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The moduli of non-differentiability for Gaussian random fields with stationary increments

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We establish the exact moduli of non-differentiability of Gaussian random fields with stationary increments. As an application of the result, we prove that the uniform Hölder condition for the maximum local times of Gaussian random fields with stationary increments obtained in Xiao (1997) is optimal. These results are applicable to fractional Riesz–Bessel processes and stationary Gaussian random fields in the Matérn and Cauchy classes.

Keywords: Cauchy class; fractional Riesz–Bessel process; Gaussian random field; local time; modulus of non-differentiability; strong local nondeterministism

1. Introduction

Lévy's uniform modulus of continuity, Khinchin's law of the iterated logarithm (also known as local modulus of continuity), Chung's law of the iterated logarithm, and Csörgő-Révész's modulus of non-differentiability, describe precisely the uniform and local maximal and minimal oscillations of the sample functions of Brownian motion. See Csörgő and Révész [8] for more information. Many authors have extended some of these results to more general Gaussian processes and fields. We refer to the monographs of Marcus and Rosen [25] and Adler and Taylor [1] for comprehensive and historical accounts on local and uniform moduli of continuity for Gaussian processes, and to Meerschaert *et al.* [26] for a general method based on the property of sectorial local nondeterminism for proving exact uniform moduli of continuity of Gaussian processes and random fields, and to Kuelbs *et al.* [20], Monrad and Rootzén [27] for Chung's laws of the iterated logarithm (LIL) for a large class of Gaussian processes. We also mention that Talagrand [35] refined the results on Chung's LIL by characterizing the lower functions for fractional Brownian motion. The results in the last three references are naturally related to the small ball probability of Gaussian processes, which is an important topic on its own right and has many important applications; see, for example, Kuelbs and Li [19], Talagrand [34], Li and Linde [21], Li and Shao [22], Shao [30] for further information.

In contrast, the problem on the modulus of non-differentiability of Gaussian processes (fields) has received much less attention. Berman [4], Geman and Horowitz [11], Csörgő and Shao [9],

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and Xiao [38] provided some sufficient conditions for the sample functions of a Gaussian process to be nowhere differentiable. In Xiao [38], a conjecture on the modulus of non-differentiability for Gaussian random fields was given. Recently Wang and Xiao [37] verified this conjecture for fractional Brownian motion $B^H = \{B_t^H, t \in \mathbb{R}_+\}$. The purpose of the present paper is to extend the result in Wang and Xiao [37] to a large class of Gaussian random fields with stationary increments considered in Xiao [38,39]. Our main result is Theorem 1.2 below. Its proof is built upon the general framework on limsup random fractals in Khoshnevisan *et al.* [17], as well as technical ingredients such as the small ball probability estimates, a correlation inequality of Shao [30], and the property of strong local nondeterminism.

As pointed out by Berman [4], the modulus of non-differentiability of a Gaussian random field is closely related to the Hölder continuity of its local times. By using this connection and Theorem 1.2, we prove in Theorem 4.1 that the uniform Hölder condition for the maximum local times of a Gaussian random field obtained in Xiao [38] is optimal. To the best of our knowledge, this is the only approach for determining the exact uniform Hölder condition for the local times of Gaussian random fields.

Now we specify the Gaussian random fields under investigation in this paper. Let $Y = \{Y(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with $Y(0) = 0$. We assume that Y has stationary increments with variance function $\sigma^2(h) = \mathbb{E}[(Y(t+h) - Y(t))^2]$. We will consider the (N, d)-Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \dots, X_d(t)\big)', \quad \forall t \in \mathbb{R}^N, \tag{1.1}$$

where the coordinator processes $X_1, \dots, X_d$ are independent copies of Y. We call Y the associated random field of X.

Let $\phi(x) = x^{2\nu} L(x)$ for $x \in [0, 1]$, where $\nu \in (0, 1)$ is a constant and $L : [0, 1] \mapsto \mathbb{R}_+$ is a monotone function and slowly varying at zero, that is, $\lim_{x \to 0} L(\xi x)/L(x) = 1$ for every constant $\xi > 0$. We will assume that Y satisfies the following Condition (C):

(C1) There exist positive constants $c_{1,1}$ and $c_{1,2}$ such that for all $h \in B(0, \delta_0)$ with some $\delta_0 > 0$,

$$c_{1,1}\phi\big(\|h\|\big) \le \sigma^2(h) \le c_{1,2}\phi\big(\|h\|\big), \tag{1.2}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^N$ and $B(0, \delta_0) = \{h \in \mathbb{R}^N : \|h\| \le \delta_0\}$.

(C2) There exists a positive constant $c_{1,3}$ such that for all $h = (h_1, \dots, h_N) \in B(0, \delta_0)\backslash\{0\}$,

$$\max_{1 \le l,m \le N} \big|\partial^2\sigma^2(h)/\partial h_l \partial h_m\big| \le c_{1,3}\sigma^2(h)/\|h\|^2. \tag{1.3}$$

(C3) For $T > 0$, Y is strongly locally ϕ-nondeterministic on $[-T, T]^N$, namely, there exist positive constants $c_{1,4}$ and $r_0 > 0$ such that for all $t \in [-T, T]^N$ and all $0 < r \le \min\{\|t\|, r_0\}$,

$$\operatorname{Var}\big(Y(t) | Y(s) : s \in [-T, T]^N, r \le \|s - t\| \le r_0\big) \ge c_{1,4}\phi(r).$$

Conditions (C1) and (C3) are the same as in Xiao [38] and they indicate that the random field Y is approximately isotropic in the time-variable. Xiao [39] gave a sufficient condition in terms of the tail behavior at infinity of the spectral measure of Y for (C1) and (C3) to hold. (C2) is a mild regularity condition on $\sigma^2(h)$ and can usually be verified if one has information on the spectral measure of Y. When Y is isotropic in the sense that $\sigma^2(h) = \psi(\|h\|)$ for some non-negative function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, then (C2) holds provided $|\psi''(x)| \le c_{1,3}\psi(x)/x^2$ for $x \in (0, \delta_0]$.

Clearly the multiparameter fractional Brownian motion of index H satisfies Condition (C) with $\sigma^2(h) = \|h\|^{2H}$ for all $h \in \mathbb{R}^N$. Further examples of Gaussian random fields that satisfy (C) include the fractional Riesz–Bessel processes and stationary Gaussian random fields in the Matérn or Cauchy classes. The latter Gaussian fields play important roles in statistics. See Section 5 for details.

For an illustration purpose, we first mention Chung's LIL of X due to Monrad and Rootzén [27], Talagrand [34,35], Li and Shao [22], Xiao [39], in increasing generality. In fact, the following Chung's LIL improves that in Xiao [39] by removing some conditions on spectral measure of X in that paper. This can be seen from the proof of Theorem 1.2 (see Section 2). For $t \in \mathbb{R}^N$ and $r > 0$, we set

$$M(t, r) = \sup_{s\in[0,r]^N} \big\|X(t + s) - X(t)\big\|.$$

Theorem 1.1 (Chung's LIL). *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field defined by (1.1). Suppose that the associated Gaussian random field Y has stationary increments and satisfies Condition (C). Then, there exists a positive constant $c_{1,5}$ such that*

$$\liminf_{r\to 0+} \gamma(r)M(0, r) = c_{1,5} \quad a.s., \tag{1.4}$$

where $\gamma(r) = [\phi(r/(\log\log(1/r))^{1/N})]^{-1/2}$.

The main purpose of the present paper is to establish the following modulus of non-differentiability in the sense of Csörgő-Révész [7].

Theorem 1.2. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field defined by (1.1). Suppose that the associated Gaussian random field Y has stationary increments and satisfies Condition (C). Put*

$$\beta(r) = \big[\phi\big(r/\big(\log(1/r)\big)^{1/N}\big)\big]^{-1/2}. \tag{1.5}$$

Then, there exists a positive constant $c_{1,6}$ such that for all compact rectangles $I \subseteq \mathbb{R}^N$,

$$\liminf_{r\to 0+} \beta(r) \inf_{t\in I} M(t, r) = c_{1,6} \quad a.s. \tag{1.6}$$

Consequently, the sample paths of X and its associated Gaussian field Y are almost surely nowhere differentiable.

Remark 1.1. The following are some remarks on Theorems 1.1 and 1.2.

- (1.4) and (1.6) describe precisely the local minimal oscillations and the uniform minimal oscillations of the sample functions of X, respectively. Together with the Khinchin-type law of the iterated logarithm and the uniform modulus of continuity, they provide complete information on the regularity properties of X.

- It can be seen from the proof of Theorem 1.2 that $c_{1,6} \in [(c_{2,3}/N)^{\nu/N}, (dc_{2,2}/N)^{\nu/N}]$, where $c_{2,2}$ and $c_{2,3}$ are given in (2.12) below. For Brownian motion in $\mathbb{R}$, $c_{1,5} = c_{1,6} = \pi/\sqrt{8}$; see Csörgő and Révész [7]. For fractional Brownian motion $B^H = \{B_t^H, t \in \mathbb{R}_+\}$, Wang and Xiao [37] showed that $c_{1,6}$ is the small ball constant of B^H (thus $c_{1,5} = c_{1,6}$), whose existence was established by Li and Linde [21] and Shao [30] independently. But the exact value of $c_{1,6}$ for B^H is unknown. For a general Gaussian process (field), little is known about the existence of the small ball constant nor the exact value of $c_{1,6}$.

