1) *K is an A-RN set if and only if* $\overline{\mathrm{co}}^*(K)$ *is convex subset-A-weak*-dentable* (*that is, for every nonempty convex subset C of* $\overline{\mathrm{co}}^*(K)$ *and any positive number* η, *there exists a weak*-open slice S of C such that* $\mathrm{diam}_A(S) < \eta$).

(2) *K is an A-Pettis set if and only if* $\overline{\mathrm{co}}^*(K)$ *is A-weak*-strongly regular.*

(II) Trees and martingales.

Proposition 9. *Let A be a bounded subset of X and K a weak*-compact subset of* X^*. *Then*

(1) *K is an A-RN set if and only if for every weak*-measurable function* $f : I \to K$, *the tree associated with f is not an A-*δ*-tree.*

(2) *K is an A-Pettis set if and only if for every weak*-measurable function* $f : I \to K$, *the tree associated with f is not an A-separated* δ*-tree.*

Proposition 10. *Let A be a bounded subset of X and K a weak*-compact subset of* X^*. *Then*

(1) *K is an A-RN set if and only if for every weak*-measurable function* $f : I \to K$, *it holds that*

$$\inf_{n \geqq 1} \int_I q_A(f_n(t) - f_{n+1}(t))\, d\lambda(t) = 0.$$

(2) *K is an A-Pettis set if and only if for every weak*-measurable function* $f : I \to K$, *it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} \|x \circ f_n - x \circ f_{n+1}\|_1 \right\} = 0.$$

(2′) *K is an A-Pettis set if and only if for every weak*-measurable function* $f: I \to K$, *it holds that*

$$\inf_{n \geqq 1}\left\{\sup_{x \in A} |(x, T_f^*(r_n))|\right\} = 0.$$

(III) Operators into L_∞.

PROPOSITION 11. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then*

(1) *K is an A-RN set if and only if for every weak*-measurable function* $f: I \to \overline{\mathrm{co}}^*(K)$, $S_f(A)$ *is equimeasurable in* L_∞.

(2) *K is an A-Pettis set if and only if for every weak*-measurable function* $f: I \to \overline{\mathrm{co}}^*(K)$, $S_f(A)$ *is a set of small oscillation with respect to* λ.

(IV) Radon-Nikodym properties.

PROPOSITION 12. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then*

(1) *K is an A-RN set if and only if $\overline{\mathrm{co}}^*(K)$ has the A-RNP.*

(2) *K is an A-Pettis set if and only if $\overline{\mathrm{co}}^*(K)$ has the $\bar{A}^*$-WRNP.*

(V) Differentiability of support functions.

PROPOSITION 13. *Let A be a bounded subset of X and K a weak*-compact subset of X^*.*

(1) *The following statements are equivalent.*

(a) *K is an A-RN set.*

(b) *For every nonempty subset G of K and every sequence $\{x_n\}_{n \geqq 1}$ in A, there exists a dense G_δ-subset W of Y (the closed linear span of $\Psi = \{x_n : n \geqq 1\}$) such that c_G is Ψ-uniformly Gateaux differentiable at each $y \in W$.*

(c) *For every nonempty subset G of K and every sequence $\{x_n\}_{n \geqq 1}$ in A, there exists a point y of Y such that c_G is Ψ-uniformly Gateaux differentiable at y.*

(2) *The following statements are equivalent.*

(a) *K is an A-Pettis set.*

(b) *For every nonempty subset H of K and every sequence $\{x_n\}_{n \geqq 1}$ in A, there exist a dense G_δ-subset W of Y and a subsequence $\{x_{n(k)}\}_{k \geqq 1}$ of $\{x_n\}_{n \geqq 1}$ such that c_H is Φ-uniformly Gateaux differentiable at each $y \in W$ (Here, $\Phi = \{x_{n(k)} : k \geqq 1\}$).*

(c) *For every nonempty subset H of K and every sequence $\{x_n\}_{n \geqq 1}$ in A, there exist a point y of Y and a subsequence $\{x_{n(k)}\}_{k \geqq 1}$ of $\{x_n\}_{n \geqq 1}$ such that c_H is Φ-uniformly Gateaux differentiable at y.*

In order to show the equivalence of A-RN sets stated in Propositions $6 \sim 13$, we have only to show the following Theorem 1 by combining with Proposition 5.

