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Perturbation analysis of Poisson processes

GÜNTER LAST

Karlsruhe Institute of Technology, Institut für Stochastik, Kaiserstraße 89, D-76128 Karlsruhe, Germany. E-mail: guenter.last@kit.edu

We consider a Poisson process Φ on a general phase space. The expectation of a function of Φ can be considered as a functional of the intensity measure λ of Φ. Extending earlier results of Molchanov and Zuyev [*Math. Oper. Res.* **25** (2010) 485–508] on finite Poisson processes, we study the behaviour of this functional under signed (possibly infinite) perturbations of λ. In particular, we obtain general Margulis–Russo type formulas for the derivative with respect to non-linear transformations of the intensity measure depending on some parameter. As an application, we study the behaviour of expectations of functions of multivariate Lévy processes under perturbations of the Lévy measure. A key ingredient of our approach is the explicit Fock space representation obtained in Last and Penrose [*Probab. Theory Related Fields* **150** (2011) 663–690].

Keywords: Fock space representation; Lévy process; Margulis–Russo type formula; perturbation; Poisson process; variational calculus

1. Introduction

The aim of this paper is to advance the perturbation analysis of a *Poisson process* Φ on a general measurable space $(\mathbb{X}, \mathcal{X})$. For any σ-finite measure λ on $(\mathbb{X}, \mathcal{X})$, we let Π_λ denote the distribution of a Poisson process with *intensity measure* λ, see, for example, [12], Chapter 12. Further we let $\mathbb{P}_\lambda$ be a probability measure on some fixed measurable sample space such that $\mathbb{P}_\lambda(\Phi \in \cdot) = \Pi_\lambda$. We let $\mathbb{E}_\lambda$ denote the expectation operator with respect to $\mathbb{P}_\lambda$. Let $f(\Phi)$ be some (measurable) function of Φ. Under certain assumptions on f, Molchanov and Zuyev [18] showed for finite measures λ and ν the variational formula

$$\mathbb{E}_\nu f(\Phi) = \mathbb{E}_\lambda f(\Phi) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \bigl(\mathbb{E}_\lambda D^n_{x_1,\ldots,x_n} f(\Phi)\bigr)(\nu - \lambda)^n \bigl(\mathrm{d}(x_1, \ldots, x_n)\bigr), \tag{1.1}$$

where

$$D^n_{x_1,\ldots,x_n} f(\Phi) = \sum_{J \subset \{1,2,\ldots,n\}} (-1)^{n-|J|} f\biggl(\Phi + \sum_{j \in J} \delta_{x_j}\biggr), \qquad x_1, \ldots, x_n \in \mathbb{X}, n \in \mathbb{N}. \tag{1.2}$$

Here, $|J|$ denotes the number of elements of J, while δ_x is the Dirac measure located at a point $x \in \mathbb{X}$. It is common to say that ν results from λ by adding the *perturbation* $\nu - \lambda$.

In this paper, we shall extend (1.1) to σ-finite measures λ and ν. One can use a pathwise defined thinning and superposition construction to move from $\mathbb{P}_\lambda$ to $\mathbb{P}_\nu$, see Remark 4.2. In general, $\nu - \lambda$ is a *signed measure* that cannot be defined on the whole σ-field $\mathcal{X}$. Integration

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with respect to $\nu - \lambda$, however, is well defined via (4.1) below. Under an additional assumption on ν and λ (satisfied for positive, negative and many other perturbations of λ), we shall establish a condition that is necessary and sufficient for (1.1) to hold for all bounded functions of Φ. If, for instance, $\lambda \le \nu$, this condition is equivalent to the absolute continuity $\Pi_\lambda \ll \Pi_\nu$. The variational formula does not only hold for bounded functions but under a more general second moment assumption on f.

A consequence of (1.1) are derivative formulas of the form

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_{\lambda+\theta(\nu-\lambda)} f(\Phi) = \int \mathbb{E}_{\lambda+\theta(\nu-\lambda)} D_x f(\Phi)(\nu-\lambda)(\mathrm{d}x), \qquad \theta \in [0,1], \tag{1.3}$$

where $D_x := D_x^1$ is the first order difference (or add one cost) operator. This can be generalized to non-linear perturbations of λ and to more than one parameter. Such formulas are useful in the performance evaluation, optimization and simulation of discrete event systems [1,10]. Applications in a spatial setting can be found in [2,18]. Equation (1.3) can be seen as a Poisson version of the *Margulis–Russo formula* for Bernoulli random fields (see, e.g., [5]). Such formulae are, for instance, an important tool in both discrete and continuum percolation theory.