We now comment on the method for proving Theorem 1.2. We first prove a zero-one law for the modulus of non-differentiability of Gaussian random fields which implies that the limit inferior in (1.6) is a constant and we denote it by $c_{1,6}$. By using the small ball probability estimate for X and a standard Borel–Cantelli argument, we can see that $c_{1,6} > 0$. However, it is more difficult to prove that $c_{1,6} < \infty$ and the method for proving the Chung's LIL of X at a fixed point is not of much use anymore. To be more precise, we recall that a key step for proving Chung's LIL of X at a fixed point, say $s = 0$, is to approximate X by a sequence of independent Gaussian processes that are obtained by decomposing X as the sums of stochastic integrals over disjoint sets. See Monrad and Rootzén [27], Talagrand [34], Li and Shao [22]. This argument is not effective anymore for proving the upper bound in (1.6) because one has to consider all $s \in I \subseteq \mathbb{R}^N$. Our proof is based on a different approach. The new ingredient of our present paper is to make use of the theory on limsup random fractals of Khoshnevisan *et al.* [17]. In order to apply their results, the Gaussian correlation inequality in Shao [30] will also play an important role.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries about Gaussian random fields with stationary increments and prove a zero-one law for the modulus of non-differentiability of such a Gaussian field by using its stochastic integral representation. We also recall some important inequalities for Gaussian fields. In Section 3, we prove Theorem 1.2. In Section 4, we explore the connection between the roughness of the sample functions of X and the regularity of its local times. As an application to Theorem 1.2, we prove a lower bound for the uniform Hölder condition for the maximum local times of X. This solves an open problem in Xiao [38]. Finally in Section 5 we show that the results in this paper are applicable to the fractional Riesz–Bessel processes introduced by Anh *et al.* [2] and stationary Gaussian random fields in the Matérn and Cauchy class. These Gaussian random fields play important roles in statistics. See Chilés and Delfiner [6] and Gneiting and Schlather [13].

Throughout this paper, for $s, t \in \mathbb{R}^N$, $s \le t$ (resp. $s < t$) means that $s_i \le t_i$ (resp. $s_i < t_i$) for all $1 \le i \le N$. $[s,t]^N = [s_1, t_1] \times \cdots \times [s_N, t_N]$ is called an interval or a rectangle. Positive and finite constants in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$.

2. Preliminaries

Let $Y = \{Y(t), t \in \mathbb{R}^N\}$ be a real-valued, centered Gaussian random field with stationary increments, $Y(0) = 0$, and continuous covariance function $R(s,t) = \mathbb{E}[Y(s)Y(t)]$. According to

Yaglom [40] (see also [10]), $R(s,t)$ can be represented as

$$R(s,t) = \int_{\mathbb{R}^N} \big(e^{i\langle s,\lambda\rangle} - 1\big)\big(e^{-i\langle t,\lambda\rangle} - 1\big) F(d\lambda) + \langle s, Qt\rangle, \tag{2.1}$$

where $\langle x, y\rangle$ is the ordinary scalar product in $\mathbb{R}^N$, Q is an $N \times N$ non-negative definite matrix and $F(d\lambda)$ is a nonnegative symmetric measure on $\mathbb{R}^N \backslash \{0\}$ satisfying

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1+\|\lambda\|^2} F(d\lambda) < \infty. \tag{2.2}$$

The measure F is called the *spectral measure* of Y. The density function of F (if exists) is called the spectral density of Y.

It follows from (2.1) that Y has the following stochastic integral representation:

$$Y(t) = \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big) \mathcal{M}(d\lambda) + \langle Z, t\rangle, \tag{2.3}$$

where Z is an N-dimensional Gaussian random vector with mean 0 and $\mathcal{M}$ is a centered complex-valued Gaussian random measure which is independent of Z and satisfies

$$\mathbb{E}\big(\mathcal{M}(A)\overline{\mathcal{M}(B)}\big) = F(A \cap B) \quad \text{and} \quad \mathcal{M}(-A) = \overline{\mathcal{M}(A)}$$

for all Borel sets $A, B \subseteq \mathbb{R}^N$ with finite F-measure. Since the linear term $\langle Z, t\rangle$ in (2.3) will not have any effect on the problems considered in the present paper, we will assume $Z = 0$. This is equivalent to assuming $Q = 0$ in (2.1). Consequently, we have

$$\sigma^2(h) = \mathbb{E}\big[\big(Y(t+h) - Y(t)\big)^2\big] = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big) F(d\lambda). \tag{2.4}$$

To prove Theorem 1.2, we need some technical lemmas. The first lemma establishes a zero-one law for the modulus of non-differentiability of Gaussian random fields.

Lemma 2.1 (Zero-one law). *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by (1.2). Suppose that the associated Gaussian random field Y has representation (2.3). Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a function and $I \subseteq \mathbb{R}^N$ be a compact set. If*

$$\lim_{\delta \to 0+} \delta\big(\log(1/\delta)\big)^{1/2} \omega(\delta) = 0, \tag{2.5}$$

then we have

$$\liminf_{\delta \to 0+} \omega(\delta) \inf_{t \in I} M(t,\delta) = C' \quad \textit{a.s. for some } 0 \le C' \le \infty. \tag{2.6}$$

Proof. Denote by $B(\lambda, r) = \{x \in \mathbb{R}^N : \|x - \lambda\| \le r\}$ the ball in $\mathbb{R}^N$ with center λ and radius r. Let $U_1 = B(0,1)$ and $U_n = B(0,n) \setminus B(0,n-1)$ for for $n \ge 2$. Then $U_1, U_2, \ldots,$ are mutually disjoint. For $n \ge 1$ and $t \in \mathbb{R}^N$, let $\xi_n(t) := \int_{U_n} (e^{i\langle t,\xi\rangle} - 1)\mathcal{M}(d\xi)$, where $\mathcal{M}(d\xi)$ is a centered complex-valued Gaussian random measure defined in (2.3). Then $\xi_n = \{\xi_n(t), t \in \mathbb{R}^N\}$, $n = 1, 2, \dots$, are independent Gaussian fields. By (2.3), we express

$$Y(t) = \sum_{n=1}^{\infty} \xi_n(t), \quad t \in \mathbb{R}^N.$$

Equip $I = [0, 1]^N$ with the canonical metric

$$d_{\xi_n}(s, t) = \sqrt{\mathbb{E}\big[\big(\xi_n(s) - \xi_n(t)\big)^2\big]}, \quad s, t \in I, \tag{2.7}$$

and denote by $N(d_{\xi_n}, I, \varepsilon)$ the smallest number of d_{ξ_n}-balls of radius $\varepsilon > 0$ needed to cover I. Note

$$\begin{aligned} d_{\xi_n}(s, t) &= \left(2 \int_{U_n} \big(1 - \cos\big(\langle (s-t), \xi\rangle\big)\big) F(d\xi)\right)^{1/2} \\ &\le \|s - t\| \left(\int_{U_n} |\xi|^2 F(d\xi)\right)^{1/2} \\ &=: \|s - t\| K_n, \quad s, t \in \mathbb{R}^N. \end{aligned} \tag{2.8}$$

In the above, we have bounded $1 - \cos(\langle t, x\rangle)$ by $|t|^2|x|^2/2$ to obtain the inequality in (2.8). Then it is easy to see from (2.8) that for all $\varepsilon \in (0, 1)$,

$$N(d_{\xi_n}, I, \varepsilon) \le (K_n/\varepsilon)^N.$$

Hence, by making use of Theorem 1.3.5 in Adler and Taylor [1], we have

$$\sup_{\substack{s,t \in I \\ \|s-t\| \le \delta}} \big|\xi_n(s) - \xi_n(t)\big| \le c_{2,1} \delta \Big(\big(\log(K_n/\delta)\big)^{1/2} + \big(\log(K_n/\delta)\big)^{-1/2}\Big). \tag{2.9}$$

Let $\xi_n(t) = (\xi_n^1(t), \dots, \xi_n^d(t))'$, $t \in \mathbb{R}^N$, where $\xi_n^1, \dots, \xi_n^d$ are independent copies of $\xi_n = \{\xi_n(t), t \in \mathbb{R}^N\}$. Set

$$Y_M^i(t) = \sum_{n=1}^{M} \xi_n^i(t), \quad 1 \le i \le d, t \in \mathbb{R}^N.$$

Let $X_M(t) = (Y_M^1(t), \dots, Y_M^d(t))'$, $t \in \mathbb{R}^N$, where the coordinator processes $Y_M^1, \dots, Y_M^d$ are independent copies of the Gaussian random field $Y_M = \{Y_M(t), t \in \mathbb{R}^N\}$. Noting $\|\mathbf{x}\| \le d \max_{1 \le i \le d} |x_i|$ for all $\mathbf{x} \in \mathbb{R}^d$, by (2.5) and (2.9), we have

$$\lim_{\delta \to 0+} \omega(\delta) \sup_{t \in I} \sup_{s \in [0,\delta]^N} \big\|X_M(t+s) - X_M(t)\big\| = 0 \quad \text{a.s.} \tag{2.10}$$

Therefore, the random variable

$$\liminf_{\delta \to 0+} \omega(\delta) \inf_{t \in I} M(t, \delta) \tag{2.11}$$

is measurable with respect to the tail field of $\{\xi_n\}_{n=1}^{\infty}$ and hence is constant almost surely. This implies (2.6). $\square$

We will make use of the following small ball probability estimate which is a consequence of Theorem 3.1 in Xiao [39], where the lower bound follows from a result of Talagrand [33] and the upper bound was proved by applying (C1) and (C3). See also Theorem 1.1 in Shao and Wang [31] for a lower bound under a different condition.