THEOREM 1. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then the following statements are equivalent.*

(a) *K is an A-RN set.*

(b) *$\overline{\mathrm{co}}^*(K)$ is an A-RN set.*

(c) *Every nonempty weak*-compact convex subset of $\overline{\mathrm{co}}^*(K)$ is A-weak*-dentable.*

(d) *$\overline{\mathrm{co}}^*(K)$ is convex subset-A-weak*-dentable.*

(e) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is A-weak*-dentable.*

(f) *For every weak*-measurable function $f : I \to K$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ has the property* (P_1).

(g) *$\overline{\mathrm{co}}^*(K)$ has the A-RNP.*

(h) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$, $S_f(A)$ is equi-measurable in L_∞.*

(i) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$, it holds that*

$$\inf_{n \geqq 1} \int_I q_A(f_n(t) - f_{n+1}(t)) d\lambda(t) = 0.$$

(j) *For every weak*-measurable function $f : I \to K$, the tree associated with f is not an A-δ-tree.*

(k) *For every nonempty subset G of K and every sequence $\{x_n\}_{n \geqq 1}$ in A, there exists a dense G_δ-subset W of Y such that c_G is Ψ-uniformly Gateaux differentiable at each $y \in W$.*

(l) *For every nonempty subset G of K and every sequence $\{x_n\}_{n \geqq 1}$ in A, there exists a point y of Y such that c_G is Ψ-uniformly Gateaux differentiable at y.*

Assume further that $A = -A$. *Then each of above statements is equivalent to*

(m) *For every nonempty subset G of K, there exists a dense G_δ-subset V of X such that c_G is A-differentiable at each $x \in V$.*

We are going to prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a), (e) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (i) $\Rightarrow$ (j) $\Rightarrow$ (a), (c) $\Rightarrow$ (k) $\Rightarrow$ (l) $\Rightarrow$ (a) and (c) $\Rightarrow$ (m) $\Rightarrow$ (a). The main point in Theorem 1 is that implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c), (f) $\Rightarrow$ (a), (e) $\Rightarrow$ (g), (j) $\Rightarrow$ (a), (l) $\Rightarrow$ (a) and (m) $\Rightarrow$ (a).

PROOF OF THEOREM 1. (a) $\Rightarrow$ (b) $\Rightarrow$ (c). This has been proved in [13].

(c) $\Rightarrow$ (d). Let C be a convex subset of $\overline{\mathrm{co}}^*(K)$ and $D =$ the weak*-closure of C. Then, D being A-weak*-dentable, D has weak*-open slices of arbitrary small q_A-diameter, and so does C.

(d) $\Rightarrow$ (e) $\Rightarrow$ (f). This is trivial.

(f) $\Rightarrow$ (a). This has been stated in Remark 1 (cf. the former part of (1) of (IV) in the proof of Proposition 5).

(e) $\Rightarrow$ (g). This follows from the statement (3) of Proposition 3.

(g) $\Rightarrow$ (h) $\Rightarrow$ (i). This follows from the statement (1) of Proposition 4.

(i) $\Rightarrow$ (j). The same argument as in (2) of (IV) of the proof of Proposition 5 deduces this result.

(j) $\Rightarrow$ (a). This follows from Proposition 5.

(c) $\Rightarrow$ (k). For every nonempty subset G of K and every sequence $\{x_n\}_{n\geqq 1}$ in A, $\partial c_G(y) \subset \overline{\mathrm{co}}^*(j^*(K))$ for every $y \in Y$ by virtue of Proposition 1. So, the same argument as in Theorem 3.14 and Proposition 3.15 of [4] deduces that c_G is $\mathrm{aco}(\Psi)$ (the absolutely convex hull of Ψ)-differentiable at each y in a dense G_δ-subset W of Y, whence (k) holds.

(k) $\Rightarrow$ (l). This is trivial.

(c) $\Rightarrow$ (m). This is the same as in the proof of the implication (c) $\Rightarrow$ (k).

(l) $\Rightarrow$ (a). This follows from Proposition 5.

(m) $\Rightarrow$ (a). This follows from Proposition 5.

On the other hand, in order to show the equivalence of A-Pettis sets stated in Propositions $6\sim 13$, we have only to show the following Theorem 2 by combining with Proposition 5.