The extension of the identity (1.1) from finite to σ-finite measures is a non-trivial task. Our approach is based on a combination of the recent Fock space representation in [13] with classical results in [7] on the absolute continuity of Poisson process distributions. A related approach to derivatives of the type (1.3) for marked point processes on the real line was taken in [9]. For Poisson processes on the line and under a (rather strong) continuity assumptions on f the result (1.1) can be considered as a special case of the main result in [4].

The paper is organized as follows. In Section 2, we introduce some basic notation and recall facts about the Fock space representation and likelihood functions of Poisson processes. In Section 3, we use an elementary but illustrative argument to prove a simple version of (1.1). In Section 4, we prove and discuss Theorem 4.1, which is the main result of this paper. In Section 5, we derive conditions on λ and ν that are necessary for (1.1) to hold for all bounded functions f. In some cases these conditions are also sufficient. Section 6 gives general Margulis–Russo type formulas for derivatives. The final Section 7 treats perturbations of the Lévy measure of a Lévy process in $\mathbb{R}^d$.

2. Preliminaries

Let $\mathbf{N}$ be the space of integer-valued σ-finite measures φ on $\mathbb{X}$ equipped with the smallest σ-field $\mathcal{N}$ making the mappings $\varphi \mapsto \varphi(B)$ measurable for all $B \in \mathcal{X}$. We fix a measurable mapping $\Phi : \Omega \to \mathbf{N}$, where $(\Omega, \mathcal{F})$ is some abstract measurable (sample) space. For any σ-finite measure λ on $(\mathbb{X}, \mathcal{X})$ we let $\mathbb{P}_\lambda$ be a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}_\lambda(\Phi \in \cdot) = \Pi_\lambda$ is the distribution of a Poisson process with intensity measure λ.

For any measurable $f : \mathbf{N} \to \mathbb{R}$ and $x \in \mathbb{X}$ the function $D_x f$ on $\mathbf{N}$ is defined by

$$D_x f(\varphi) := f(\varphi + \delta_x) - f(\varphi), \qquad \varphi \in \mathbf{N}. \tag{2.1}$$

The *difference operator* D_x and its iterations play a central role in the variational analysis of Poisson processes. For $n \ge 2$ and $(x_1, \dots, x_n) \in \mathbb{X}^n$ we define a function $D^n_{x_1,\dots,x_n} f : \mathbf{N} \to \mathbb{R}$

inductively by

$$D^n_{x_1,\dots,x_n} f := D^1_{x_1} D^{n-1}_{x_2,\dots,x_n} f, \tag{2.2}$$

where $D^1 := D$ and $D^0 f = f$. Note that

$$D^n_{x_1,\dots,x_n} f(\varphi) = \sum_{J\subset\{1,2,\dots,n\}} (-1)^{n-|J|} f\Big(\varphi + \sum_{j\in J} \delta_{x_j}\Big), \tag{2.3}$$

where $|J|$ denotes the number of elements of J. This shows that the operator $D^n_{x_1,\dots,x_n}$ is symmetric in $x_1,\dots,x_n$, and that $(\varphi, x_1,\dots,x_n) \mapsto D^n_{x_1,\dots,x_n} f(\varphi)$ is measurable.

From [13], Theorem 1.1, we obtain for any measurable $f, g : \mathbf{N} \to \mathbb{R}$ satisfying $\mathbb{E}_\lambda f(\Phi)^2 < \infty$ and $\mathbb{E}_\lambda g(\Phi)^2 < \infty$ that

$$\mathbb{E}_\lambda f(\Phi)g(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_\lambda D^n_{x_1,\dots,x_n} f(\Phi)\big)\big(\mathbb{E}_\lambda D^n_{x_1,\dots,x_n} g(\Phi)\big)\lambda^n\big(\mathrm{d}(x_1,\dots,x_n)\big), \tag{2.4}$$

where the summand for $n = 0$ has to be interpreted as $(\mathbb{E}_\lambda f(\Phi))(\mathbb{E}_\lambda g(\Phi))$. (The integral of a constant c with respect to λ^0 is interpreted as c.)