Lemma 2.2. *Let $G = \{G(t), t \in \mathbb{R}^N\}$ be a Gaussian random field valued in $\mathbb{R}$ satisfying Conditions* (*C*1) *and* (*C*3). *Then, for all $r > 0$ and $x \in (0, 1)$,*

$$\exp\left(-\frac{c_{2,2} r^N}{(\phi^{-1}(x^2))^N}\right) \le \mathbb{P}\Big(\max_{t\in[0,r]^N} \big|G(t)\big| \le x\Big) \le \exp\left(-\frac{c_{2,3} r^N}{(\phi^{-1}(x^2))^N}\right), \tag{2.12}$$

where $\phi^{-1}(x) = \inf\{y : \phi(y) > x\}$ is the right-continuous inverse function of ϕ.

The following lemma from Talagrand [34] is needed.

Lemma 2.3. *Let $G = \{G(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field with values in $\mathbb{R}$ and let $\mathcal{S} \subset \mathbb{R}^N$ be a compact set equipped with the canonical metric $d_G(s, t)$. Then, for all $u > 0$,*

$$\mathbb{P}\left(\sup_{s,t\in\mathcal{S}} \big|G(s) - G(t)\big| \ge c_{2,4}\left(u + \int_0^D \sqrt{\log N(d_G, \mathcal{S}, \varepsilon)}\, d\varepsilon\right)\right) \le \exp\left(-\frac{u^2}{D^2}\right), \tag{2.13}$$

where $D = \sup\{d_G(s, t) : s, t \in \mathcal{S}\}$ is the diameter of $\mathcal{S}$.

We will also need the following Fernique-type inequality for Gaussian random fields.

Lemma 2.4. *Let $G = \{G(t), t \in \mathbb{R}^N\}$ be a Gaussian random field valued in $\mathbb{R}$ satisfying Condition* (*C*1). *Then, for all $\delta > 0$, $0 < a \le \delta$ and $u \ge u_0$ with some $u_0 > 0$,*

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[-\delta,\delta]^N} \sup_{s\in[0,a]^N} \big|G(t+s) - G(t)\big| \ge u\left(\sqrt{\log(2\delta/a)} + \frac{1}{\sqrt{\log(2\delta/a)}}\right)\phi^{1/2}(a)\right) \\
&\quad \le e^{-c_{2,5}u^2}.
\end{aligned} \tag{2.14}$$

Proof. We apply Lemma 2.3 to prove (2.14). Define the Gaussian random field $Z = \{Z(t, s), t, s \in \mathbb{R}^N\}$ by $Z(t, s) = G(t + s) - G(t)$, $\forall t, s \in \mathbb{R}^N$.

By the Minkowski inequality and Condition (C1), there exists a positive constant $c_{2,6}$ such that

$$\begin{aligned}
d_Z\big((t_1, s_1), (t_2, s_2)\big) &= \Big(\mathbb{E}\Big[\big(Z(t_1, s_1) - Z(t_2, s_2)\big)^2\Big]\Big)^{1/2} \\
&\le \Big(\mathbb{E}\Big[\big(G(t_2 + s_2) - G(t_1 + s_1)\big)^2\Big]\Big)^{1/2} + \Big(\mathbb{E}\Big[\big(G(t_2) - G(t_1)\big)^2\Big]\Big)^{1/2} \\
&\le c_{2,6}\phi^{1/2}\big(\big\|(t_2, s_2) - (t_1, s_1)\big\|\big)
\end{aligned}$$

for all $(t_1, s_1), (t_2, s_2) \in \mathcal{S} := [-\delta, \delta]^N \times [0, a]^N$. Thus,

$$N(d_Z, \mathcal{S}, \varepsilon) \le \frac{a^N (2\delta)^N}{[\phi^{-1}(\varepsilon^2/c_{2,6}^2)]^{2N}},$$

where $\phi^{-1}(x) = \inf\{y : \phi(y) > x\}$ is the right-continuous inverse function of ϕ. By Condition (C1) again, $d_Z(0, t) \le c_{2,6}\phi^{1/2}(a)$. It follows that the diameter D of $\mathcal{S}$ is less than $c_{2,6}\phi^{1/2}(a)$. Thus, noting $\phi^{1/2}(x)/x^\nu$ is non-decreasing on $[0, \infty)$, some simple calculations yield

$$\begin{aligned}
\int_0^D \sqrt{\log N(d_Z, \mathcal{S}, \varepsilon)}\, d\varepsilon &\le \int_0^{c_{2,6}\phi^{1/2}(a)} \sqrt{\log \frac{(2a\delta)^{1/2}}{\phi^{-1}(\varepsilon^2/c_{2,6}^2)}}\, d\varepsilon \\
&= \int_0^a \sqrt{\log\bigl((2a\delta)^{1/2}/t\bigr)}\, d\phi^{1/2}(t) \\
&= \sqrt{\log(2\delta/a)}\phi^{1/2}(a) + \int_0^a \frac{1}{t\sqrt{\log((2a\delta)^{1/2}/t)}} \phi^{1/2}(t)\, dt \\
&\le \sqrt{\log(2\delta/a)}\phi^{1/2}(a) + \frac{1}{\sqrt{\log(2\delta/a)}} \int_0^\infty u\phi^{1/2}\bigl(ae^{-u^2}\bigr)\, du \\
&\le \sqrt{\log(2\delta/a)}\phi^{1/2}(a) + \frac{c_{2,7}}{\sqrt{\log(2\delta/a)}}\phi^{1/2}(a) \int_0^\infty ue^{-\nu u^2}\, du \\
&\le c_{2,8}\left(\sqrt{\log(2\delta/a)} + \frac{1}{\sqrt{\log(2\delta/a)}}\right)\phi^{1/2}(a).
\end{aligned} \tag{2.15}$$

Thus, by Lemma 2.3, we get (2.14). □

We also need the following lemma, which is Theorem 1.1 in Shao [30].

Lemma 2.5. *Let $G' = (G_1', G_2')$ be an $\mathbb{R}^n$-valued normal random vector with mean vector* 0, *where $G_1 = (X_1, \dots, X_k)'$, $G_2 = (X_{k+1}, \dots, X_n)'$ and $1 \le k < n$. Then $\forall x > 0$,*

$$\mathbb{P}\bigl(\|G\|_\infty \le x\bigr) \le \rho\mathbb{P}\bigl(\|G_1\|_\infty \le x\bigr)\mathbb{P}\bigl(\|G_2\|_\infty \le x\bigr), \tag{2.16}$$

where $\|\mathbf{x}\|_\infty$ denotes the maximum norm of a vector $\mathbf{x}$ *and*

$$\rho = \left(\frac{\det(\mathbb{E}[G_1G_1']) \det(\mathbb{E}[G_2G_2'])}{\det(\mathbb{E}[GG'])}\right)^{1/2}. \tag{2.17}$$

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Because of Lemma 2.1, it is sufficient to prove the following two inequalities:

$$\liminf_{r\to 0+}\beta(r)\inf_{t\in I}M(t,r)\ge (c_{2,3}/N)^{\nu/N}\quad \text{a.s.,}\tag{3.1}$$

and

$$\liminf_{r\to 0+}\beta(r)\inf_{t\in I}M(t,r)\le (dc_{2,2}/N)^{\nu/N}\quad \text{a.s.}\tag{3.2}$$

Let us first prove (3.1). Since $\|x\|\ge |x_1|$ for all $x=(x_1,\dots,x_d)'\in\mathbb{R}^d$, we prove (3.1) only for $d=1$ and $I=[0,1]^N$.

Fix an arbitrary $\theta>1$. For $n\ge 1$, let $r_n=\theta^{-n}$ and $\varrho_n=n^{1/N+1/\nu}\theta^n$. For $\mathbf{i}\in\mathbb{Z}_+^N$ and $n\ge 1$, we define two sets A_n and $A_{n,\mathbf{i}}$ as follows:

$$\begin{aligned}A_n&=\bigl\{r\in(0,1):r_{n+1}<r\le r_n\bigr\},\\ A_{n,\mathbf{i}}&=\bigl\{t=(t_1,\dots,t_N)'\in I:\mathbf{i}\varrho_n^{-1}\le t<(\mathbf{i}+\mathbf{1})\varrho_n^{-1}\bigr\},\end{aligned}$$

where $\mathbf{1}$ is a vector with elements 1. Observe that for all $r\in(0,1)$, there exists a set A_n such that $r\in A_n$. On the other hand, for all $t\in I$, there exists a set $A_{n,\mathbf{i}}$ such that $t\in A_{n,\mathbf{i}}$. Let $t_{\mathbf{i},n}:=\mathbf{i}\varrho_n^{-1}$ be a point in the set $A_{n,\mathbf{i}}$, $\mathbf{i}\in[0,\varrho_n]^N$. Noticing $\phi(c_{3,1}^{1/\nu}x)\ge c_{3,1}^2\phi(x)$ for all $x\in[0,\delta_0)$ and by Lemma 2.2, we have

$$\begin{aligned}&\mathbb{P}\Bigl(\min_{\mathbf{i}\in[0,\varrho_n]^N}\beta(r_n)M(t_{\mathbf{i},n},r_{n+1})\le c_{3,1}\Bigr)\\ &\quad\le\sum_{\mathbf{i}\in[0,\varrho_n]^N}\mathbb{P}\bigl(\beta(r_n)M(t_{\mathbf{i},n},r_{n+1})\le c_{3,1}\bigr)\\ &\quad\le n^{aN}\theta^{nN}\mathbb{P}\biggl(\sup_{s\in[0,r_{n+1}]^N}\bigl|X(s)\bigr|\le\phi^{1/2}\biggl(\frac{c_{3,1}^{1/\nu}r_n}{(\log(1/r_n))^{1/N}}\biggr)\biggr)\\ &\quad\le n^{aN}\theta^{(N-c_{2,3}\theta^{-N}/c_{3,1}^{N/\nu})n}.\end{aligned}$$