THEOREM 2. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then the following statements are equivalent.*

(a) *K is an A-Pettis set.*

(b) *$\overline{\mathrm{co}}^*(K)$ is an A-Pettis set.*

(c) *Every nonempty weak*-compact convex subset of $\overline{\mathrm{co}}^*(K)$ is weak*-$\bar{A}^*$-dentable.*

(d) *For every weak*-measurable function $f: I \to \overline{\mathrm{co}}^*(K)$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is weak*-$\bar{A}^*$-dentable.*

(e) *$\overline{\mathrm{co}}^*(K)$ is A-weak*-strongly regular.*

(f) *For every weak*-measurable function $f: I \to K$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ has the property (P_2).*

(g) *$\overline{\mathrm{co}}^*(K)$ has the $\bar{A}^*$-WRNP.*

(h) *For every weak*-measurable function $f: I \to \overline{\mathrm{co}}^*(K)$, $S_f(A)$ is a set of small oscillation with respect to λ.*

(i) *For every weak*-measurable function $f: I \to \overline{\mathrm{co}}^*(K)$, it holds that*

$$\inf_{n\geqq 1}\left\{\sup_{x\in A}\|x\circ f_n - x\circ f_{n+1}\|_1\right\} = 0.$$

(j) *For every weak*-measurable function $f: I \to \overline{\mathrm{co}}^*(K)$, it holds that*

$$\inf_{n\geqq 1}\left\{\sup_{x\in A}|(x, T_f^*(r_n))|\right\} = 0.$$

(k) *For every weak*-measurable function* $f : I \to K$, *the tree associated with* f *is not an* A*-separated* δ*-tree.*

(l) *For every nonempty subset* H *of* K *and every sequence* $\{x_n\}_{n\geqq 1}$ *in* A, *there exist a dense* G_δ*-subset* W *of* Y *and a subsequence* $\{x_{n(k)}\}_{k\geqq 1}$ *of* $\{x_n\}_{n\geqq 1}$ *such that* c_H *is* Φ*-uniformly Gateaux differentiable at each* $y \in W$.

(m) *For every nonempty subset* H *of* K *and every sequence* $\{x_n\}_{n\geqq 1}$ *in* A, *there exist a point* y *of* Y *and a subsequence* $\{x_{n(k)}\}_{k\geqq 1}$ *of* $\{x_n\}_{n\geqq 1}$ *such that* c_H *is* Φ*-uniformly Gateaux differentiable at* y.

We are going to prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (g) $\Rightarrow$ (a), (j) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (h) $\Rightarrow$ (i) $\Rightarrow$ (j) $\Rightarrow$ (k) $\Rightarrow$ (a) $\Rightarrow$ (i) and (a) $\Rightarrow$ (l) $\Rightarrow$ (m) $\Rightarrow$ (a). The main point in Theorem 2 is that implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (g) $\Rightarrow$ (a), (j) $\Rightarrow$ (e), (f) $\Rightarrow$ (h), (k) $\Rightarrow$ (a) $\Rightarrow$ (l) and (m) $\Rightarrow$ (a). Now we make use of the fact (F): K is an A-Pettis set if and only if A is K-weakly precompact (that is, every sequence $\{x_n\}_{n\geqq 1}$ in A has a pointwise convergent subsequence $\{x_{n(k)}\}_{k\geqq 1}$ on K). This is regarded as a special case of the well-known result in the theory of pointwise compactness in the space of universally measurable functions, which we state as Lemma 2 for the convenience of the readers. This also gives equivalent conditions on Pettis sets.

Lemma 2. *Let* Z *be a compact Hausdorff space,* $\mathscr{F}$ *a uniformly bounded subset of* $C(Z)$ (*the Banach space of real-valued continuous functions on* Z). *Then the following statements about* $\mathscr{F}$ *are equivalent.*

(a) *Every sequence* $\{f_n\}_{n\geqq 1}$ *in* $\mathscr{F}$ *has a pointwise convergent subsequence* $\{f_{n(k)}\}_{k\geqq 1}$ *on* Z.

(b) *For every* $f \in \tau_p$ (*pointwise convergence topology*)*-closure of* $\mathscr{F}$, *every closed subset* D *of* Z *and every* $\varepsilon > 0$, *there exists an open subset* U *such that* $U \cap D \neq \emptyset$ *and* $O(f|U \cap D) < \varepsilon$.