For any $c_{3,1}<(c_{2,3}/N)^{\nu/N}$, we choose $\theta\downarrow 1$ such that $c_{2,3}\theta^{-N}/c_{3,1}^{N/\nu}>N$. Hence, by the Borel–Cantelli lemma, we have

$$\liminf_{n\to\infty}\min_{\mathbf{i}\in[0,\varrho_n]^N}\beta(r_n)M(t_{\mathbf{i},n},r_{n+1})\ge c_{3,1}\quad\text{a.s.}\tag{3.3}$$

By Lemma 2.4 and a Borel–Cantelli lemma argument, we have

$$\limsup_{n\to\infty}\sup_{t\in[-2,2]^N}\sup_{s\in[0,\varrho_n^{-1}]^N}\beta(r_n)\bigl|X(t+s)-X(t)\bigr|=0\quad\text{a.s.}\tag{3.4}$$

Note that

$$
\begin{aligned}
&\liminf_{r\to 0+}\beta(r)\inf_{t\in I} M(t,r)\\
&\quad\geq \liminf_{n\to\infty}\inf_{r\in A_n}\min_{\mathbf{i}\in[0,\varrho_n]^N}\inf_{t\in A_{n,\mathbf{i}}}\beta(r)M(t,r)\\
&\quad\geq \liminf_{n\to\infty}\min_{\mathbf{i}\in[0,\varrho_n]^N}\inf_{t\in A_{n,\mathbf{i}}}\beta(r_n)M(t,r_{n+1})\\
&\quad\geq \liminf_{n\to\infty}\min_{\mathbf{i}\in[0,\varrho_n]^N}\beta(r_n)M(t_{\mathbf{i},n},r_{n+1})\\
&\qquad -2\limsup_{n\to\infty}\max_{\mathbf{i}\in[0,\varrho_n]^N}\sup_{t\in A_{n,\mathbf{i}}}\sup_{s\in[0,r_{n+1}]^N}\beta(r_n)\big|X(t_{\mathbf{i},n}+s)-X(t+s)\big|\\
&\quad\geq \liminf_{n\to\infty}\min_{\mathbf{i}\in[0,\varrho_n]^N}\beta(r_n)M(t_{\mathbf{i},n},r_{n+1})\\
&\qquad -2\limsup_{n\to\infty}\sup_{t\in[-2,2]^N}\sup_{s\in[0,\varrho_n^{-1}]^N}\beta(r_n)\big|X(t+s)-X(t)\big|.
\end{aligned}
\tag{3.5}
$$

It follows from (3.3)–(3.5) and the arbitrariness of $c_{3,1}$ that (3.1) holds.

Next, we prove (3.2). We will make use of the approach for limsup random fractals in Khoshnevisan *et al.* [17]. Without loss of generality, we assume $I=[0,1]^N$. For each $n\geq 1$, we set $r_n=2^{-n}$, $M_n:=2^n n$, $\mathcal{D}_n:=\{\mathbf{i}=(i_1,\dots,i_N)'\in\mathbb{Z}_+^N: i_k\in\{1,\dots,M_n\}, k=1,\dots,N\}$, and $\mathcal{N}_n:=\{\mathbf{i}=(i_1,\dots,i_N)'\in\mathbb{Z}_+^N: i_k\in\{1,\dots,n\}, k=1,\dots,N\}$. For each $\mathbf{i}=(i_1,\dots,i_N)'\in\mathcal{D}_n$, we define

$$t_{\mathbf{i},n}=\mathbf{i}r_n n^{-1}.$$

Fix an arbitrary constant $c_{3,2}>(dc_{2,2}/N)^{\nu/N}$. For each $\mathbf{i}\in\mathcal{D}_n$, we define a Bernoulli random variable $\eta_{\mathbf{i},n}$ that takes value 1 or 0 according as

$$\beta(r_n)\max_{1\leq\ell\leq d}\max_{\mathbf{k}\in\mathcal{N}_n}\big|X_\ell\big(t_{\mathbf{i},n}+\mathbf{k}r_n n^{-1}\big)-X_\ell(t_{\mathbf{i},n})\big|\leq c_{3,2} \tag{3.6}$$

or not. Define $S_n:=\sum_{\mathbf{i}\in\mathcal{D}_n}\eta_{\mathbf{i},n}$, the total number of $\mathbf{i}\in\mathcal{D}_n$ such that (3.6) holds. It follows from the stationarity of increments of X that, for each $n\geq 1$, the mean $p_n:=\mathbb{E}(\eta_{\mathbf{i},n})$ is the same for all $\mathbf{i}\in\mathcal{D}_n$. Moreover, by (2.12), we have uniformly over $\mathbf{i}\in\mathcal{D}_n$ (where n is large enough),

$$
\begin{aligned}
p_n&=\mathbb{P}(\eta_{\mathbf{i},n}=1)\\
&=\prod_{\ell=1}^d\mathbb{P}\Big(\beta(r_n)\max_{\mathbf{k}\in\mathcal{N}_n}\big|X_\ell\big(t_{\mathbf{i},n}+\mathbf{k}r_n n^{-1}\big)-X_\ell(t_{\mathbf{i},n})\big|\leq c_{3,2}\Big)\\
&\geq\prod_{\ell=1}^d\mathbb{P}\Big(\beta(r_n)\sup_{s\in[0,1]^N}\big|X_\ell(t_{\mathbf{i},n}+sr_n)-X_\ell(t_{\mathbf{i},n})\big|\leq c_{3,2}\Big)
\end{aligned}
$$

$$= \left(\mathbb{P}\left(\beta(r_n) \sup_{s\in[0,1]^N} \left|Y(sr_n)\right| \le c_{3,2}\right)\right)^d$$
$$\ge \exp\left(\left(c_{2,2}/c_{3,2}^{N/\nu}\right) d \log(r_n)\right). \tag{3.7}$$

In order to prove (3.2), we show that $S_n > 0$ for infinitely many n's. For this purpose, we estimate

$$\mathrm{Var}(S_n) = \sum_{\mathbf{i},\mathbf{j}\in\mathcal{D}_n} \mathrm{Cov}(\eta_{\mathbf{i},n}, \eta_{\mathbf{j},n}). \tag{3.8}$$

Set $\tau_n = n^{b/(2-2\nu)+1}[\min(L(1), L(r_n n^{-1}))]^{-1/(2-2\nu)}$, where $b > \max(1, N/2 + 3\nu)$ is a constant. We make the following claim: $\forall\delta > 0$, whenever $\|\mathbf{i} - \mathbf{j}\| \ge \tau_n$,

$$\mathbb{P}(\eta_{\mathbf{i},n} = 1, \eta_{\mathbf{j},n} = 1) \le (1 + \delta)\left(\mathbb{P}(\eta_{1,n} = 1)\right)^2. \tag{3.9}$$

Before we prove (3.9), let us complete the proof of (3.2).

It follows from (3.9) that for all $\delta > 0$, whenever $\mathbf{i}, \mathbf{j} \in \mathcal{D}_n$ satisfy $\|\mathbf{j} - \mathbf{i}\| \ge \tau_n$, then $\mathrm{Cov}(\eta_{\mathbf{i},n}, \eta_{\mathbf{j},n}) \le \delta\mathbb{E}(\eta_{\mathbf{i},n})\mathbb{E}(\eta_{\mathbf{i},n})$. Thus, by (3.8),

$$\mathrm{Var}(S_n) \le \delta M_n^{2N} p_n^2 + \sum_{\mathbf{i},\mathbf{j}\in\mathcal{D}_n:\|\mathbf{i}-\mathbf{j}\|\le\tau_n} \mathrm{Cov}(\eta_{\mathbf{i},n}, \eta_{\mathbf{j},n}).$$

For the remaining covariance, we use the fact that all $\eta_{\mathbf{i},n}$'s are either 0 or 1 to see that $\mathrm{Cov}(\eta_{\mathbf{i},n}, \eta_{\mathbf{j},n}) \le \mathbb{E}(\eta_{\mathbf{i},n}) = p_n$. Thus,

$$\mathrm{Var}(S_n) \le \delta M_n^{2N} p_n^2 + M_n^N p_n \tau_n^N.$$

It follows from the Paley-Zygmund inequality (see [16], p.8) that

$$\mathbb{P}(S_n > 0) \ge \frac{[\mathbb{E}(S_n)]^2}{\mathbb{E}(S_n^2)}.$$

Thus, we have

$$\mathbb{P}(S_n = 0) \le \frac{\mathrm{Var}(S_n)}{[\mathbb{E}(S_n)]^2} \le \delta + \frac{\tau_n^N}{M_n^N p_n} \tag{3.10}$$

since $\mathbb{E}(S_n) = M_n^N p_n$. On the other hand, it is known (see [5]) that for any $a > 0$, $s^a L(s) \to 0$ as $s \to 0$. Thus, by (3.7), (3.10) and the arbitrariness of δ, we see that $\mathbb{P}(S_n = 0) \to 0$ as $n \to \infty$. Finally

$$\mathbb{P}(S_n > 0 \text{ i.o.}) \ge \limsup_{n\to\infty} \mathbb{P}(S_n > 0) = 1,$$

where i.o. denotes infinitely often. This yields

$$\limsup_{n\to\infty} \beta(r_n) \min_{\mathbf{i}\in\mathcal{D}_n} \max_{1\le\ell\le d} \max_{\mathbf{k}\in\mathcal{N}_n} \left|X_\ell\left(t_{\mathbf{i},n} + \mathbf{k}r_n n^{-1}\right) - X_\ell(t_{\mathbf{i},n})\right| \le c_{3,2} \quad \text{a.s.} \tag{3.11}$$

Thus,

$$\limsup_{n\to\infty}\beta(r_n)\inf_{t\in I}\max_{1\le\ell\le d}\max_{\mathbf{k}\in\mathcal{N}_n}\bigl|X_\ell\bigl(t+\mathbf{k}r_nn^{-1}\bigr)-X_\ell(t)\bigr|\le c_{3,2}\quad\text{a.s.}\tag{3.12}$$

Noting $\|x\|\le d\max_{1\le\ell\le d}|x_\ell|$ for all $x=(x_1,\dots,x_d)'\in\mathbb{R}^d$, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\beta(r_n)\inf_{t\in I}M(t,r_n)\\
&\quad=\liminf_{n\to\infty}\beta(r_n)\inf_{t\in I}\sup_{u\in[0,1]^N}\bigl\|X(t+ur_n)-X(t)\bigr\|\\
&\quad\le\liminf_{n\to\infty}\beta(r_n)\inf_{t\in I}\max_{\mathbf{k}\in\mathcal{N}_n}\sup_{\substack{(k_i-1)n^{-1}\le u_i\le k_in^{-1}\\1\le i\le N}}\bigl\|X(t+ur_n)-X(t)\bigr\|\\
&\quad\le\liminf_{n\to\infty}\beta(r_n)\inf_{t\in I}\max_{\mathbf{k}\in\mathcal{N}_n}\bigl\|X\bigl(t+\mathbf{k}r_nn^{-1}\bigr)-X(t)\bigr\|\\
&\qquad+\limsup_{n\to\infty}\beta(r_n)\sup_{t\in I}\max_{\mathbf{k}\in\mathcal{N}_n}\sup_{\substack{(k_i-1)n^{-1}\le u_i\le k_in^{-1}\\1\le i\le N}}\bigl\|X(t)-X\bigl(t+\mathbf{k}r_nn^{-1}\bigr)\bigr\|\\
&\quad\le d\liminf_{n\to\infty}\beta(r_n)\inf_{t\in I}\max_{1\le\ell\le d}\max_{\mathbf{k}\in\mathcal{N}_n}\bigl|X_\ell\bigl(t+\mathbf{k}r_nn^{-1}\bigr)-X_\ell(t)\bigr|\\
&\qquad+\limsup_{n\to\infty}\beta(r_n)\sup_{t\in[0,2]^N}\max_{s\in[0,r_nn^{-1}]^N}\bigl\|X(t+s)-X(t)\bigr\|.
\end{aligned}\tag{3.13}$$

By making use of Lemma 2.4 and a standard Borel–Cantelli lemma argument, we have

$$\limsup_{n\to\infty}\beta(r_n)\sup_{t\in[0,2]^N}\max_{s\in[0,r_nn^{-1}]^N}\bigl\|X(t+s)-X(t)\bigr\|=0\quad\text{a.s.}\tag{3.14}$$

Hence, by (3.12)–(3.14) and the arbitrariness of $c_{3,2}$, (3.2) holds.

Therefore, it remains to prove (3.9). Write $Z_{\mathbf{i},n}(t)=Y(t_{\mathbf{i},n}+t)-Y(t_{\mathbf{i},n})$, $t\in I$. Then for $\mathbf{i},\mathbf{j}\in\mathcal{D}_n$ and $s,t\in[0,1]^N$, we have

$$\begin{aligned}
\mathbb{E}\bigl[Z_{\mathbf{i},n}(s)Z_{\mathbf{j},n}(t)\bigr]=\frac{1}{2}\bigl[&-\sigma^2\bigl((t_{\mathbf{j},n}-t_{\mathbf{i},n})+(t-s)\bigr)+\sigma^2\bigl((t_{\mathbf{j},n}-t_{\mathbf{i},n})-s\bigr)\\
&+\sigma^2\bigl((t_{\mathbf{j},n}-t_{\mathbf{i},n})+t\bigr)-\sigma^2(t_{\mathbf{j},n}-t_{\mathbf{i},n})\bigr].
\end{aligned}\tag{3.15}$$

Let $K(h)=\sigma^2(h)$ $(h\in\mathbb{R}_+^N)$, $\nabla K=(\partial K/\partial u_1,\dots,\partial K/\partial u_N)'$ and $H=(a_{km})_{N\times N}$ be the matrix with entries

$$a_{km}=\frac{\partial^2K}{\partial u_k\partial u_m}\bigl((t_{\mathbf{j},n}-t_{\mathbf{i},n})+\theta_3(t-\theta_1s+\theta_2s)\bigr),$$

where $\theta_1,\theta_2,\theta_3\in[0,1]$. We use Taylor's expansion in (3.15) to see that

$$\begin{aligned}
\mathbb{E}\bigl[Z_{\mathbf{i},n}(s)Z_{\mathbf{j},n}(t)\bigr]&=\frac{1}{2}\bigl(\nabla K\bigl((t_{\mathbf{j},n}-t_{\mathbf{i},n})+(t-\theta_1s)\bigr)-\nabla K\bigl((t_{\mathbf{j},n}-t_{\mathbf{i},n})-\theta_2s\bigr)\bigr)'s\\
&=\frac{1}{2}(t-\theta_1s+\theta_2s)'Hs.
\end{aligned}\tag{3.16}$$

Let $T_{\mathbf{i},n}(t)=(Z^1_{\mathbf{i},n}(t),\dots,Z^d_{\mathbf{i},n}(t))'$, $t\in\mathbb{R}^N$, where the coordinate processes $Z^1_{\mathbf{i},n},\dots,Z^d_{\mathbf{i},n}$ are independent copies of the Gaussian random field $Z_{\mathbf{i},n}=\{Z_{\mathbf{i},n}(t),t\in\mathbb{R}^N\}$. In the sequel, for ease of exposition, we order the points in $\mathcal{N}_n$ according to the following rule: for $\mathbf{i}=(i_1,\dots,i_N)$, $\mathbf{j}=(j_1,\dots,j_N)\in\mathcal{N}_n$, we define $\mathbf{i}\prec\mathbf{j}$ if there exists $1\le k\le N$ such that $i_1=j_1,\dots,i_{k-1}=j_{k-1},i_k<j_k$ with convention $i_0=j_0=0$. Consider the Gaussian random vectors $X'_{1,\ell}:=(Z^\ell_{\mathbf{i},n}(\mathbf{k}r_nn^{-1}),\mathbf{k}\in\mathcal{N}_n)$ and $X'_{2,\ell}:=(Z^\ell_{\mathbf{j},n}(\mathbf{m}r_nn^{-1}),\mathbf{m}\in\mathcal{N}_n)$, $1\le\ell\le d$. Set $X'_1=(X'_{1,1},\dots,X'_{1,d})$, $X'_2=(X'_{2,1},\dots,X'_{2,d})$ and $X'=(X'_1,X'_2)$. Notice that X is a $(2n^Nd)$-dimensional Gaussian random vector and its covariance matrix Σ is

$$\Sigma=\begin{pmatrix}\Sigma_1 & \Sigma_2\\ \Sigma_2' & \Sigma_1\end{pmatrix},$$

where $\Sigma_1=\operatorname{diag}(\mathbb{E}[X_{1,1}X'_{1,1}],\dots,\mathbb{E}[X_{1,d}X'_{1,d}])=\operatorname{diag}(\mathbb{E}[X_{2,1}X'_{2,1}],\dots,\mathbb{E}[X_{2,d}X'_{2,d}])$ and $\Sigma_2=\operatorname{diag}(\mathbb{E}[X_{1,1}X'_{2,1}],\dots,\mathbb{E}[X_{1,d}X'_{2,d}])$. Set $g(r_n)=c_{3,2}\beta^{-1}(r_n)$. By Lemma 2.5, we have

$$\begin{aligned}&\mathbb{P}\Big(\max_{1\le\ell\le d}\max_{\mathbf{k}\in\mathcal{N}_n}\big|Z^\ell_{\mathbf{i},n}\big(\mathbf{k}r_nn^{-1}\big)\big|\le g(r_n),\ \max_{1\le\ell\le d}\max_{\mathbf{m}\in\mathcal{N}_n}\big|Z^\ell_{\mathbf{j},n}\big(\mathbf{m}r_nn^{-1}\big)\big|\le g(r_n)\Big)\\&\quad\le\rho\mathbb{P}\Big(\max_{1\le\ell\le d}\max_{\mathbf{k}\in\mathcal{N}_n}\big|Z^\ell_{\mathbf{i},n}\big(\mathbf{k}r_nn^{-1}\big)\big|\le g(r_n)\Big)\\&\qquad\times\mathbb{P}\Big(\max_{1\le\ell\le d}\max_{\mathbf{m}\in\mathcal{N}_n}\big|Z^\ell_{\mathbf{j},n}\big(\mathbf{m}r_nn^{-1}\big)\big|\le g(r_n)\Big),\end{aligned}\tag{3.17}$$

where

$$\rho=\left(\frac{\det(\Sigma_1)\det(\Sigma_1)}{\det(\Sigma)}\right)^{1/2}.$$

The determinant of Σ is

$$\det(\Sigma)=\big[\det(\Sigma_1)\big]^2-\big[\det(\Sigma_2)\big]^2=\big[\det(\Sigma_{1,1})\big]^{2d}-\big[\det(\Sigma_{2,1})\big]^{2d},\tag{3.18}$$

where $\Sigma_{1,1}=\mathbb{E}[X_{1,1}X'_{1,1}]$ and $\det(\Sigma_{2,1})=\mathbb{E}[X_{1,1}X'_{2,1}]$.