(c) *For every* $f \in \tau_p$*-closure of* $\mathscr{F}$, f *is universally measurable on* Z.

Now, let us prove Theorem 2.

Proof of Theorem 2. (a) $\Rightarrow$ (b). Since it easily follows that A is K-weakly precompact if and only if A is $\overline{\mathrm{co}}^*(K)$-weakly precompact, we can get this implication, by virtue of the fact (F) above.

(b) $\Rightarrow$ (c). This can be shown by the same argument as in (iii) of the proof of Theorem in [9].

(c) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (g). This has been proved in Corollary 2 of [10].

(g) $\Rightarrow$ (a). This follows from Proposition 5 (or, Remark 2).

(j) $\Rightarrow$ (e). This has been proved in Proposition 5 of [11].

(e) ⇒ (f). This is trivial.

(f) ⇒ (h). This can be shown by the almost same argument as in (i) of the proof of Proposition 5 of [11].

(h) ⇒ (i). Suppose that (h) holds. Then $T_f(A)$ is relatively norm compact in L_1 and so, (i) follows from Lemma 2 in [11].

(i) ⇒ (j). This follows from Lemma 1 in [11] (cf. the statement (3) of (S) in Proposition 5).

(j) ⇒ (k). This also follows from Lemma 1 in [11].

(k) ⇒ (a). This follows from Proposition 5.

(a) ⇒ (i). Suppose that (a) holds. Then A is $\overline{\mathrm{co}}^*(K)$-weakly precompact, and so, by the dominated convergence theorem, $T_f(A)$ is relatively norm compact in L_1 and hence, (i) follows from Lemma 2 in [11].

(a) ⇒ (l). In view of Lemma 2, this has been shown in Theorem of [12].

(l) ⇒ (m). This is trivial.

(m) ⇒ (a). This follows from Proposition 5.

6. RN sets, GSP sets, Pettis sets and weakly precompact sets

Before closing the paper, let us collect results which can be obtained immediately as corollaries to Theorems 1 and 2. These are concerned with RN sets, GSP sets, Pettis sets or weakly precompact sets.

First, letting $A = B(X)$ in Theorem 1, we have:

COROLLARY 1 (Characterizations of RN sets). *Let K be a weak*-compact subset of X^*. Then the following statements are equivalent.*

(a) *K is a RN sets.*

(b) *Every nonempty weak*-compact convex subset of $\overline{\mathrm{co}}^*(K)$ is weak*-dentable.*

(c) *For every weak*-measurable function $f : I \to K$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is weak*-dentable.*

(d) *$\overline{\mathrm{co}}^*(K)$ has the RNP.*

(e) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$, $S_f(B(X))$ is equimeasurable in L_∞.*

(f) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$, it holds that*

$$\inf_{n \geqq 1} \int_I \|f_n(t) - f_{n+1}(t)\| d\lambda(t) = 0.$$

(g) *For every weak*-measurable function $f : I \to K$, the tree associated with f is not a δ-tree.*

(h) *For every nonempty subset G of K and every bounded sequence $\{x_n\}_{n \geqq 1}$, there exists a dense G_δ-subset W of Y such that c_G is Ψ-uniformly Gateaux differentiable at each $y \in W$.*

(i) *For every nonempty subset G of K, there exists a dense G_δ-subset V of X such that c_G is Frechet differentiable at each $x \in V$.*

Letting $K = B(X^*)$ in Theorem 1, we have:

Corollary 2 (Characterizations of GSP sets). *Let A be a bounded subset of X. Then the following statements are equivalent.*

(a) *$B(X^*)$ is an A-RN set.*

(b) *Every nonempty weak*-compact convex subset of X^* is A-weak*-dentable.*

(c) *For every bounded weak*-measurable function $f : I \to X^*$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is A-weak*-dentable.*

(d) *$B(X^*)$ has the A-RNP.*

(e) *For every bounded weak*-measurable function $f : I \to X^*$, $S_f(A)$ is equimeasurable in L_∞.*