We will make use of the following three facts:

(i) The determinant of the covariance matrix Σ_1 can be expressed as

$$\det(\Sigma_{1,1})=\operatorname{Var}\big(Z^1_{\mathbf{i},n}\big(\mathbf{1}r_nn^{-1}\big)\big)\prod_{\mathbf{k}\in\mathcal{N}_n\setminus\{\mathbf{1}\}}\operatorname{Var}\big(Z^1_{\mathbf{i},n}\big(\mathbf{k}r_nn^{-1}\big)|Z^1_{\mathbf{i},n}\big(\mathbf{m}r_nn^{-1}\big),\mathbf{m}\prec\mathbf{k}\big),$$

where $\operatorname{Var}U$ and $\operatorname{Var}(U|V)$ denote the variance of U and the conditional variance of U given V, respectively.

(ii) The property of strong local ϕ-nondeterminism of X (i.e., (C3)) implies

$$\begin{aligned}&\operatorname{Var}\big(Z^1_{\mathbf{i},n}\big(\mathbf{k}r_nn^{-1}\big)|Z^1_{\mathbf{i},n}\big(\mathbf{m}r_nn^{-1}\big),\mathbf{m}\prec\mathbf{k}\big)\\&\quad=\operatorname{Var}\big(Y\big(\mathbf{k}r_nn^{-1}\big)|Y\big(\mathbf{m}r_nn^{-1}\big),\mathbf{m}\prec\mathbf{k}\big)\ge c_{3,3}\phi\big(r_nn^{-1}\big).\end{aligned}$$

(iii) Hadamard's inequality (see [15], p. 506) implies

$$\det(\Sigma_{2,1}) \le \left(\prod_{\mathbf{k}\in\mathcal{N}_n}\left(\sum_{\mathbf{m}\in\mathcal{N}_n}|a_{\mathbf{km}}|^2\right)\right)^{1/2},$$

where $a_{\mathbf{km}} = \mathbb{E}[Z_{\mathbf{i},n}(\mathbf{k}r_n n^{-1})Z_{\mathbf{j},n}(\mathbf{m}r_n n^{-1})]$, $\mathbf{i},\mathbf{j}\in\mathcal{D}_n$ with $\mathbf{i}\neq\mathbf{j}$, and $\mathbf{k},\mathbf{m}\in\mathcal{N}_n$.

It follows from (i) and (ii) that

$$\det(\Sigma_{1,1}) \ge \left[c_{3,3}\phi\left(r_n n^{-1}\right)\right]^{n^N} \ge \left[r_n^{2\nu}n^{-3\nu}\right]^{n^N}\left[L\left(r_n n^{-1}\right)\right]^{n^N} \tag{3.19}$$

for all n large enough.

It follows from (3.16) that for all $\mathbf{i},\mathbf{j}\in\mathcal{D}_n$ with $\mathbf{i}\neq\mathbf{j}$, and $\mathbf{k},\mathbf{m}\in\mathcal{N}_n$,

$$\begin{aligned}
&\left|\mathbb{E}\left[Z_{\mathbf{i},n}\left(\mathbf{k}r_n n^{-1}\right)Z_{\mathbf{j},n}\left(\mathbf{m}r_n n^{-1}\right)\right]\right| \\
&\quad = (\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k})'H\mathbf{k}r_n^2 n^{-2} \\
&\quad \le c_{2,2}r_n^2 K\left(r_n n^{-1}\left(\mathbf{j}-\mathbf{i}+\theta_3(\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k})\right)\right) \\
&\qquad /(r_n n^{-1}\|\mathbf{j}-\mathbf{i}+\theta_3(\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k}\|)^2 \\
&\quad \le c_{2,3}\phi\left(r_n n^{-1}\left\|\mathbf{j}-\mathbf{i}+\theta_3(\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k})\right\|\right)/(n^{-1}\|\mathbf{j}-\mathbf{i}+\theta_3(\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k}\|)^2 \\
&\quad \le c_{2,3}r_n^{2\nu}L\left(r_n n^{-1}\left\|\mathbf{j}-\mathbf{i}+\theta_3(\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k})\right\|\right) \\
&\qquad /(n^{-1}\|\mathbf{j}-\mathbf{i}+\theta_3(\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k}\|)^{2-2\nu} \\
&\quad \le c_{2,3}r_n^{2\nu}\max\left(L(1), L\left(r_n n^{-1}\right)\right)/(n^{-1}\|\mathbf{j}-\mathbf{i}+\theta_3(\mathbf{m}-\theta_1\mathbf{k}+\theta_2\mathbf{k}\|)^{2-2\nu} \\
&\quad \le r_n^{2\nu}n^{-b}L\left(r_n n^{-1}\right)
\end{aligned} \tag{3.20}$$

if $\|\mathbf{j}-\mathbf{i}\|\ge\tau_n$. Thus, by (3.20) and (iii),

$$\det(\Sigma_{2,1}) \le \left[r_n^{2\nu}n^{-b}\right]^{n^N}\left[L\left(r_n n^{-1}\right)\right]^{n^N} n^{Nn^N/2}. \tag{3.21}$$

It follows from (3.19) and (3.21) that there exists a sequence $\{a_n\}$ of constants such that $a_n\to 0$ and

$$\det(\Sigma_{2,1}) \le a_n\det(\Sigma_{1,1}). \tag{3.22}$$

Thus, by (3.18) and (3.22),

$$\det(\Sigma) \ge \left(1-a_n^{2d}\right)\det(\Sigma_1)\det(\Sigma_1). \tag{3.23}$$

It follows that

$$\rho \le \left(1-a_n^{2d}\right)^{-1/2}.$$

Thus, by (3.17),

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\le\ell\le d}\max_{\mathbf{k}\in\mathcal{N}_n}\big|Z^{\ell}_{\mathbf{i},n}\big(\mathbf{k}r_n n^{-1}\big)\big|\le g(r_n),\ \max_{1\le\ell\le d}\max_{\mathbf{m}\in\mathcal{N}_n}\big|Z^{\ell}_{\mathbf{j},n}\big(\mathbf{m}r_n n^{-1}\big)\big|\le g(r_n)\Big)\\
&\quad\le\big(1-a_n^{2d}\big)^{-1/2}\mathbb{P}\Big(\max_{1\le\ell\le d}\max_{\mathbf{k}\in\mathcal{N}_n}\big|Z^{\ell}_{\mathbf{i},n}\big(\mathbf{k}r_n n^{-1}\big)\big|\le g(r_n)\Big)\\
&\qquad\times\mathbb{P}\Big(\max_{1\le\ell\le d}\max_{\mathbf{m}\in\mathcal{N}_n}\big|Z^{\ell}_{\mathbf{j},n}\big(\mathbf{m}r_n n^{-1}\big)\big|\le g(r_n)\Big).
\end{aligned}\tag{3.24}$$

Noting $(1-a_n^{2d})^{-1/2}=1+o(1)$ as $n\to\infty$, by (3.24), we have (3.9) holds. The proof of Theorem 1.2 is completed. □

4. Hölder conditions of the maximum local time

As Berman [4] pointed out, the irregularity of the sample paths of a stochastic process is closely related to the Hölder conditions for the local times. In this section, we apply Berman's observation and Theorem 1.2 to study the uniform Hölder conditions for the local times of Gaussian field X. Our Theorem 4.1 below solves a problem in Xiao [38].

For any rectangle $I\subset\mathbb{R}^N$ and $x\in\mathbb{R}^d$, the local time $L(x,I)$ of Gaussian field X is defined as the density of the occupation measure μ_I which is defined by

$$\mu_I(A)=\int_I\mathbb{I}_A\big(X(s)\big)\,ds,\qquad A\in\mathcal{B}\big(\mathbb{R}^d\big),$$

where $\mathbb{I}_A$ denotes the indicator function of A. It can be shown (see, e.g., [11], Theorem 6.4) that the following occupation density formula holds: for every Borel function $g(t,x)\ge 0$ on $I\times\mathbb{R}^d$,

$$\int_I g\big(t,X(t)\big)\,dt=\int_{\mathbb{R}^d}\int_I g(t,x)L(x,dt)\,dx.$$

For any rectangle $I\subseteq\mathbb{R}^N$, it follows from Pitt [28] or Kahane [16] that if

$$\int_I\int_I\frac{dt\,ds}{\sigma(\|t-s\|)}<\infty,$$

then almost surely the local time $L(x,I)$ of X exists and is square integrable. For $I\in\mathcal{B}(\mathbb{R}^N)$, define by $L^*(I)=\sup_{x\in\mathbb{R}^d}L(x,I)$ the maximum local time. Xiao [38] proved that if $N>\nu d$, then for any rectangle $I\subseteq\mathbb{R}^N$, there exists a positive constant $c_{4,1}$ such that

$$\limsup_{r\to0+}\sup_{t\in I}\frac{L^*(B(t,r))}{\Gamma(r)}\le c_{4,1}\quad\text{a.s.},\tag{4.1}$$

where $B(t,r)=\{x\in\mathbb{R}^N:\|x-t\|\le r\}$ and $\Gamma(r)=r^N[\phi(r/(\log(1/r))^{1/N})]^{-d/2}$.

The proof of (3.1) in Xiao [38] replies on the moment estimates for $L(x,B)$ and a chaining argument. However, the problem for proving an optimal lower bound corresponding to (4.1) (with possibly a different constant) had remained open. It turns out that the key for solving this problem is the modulus of non-differentiability of X. The result reads as follows.

Theorem 4.1. *Assume the conditions of Theorem 1.2 hold and $N > \nu d$. Then for any compact rectangle $I \subseteq \mathbb{R}^N$,*

$$c_{4,2} \le \limsup_{r\to 0+} \sup_{t\in I} \frac{L^*(B(t,r))}{\Gamma(r)} \le c_{4,1} \quad a.s. \tag{4.2}$$

Proof. As alluded to earlier, we need only to prove the left-hand side of (4.2). By the definition of local times, we have that for any Borel set $Q \subseteq \mathbb{R}^N$,

$$\begin{aligned} |Q| &= \int_{\overline{X(Q)}} L(x, Q)\, dx \\ &\le L^*(Q) \cdot \Big(\sup_{s,t\in Q} \big\| X(s) - X(t) \big\| \Big)^d, \end{aligned} \tag{4.3}$$

where $|Q|$ is the Lebesgue measure of Q, $\overline{X(Q)}$ is the closure of the set $X(Q) = \{X(t), t \in Q\}$. By making use of (1.6), with slight modifications (although Theorem 1.2 is concerned with the special case $t \in I$ and $s \in [0, r]^N$, from the proof of the theorem, it is not difficult to find out the following inequality holds), for every compact interval $I \subseteq \mathbb{R}^N$,

$$\liminf_{r\to 0+} \inf_{t\in I} \sup_{s\in B(t,r)} \frac{\|X(s) - X(t)\|}{[\phi(r/(\log(1/r))^{1/N})]^{1/2}} \le c_{4,3} \quad \text{a.s.}$$

Hence, a.s. there exist $t \in I$ and a sequence $\{r_n\}$ such that $r_n \downarrow 0$ and

$$\sup_{s\in B(t,r_n)} \big\| X(s) - X(t) \big\| \le c_{4,3} \big[\phi\big(r_n / \big(\log(1/r_n)\big)^{1/N} \big) \big]^{1/2}. \tag{4.4}$$

By taking $Q = B(t, r_n)$ in (4.3) and applying (4.4), we obtain that for some constant $c > 0$,

$$c r_n^N \big[\phi\big(r_n / \big(\log(1/r_n)\big)^{1/N} \big) \big]^{-d/2} \le L^*\big(B(t, r_n)\big),$$

which implies the left-hand side of (4.2). The proof of Theorem 4.1 is completed. □

5. Examples

In this section, we provide more examples of Gaussian random fields that satisfy Condition (C). These include fractional Riesz–Bessel processes and stationary Gaussian random fields in the Matérn and Cauchy classes.

5.1. Fractional Riesz–Bessel processes

Consider an (N, d)-Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ defined by (1.1), whose associate random field Y has a representation (2.3) with spectral density

$$f_{\gamma,\beta}(\lambda) = \frac{c(\gamma, \beta, N)}{\|\lambda\|^{2\gamma}(1 + \|\lambda\|^2)^\beta},$$

where γ and β are constants satisfying

$$\beta + \gamma > N/2, \quad 0 < \gamma < 1 + N/2 \tag{5.1}$$

and $c(\gamma, \beta, N) > 0$ is a normalizing constant. Since the spectral density $f_{\gamma,\beta}$ involves both the Fourier transforms of both the Riesz kernel and the Bessel kernel, following Anh *et al.* [2] we call X an (N, d)-fractional Riesz–Bessel process with indices γ and β. Anh *et al.* [2] suggested that these Gaussian random fields might be used for modelling simultaneously long range dependence and intermittency. More precisely, the index γ decides the long range dependence and the index $\nu = \gamma + \beta - N/2$ determines fractal and other sample path properties of X. For any fixed $t_0 \in \mathbb{R}^N_+$, the tangent field of fractional Riesz–Bessel process X at t_0 is fractional Brownian motion of Hurst index ν. This explains roughly why X shares many local properties with fractional Brownian motion.

Xiao [39] studied some sample path properties of an (N, d)-fractional Riesz–Bessel process. The following result is a corollary of Theorems 1.2 and 4.1.

Corollary 5.1. *Let $\{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-fractional Riesz–Bessel process with indices γ and β satisfying (5.1) and $0 < \gamma + \beta - \frac{N}{2} < 1$. Then, the following two statements hold:*

(1) *(1.6) holds with $\phi(x) = x^{2(\gamma+\beta)-N}$.*
(2) *If $N > (\gamma + \beta - N/2)d$, then (4.2) holds with $\phi(x) = x^{2(\gamma+\beta)-N}$.*

Proof. It follows from Theorem 2.5 in Xiao [39] that (C1) and (C3) hold. We only need to verify (C2). Since the spectral density $f_{\gamma,\beta}(\lambda)$ only depends on $\|\lambda\|$, $X(t)$ is isotropic. It follows from (2.4) and a change of variables that

$$\sigma^2(h) = c\|h\|^{2\nu} \int_0^\infty (1 - \cos\rho) \frac{d\rho}{\rho^{2\gamma+1-N}(\|h\|^2 + \rho^2)^\beta}. \tag{5.2}$$

By differentiating the function on the right-hand side of (5.2) twice one can verify (C2). This completes the proof of Corollary 5.1. □

5.2. The Matérn class

The Matérn covariance functions are widely used in spatial statistics, geostatistics, machine learning, image analysis, and other scientific areas. We refer to Guttorp and Gneiting [14] for a historical account on Matérn covariance functions, and to Chilés and Delfiner [6] and Stein [32] for their applications in statistics.

In Stein's parametrization (see [32], p. 31), the Matérn covariance function $M(h|\nu, \alpha, \phi)$ on $\mathbb{R}^N$, is defined as

$$M(h|\nu, \alpha, \phi) = \frac{\pi^{1/2} 2^{1-\nu} \phi}{\Gamma(\nu + 1/2)\alpha^{2\nu}} \big(\alpha\|h\|\big)^{\nu} K_\nu\big(\alpha\|h\|\big), \quad h \in \mathbb{R}^N, \tag{5.3}$$

where ν, α, ϕ are positive constants and K_ν is a modified Bessel function of the second kind. In this parametrization, the Matérn spectral density takes the following simple form (cf. [32]):

$$f(\lambda) = \frac{\phi}{(\alpha^2 + \|\lambda\|^2)^{\nu + N/2}}. \tag{5.4}$$

The parameter ν is critical and determines the smoothness of the corresponding Gaussian random field $Y = \{Y(t), t \in \mathbb{R}\}$. If $0 < \nu < 1$, then the sample function of Y is a.s. nowhere differentiable; if $\nu > 1$, then there is a modification of Y whose sample function is a.s. continuously differentiable. We say that an (N, d)-Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ belongs to the Matérn class if its associated random field Y is stationary with covariance function (5.3).

Given the spectral density function (5.4), one can use the same method as in the proof of Corollary 4.1 to conclude the following.

Corollary 5.2. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field that belongs to the Matérn class with smoothness parameter $\nu \in (0, 1)$. Then the following two statements hold*:

(1) (1.6) *holds with $\phi(x) = x^{2\nu}$.*
(2) *If $N > \nu d$, then* (4.2) *holds with $\phi(x) = x^{2\nu}$.*

5.3. The Cauchy class

The Cauchy class consists of stationary (N, d)-Gaussian random fields $X = \{X(t), t \in \mathbb{R}^N\}$ whose associated random field Y satisfies $\mathbb{E}[Y(t)] = 0$ and $\mathbb{E}[Y^2(t)] = 1$ for $t \in \mathbb{R}^N$, and has correlation function

$$C(h) = \big(1 + \|h\|^{\gamma}\big)^{-\beta/\gamma}, \quad h \in \mathbb{R}^N. \tag{5.5}$$

Any combination of the parameters $\gamma \in (0, 2]$ and $\beta > 0$ is permissible. The Cauchy class provides flexible power-law correlations and generalizes stochastic models recently discussed and synthesized in geostatistics (see [6,36]), physics (see [24,29]), hydrology (see [18]), and time series analysis (see [3,12]).

It follows from (5.5) that

$$C(h) \sim 1 - (\beta/\gamma)\|h\|^{\gamma}, \quad \text{as } \|h\| \to 0. \tag{5.6}$$

As shown by Gneiting and Schlather [13] or Lim and Teo [23], (5.6) implies that the fractal dimension of the graph of an (N, d)-Cauchy model $X = \{X(t), t \in \mathbb{R}^N\}$ is determined by γ.

On the other hand, X has long range dependence if and only if $\gamma \le N$. Therefore, the Gaussian random fields in the Cauchy class provide examples where the fractal index and the Hurst index of long memory may vary independently of each other.

The following corollary gives precise results on the modulus of non-differentiability and local times of the Cauchy random fields.

Corollary 5.3. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ belong to the Cauchy class with indices $\gamma \in (0,2)$ and $\beta > 0$. Then the following two statements hold*:

(1) (1.6) *holds with $\phi(x) = x^\gamma$.*
(2) *If $N > \gamma d/2$, then* (4.2) *holds with $\phi(x) = x^\gamma$.*

Proof. Since $\sigma^2(h) = 2 - 2C(h)$, one can verify directly that Conditions (C1) and (C2) hold and $\phi(x) = x^\gamma$.