(f) *A is a GSP set.*

(g) *For every bounded weak*-measurable function $f : I \to X^*$, it holds that*

$$\inf_{n \geqq 1} \int_I q_A(f_n(t) - f_{n+1}(t))\,d\lambda(t) = 0.$$

(h) *For every bounded weak*-measurable function $f : I \to X^*$, the tree associated with f is not an A-δ-tree.*

(i) *For every nonempty bounded subset G of X^* and every sequence $\{x_n\}_{n \geqq 1}$ in A, there exists a dense G_δ-subset W of Y such that c_G is Ψ-uniformly Gateaux differentiable at each $y \in W$.*

Letting $A = B(X)$ in Theorem 2, we have:

Corollary 3 (Characterization of Pettis sets). *Let K be a weak*-compact subset of X^*. Then the following statements are equivalent.*

(a) *K is a Pettis set.*

(b) *Every nonempty weak*-compact convex subset of $\overline{\mathrm{co}}^*(K)$ is weak*-scalarly-dentable.*

(c) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$ and every $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ is weak*-scalarly-dentable.*

(d) *Every nonempty convex subset of $\overline{\mathrm{co}}^*(K)$ is weak*-strongly regular.*

(e) *$\overline{\mathrm{co}}^*(K)$ has the WRNP.*

(f) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$, $S_f(B(X))$ is a set of small oscillation with respect to λ.*

(g) *For every weak*-measurable function $f : I \to \overline{\mathrm{co}}^*(K)$, it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in B(X)} \|x \circ f_n - x \circ f_{n+1}\|_1 \right\} \left(= \inf_{n \geqq 1} \|f_n - f_{n+1}\|_P \right) = 0.$$

Here $\|\cdot\|_P$ denotes the Pettis-norm.

(h) *For every weak*-measurable function* $f : I \to \overline{\mathrm{co}}^*(K)$, *it holds that*

$$\inf_{n \geqq 1} \|T_f^*(r_n)\| = 0.$$

(i) *For every weak*-measurable function* $f : I \to K$, *the tree associated with* f *is not a separated* δ*-tree.*

(j) *For every nonempty subset* H *of* K *and every bounded sequence* $\{x_n\}_{n \geqq 1}$, *there exist a dense* G_δ*-subset* W *of* Y *and a subsequence* $\{x_{n(k)}\}_{k \geqq 1}$ *of* $\{x_n\}_{n \geqq 1}$ *such that* c_H *is* Φ*-uniformly Gateaux differentiable at each* $y \in W$.

Finally, letting $K = B(X^*)$ in Theorem 2, we have:

COROLLARY 4 (Characterizations of weakly precompact sets). *Let* A *be a bounded subset of* X. *Then the following statements are equivalent.*

(a) A *is weakly precompact.*

(b) *Every nonempty weak*-compact convex subset of* X^* *is weak*-*$\bar{A}^*$*-dentable.*

(c) *For every bounded weak*-measurable function* $f : I \to X^*$ *and every* $E \in \Lambda^+$, $\overline{\mathrm{co}}^*(T_f^*(\Delta(E)))$ *is weak*-*$\bar{A}^*$*-dentable.*

(d) *Every nonempty bounded convex subset of* X^* *is* A*-weak*-strongly regular.*

(e) $B(X^*)$ *has the* $\bar{A}^*$*-WRNP.*

(f) *For every bounded weak*-measurable function* $f : I \to X^*$, $S_f(A)$ *is a set of small oscillation with respect to* λ.

(g) *For every bounded weak*-measurable function* $f : I \to X^*$, *it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} \|x \circ f_n - x \circ f_{n+1}\|_1 \right\} = 0.$$

(h) *For every bounded weak*-measurable function* $f : I \to X^*$, *it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} |(x, T_f^*(r_n))| \right\} = 0.$$

(i) *For every bounded weak*-measurable function* $f : I \to X^*$, *the tree associated with* f *is not an* A*-separated* δ*-tree.*

(j) *For every nonempty bounded subset* H *of* X^* *and every sequence* $\{x_n\}_{n \geqq 1}$ *in* A, *there exist a dense* G_δ*-subset* W *of* Y *and a subsequence* $\{x_{n(k)}\}_{k \geqq 1}$ *of* $\{x_n\}_{n \geqq 1}$ *such that* c_H *is* Φ*-uniformly Gateaux differentiable at each* $y \in W$.

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