In order to verify (C3), we first consider the spectral density $f(\lambda)$ of Y given by

$$f(\lambda) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\langle \lambda, h\rangle} C(h)\, dh \tag{5.7}$$

and study its asymptotic properties as $\|\lambda\| \to \infty$. Notice that, if $\beta > N$, then $C(\cdot) \in L^1(\mathbb{R}^N)$. In this case, the spectral density f is the inverse Fourier transform of $C(\cdot)$ and is continuous. If $\beta \le N$, (5.7) can be understood in the distribution sense. By Proposition 3.2 of Lim and Teo [23], one has

$$f(\lambda) \sim c_{5,1} \|\lambda\|^{-(N+\gamma)} \quad \text{as } \|\lambda\| \to \infty. \tag{5.8}$$

It follows from (5.8) and Theorem 2.1 of Xiao [39] that (C3) is satisfied with $\phi(x) = x^\gamma$. Hence, by Theorems 1.2 and 4.1, we obtain the desired results. □

Remark 5.1. When $\gamma = 2$ and $\beta > 0$, Proposition 3.2 of Lim and Teo [23] gives

$$f(\lambda) \sim c_{5,2} \|\lambda\|^{-(N+1-\beta)/2} e^{-\|\lambda\|} \quad \text{as } \|\lambda\| \to \infty.$$

In this case, it is not known if the corresponding Gaussian random field Y has the property of strong local nondeterminism. We are not able to provide precise information on the modulus of non-differentiability of Y.

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References

[1] Adler, R.J. and Taylor, J.E. (2007). *Random Fields and Geometry*. *Springer Monographs in Mathematics*. New York: Springer. MR2319516

[2] Anh, V.V., Angulo, J.M. and Ruiz-Medina, M.D. (1999). Possible long-range dependence in fractional random fields. *J. Statist. Plann. Inference* **80** 95–110. MR1713795 https://doi.org/10.1016/S0378-3758(98)00244-4

[3] Barndorff-Nielsen, O.E. (2000). Superposition of Ornstein–Uhlenbeck type processes. *Theory Probab. Appl.* **45** 175-194. https://doi.org/10.1137/S0040585X97978166

[4] Berman, S.M. (1972). Gaussian sample functions: Uniform dimension and Hölder conditions nowhere. *Nagoya Math. J.* **46** 63–86. MR0307320

[5] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation*. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge: Cambridge Univ. Press. MR0898871 https://doi.org/10.1017/CBO9780511721434

[6] Chilès, J.-P. and Delfiner, P. (1999). *Geostatistics*, *Modeling Spatial Uncertainty*. New York: Wiley. MR1679557 https://doi.org/10.1002/9780470316993

[7] Csörgő, M. and Révész, P. (1978). How small are the increments of a Wiener process? *Stochastic Process. Appl.* **8** 119–129. MR0520824 https://doi.org/10.1016/0304-4149(78)90001-7

[8] Csörgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics*. *Probability and Mathematical Statistics*. New York-London: Academic Press. MR0666546

[9] Csörgő, M. and Shao, Q.M. (1994). On almost sure limit inferior for B-valued stochastic processes and applications. *Probab. Theory Related Fields* **99** 29–54. MR1273741 https://doi.org/10.1007/BF01199589

[10] Dudley, R.M. (1973). Sample functions of the Gaussian process. *Ann. Probab.* **1** 66–103. MR0346884 https://doi.org/10.1214aop/1176997026

[11] Geman, D. and Horowitz, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67. MR0556414

[12] Gneiting, T. (2000). Power-law correlations, related models for long-range dependence and their simulation. *J. Appl. Probab.* **37** 1104–1109. MR1808873 https://doi.org/10.1239/jap/1014843088

[13] Gneiting, T. and Schlather, M. (2004). Stochastic models that separate fractal dimension and the Hurst effect. *SIAM Rev.* **46** 269–282. MR2114455 https://doi.org/10.1137/S0036144501394387

[14] Guttorp, P. and Gneiting, T. (2006). Studies in the history of probability and statistics. XLIX. On the Matérn correlation family. *Biometrika* **93** 989–995. MR2285084 https://doi.org/10.1093/biomet/93.4.989

[15] Horn, R.A. and Johnson, C.R. (2013). *Matrix Analysis*, 2nd ed. Cambridge: Cambridge Univ. Press. MR2978290

[16] Kahane, J.-P. (1985). *Some Random Series of Functions*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **5**. Cambridge: Cambridge Univ. Press. MR0833073

[17] Khoshnevisan, D., Peres, Y. and Xiao, Y. (2000). Limsup random fractals. *Electron. J. Probab.* **5** 1–24. MR1743726 https://doi.org/10.1214/EJP.v5-60

[18] Koutsoyiannis, D. (2000). A generalized mathematical framework for stochastic simulation and forecast of hydrologic time series. *Water Resour. Res.* **36** 1519–1533. https://doi.org/10.1029/2000wr900044

[19] Kuelbs, J. and Li, W.V. (1993). Metric entropy and the small ball problem for Gaussian measures. *J. Funct. Anal.* **116** 133–157. MR1237989 https://doi.org/10.1006/jfan.1993.1107

[20] Kuelbs, J., Li, W.V. and Talagrand, M. (1994). $\liminf$ results for Gaussian samples and Chung's functional LIL. *Ann. Probab.* **22** 1879–1903. MR1331209

[21] Li, W.V. and Linde, W. (1998). Existence of small ball constants for fractional Brownian motions. *C. R. Acad. Sci. Paris Sér. I Math.* **326** 1329–1334. MR1649147 https://doi.org/10.1016/S0764-4442(98)80189-4

[22] Li, W.V. and Shao, Q.-M. (2001). Gaussian processes: Inequalities, small ball probabilities and applications. In *Stochastic Processes*: *Theory and Methods*. *Handbook of Statist*. **19** 533–597. Amsterdam: North-Holland. MR1861734 https://doi.org/10.1016/S0169-7161(01)19019-X
[23] Lim, S.C. and Teo, L.P. (2009). Gaussian fields and Gaussian sheets with generalized Cauchy covariance structure. *Stochastic Process*. *Appl*. **119** 1325–1356. MR2508576 https://doi.org/10.1016/j.spa.2008.06.011
[24] Mandelbrot, B.B. (1982). *The Fractal Geometry of Nature*. San Francisco, CA: W. H. Freeman and Co. MR0665254
[25] Marcus, M.B. and Rosen, J. (2006). *Markov Processes*, *Gaussian Processes*, *and Local Times*. *Cambridge Studies in Advanced Mathematics* **100**. Cambridge: Cambridge Univ. Press. MR2250510 https://doi.org/10.1017/CBO9780511617997
[26] Meerschaert, M.M., Wang, W. and Xiao, Y. (2013). Fernique-type inequalities and moduli of continuity for anisotropic Gaussian random fields. *Trans*. *Amer*. *Math*. *Soc*. **365** 1081–1107. MR2995384 https://doi.org/10.1090/S0002-9947-2012-05678-9
[27] Monrad, D. and Rootzén, H. (1995). Small values of Gaussian processes and functional laws of the iterated logarithm. *Probab*. *Theory Related Fields* **101** 173–192. MR1318191 https://doi.org/10.1007/BF01375823
[28] Pitt, L.D. (1978). Local times for Gaussian vector fields. *Indiana Univ*. *Math*. *J*. **27** 309–330. MR0471055 https://doi.org/10.1512/iumj.1978.27.27024
[29] Romero, A.H. and Sancho, J.-M. (1999). Generation of short and long range temporal correlated noises. *J*. *Comput*. *Phys*. **156** 1–11. https://doi.org/10.1006/jcph.1999.6347
[30] Shao, Q.-M. (2003). A Gaussian correlation inequality and its applications to the existence of small ball constant. *Stochastic Process*. *Appl*. **107** 269–287. MR1999791 https://doi.org/10.1016/S0304-4149(03)00084-X
[31] Shao, Q.-M. and Wang, D. (1995). Small ball probabilities of Gaussian fields. *Probab*. *Theory Related Fields* **102** 511–517. MR1346263 https://doi.org/10.1007/BF01198847
[32] Stein, M.L. (1999). *Interpolation of Spatial Data*. *Springer Series in Statistics*. New York: Springer. MR1697409 https://doi.org/10.1007/978-1-4612-1494-6
[33] Talagrand, M. (1993). New Gaussian estimates for enlarged balls. *Geom*. *Funct*. *Anal*. **3** 502–526. MR1233864 https://doi.org/10.1007/BF01896240
[34] Talagrand, M. (1995). Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann*. *Probab*. **23** 767–775. MR1334170
[35] Talagrand, M. (1996). Lower classes for fractional Brownian motion. *J*. *Theoret*. *Probab*. **9** 191–213. MR1371076 https://doi.org/10.1007/BF02213740
[36] Wackernagel, H. (2003). *Multivariate Geostatistics*, 3rd ed. Berlin: Springer. https://doi.org/10.1007/978-3-662-03550-4
[37] Wang, W. and Xiao, Y. (2019). The Csörgő–Révész moduli of non-differentiability of fractional Brownian motion. *Statist*. *Probab*. *Lett*. **150** 81–87. MR3924525 https://doi.org/10.1016/j.spl.2019.02.016
[38] Xiao, Y. (1997). Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probab*. *Theory Related Fields* **109** 129–157. MR1469923 https://doi.org/10.1007/s004400050128
[39] Xiao, Y. (2007). Strong local nondeterminism and the sample path properties of Gaussian random fields. In *Asymptotic Theory in Probability and Statistics with Applications* (T. L. Lai, Q. M. Shao and L. Qian, eds.) 136–176. Beijing: Higher Education Press.
[40] Yaglom, A.M. (1957). Some classes of random fields in n-dimensional space, related to stationary random processes. *Theory Probab*. *Appl*. **2** 273-320. https://doi.org/10.1137/1102021

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