Next, we recall a result from [7] in a slightly modified form. Consider two σ-finite measures ν, ρ on $\mathbb{X}$ such that $\nu \ll \rho$, that is, ν is absolutely continuous with respect to ρ. Let $h := \mathrm{d}\nu/\mathrm{d}\rho$ be the corresponding density (Radon–Nikodym derivative) and assume that

$$\int (h-1)^2 \,\mathrm{d}\rho < \infty. \tag{2.5}$$

This implies that the sets $C_n := \{|h-1| \ge 1/n\}$, $n \in \mathbb{N}$, have finite measure with respect to both ν and ρ, cf. also [7]. Define measurable functions $L_n : \mathbf{N} \to [0,\infty)$ by

$$L_n(\varphi) := \mathbf{1}\big\{\varphi(C_n) < \infty\big\} \mathrm{e}^{\rho(C_n)-\nu(C_n)} \prod_{y\in\varphi_{C_n}} h(y), \tag{2.6}$$

where φ_B is the restriction of $\varphi \in \mathbf{N}$ to a measurable set $B \subset \mathbb{X}$ and the product is over all points of the support of φ_{C_n} taking into account the multiplicities, that is,

$$\prod_{y\in\varphi_{C_n}} h(y) := \exp\bigg[\int_{C_n} \ln h(y)\varphi(\mathrm{d}y)\bigg],$$

where $\ln 0 := -\infty$.

Proposition 2.1. *With ν and ρ as above we have for any measurable $g : \mathbf{N} \to \mathbb{R}$ that*

$$\mathbb{E}_\nu g(\Phi) = \mathbb{E}_\rho L_{\nu,\rho}(\Phi) g(\Phi), \tag{2.7}$$

where

$$L(\varphi) := L_{\nu,\rho}(\varphi) := \liminf_{n\to\infty} L_n(\varphi) \tag{2.8}$$

if this limit inferior is finite and $L(\varphi) := L_{\nu,\rho}(\varphi) := 0$ *otherwise. Furthermore,*

$$\mathbb{E}_\rho L_{\nu,\rho}(\Phi)^2 < \infty. \tag{2.9}$$

Proof. It follows as in the proof of Theorem 1 in [7] that $L_n(\Phi)$ converges $\mathbb{P}_\rho$-a.s. to a random variable Y such that $\mathbb{E}_\nu g(\Phi) = \mathbb{E}_\rho Y g(\Phi)$ for all measurable $g : \mathbf{N} \to \mathbb{R}$. Hence, (2.7) holds. Furthermore, we have for any $n \in \mathbb{N}$ that

$$\begin{aligned}
\mathbb{E}_\rho L_n(\Phi)^2 &= \exp\bigl[2\rho(C_n) - 2\nu(C_n)\bigr]\mathbb{E}_\rho\Biggl[\prod_{y\in\Phi\cap C_n} h(y)^2\Biggr] \\
&= \exp\biggl[2\rho(C_n) - 2\int_{C_n} h\,\mathrm{d}\rho\biggr]\exp\biggl[\int_{C_n}\bigl(h^2-1\bigr)\,\mathrm{d}\rho\biggr] \\
&= \exp\biggl[\int_{C_n}(h-1)^2\,\mathrm{d}\rho\biggr],
\end{aligned}$$

where we have used a well-known property of Poisson processes to obtain the second equality. (Because $\rho(C_n) < \infty$ one can use a direct calculation based on the mixed sample representation or take $f := -\ln h^2$ in [12], Lemma 12.2(i), see also [16], 1.5.6.) Fatou's lemma implies that

$$\mathbb{E}_\rho L(\Phi)^2 \le \exp\biggl[\int (h-1)^2\,\mathrm{d}\rho\biggr],$$

which is finite by assumption (2.5). □

Remark 2.2. As noted above, (2.5) implies that $\Pi_\nu \ll \Pi_\rho$. The converse is generally not true. However, if h is bounded then (2.5) is necessary and sufficient for $\Pi_\nu \ll \Pi_\rho$. This follows from the main result in [7], see also Theorem 1.5.12 in [16].

3. Finite non-negative perturbations

In this section, we fix a σ-finite measure λ on $\mathbb{X}$ and a finite measure μ on $\mathbb{X}$. In this case, we can derive the variational formula (1.1) for $\nu := \lambda + \mu$ under a minimal integrability assumption on the function f. Our proof (basically taken from [18]) is elementary but instructive.

Theorem 3.1. *Let* $f : \mathbf{N} \to \mathbb{R}$ *be a measurable function such that* $\mathbb{E}_{\lambda+\mu}|f(\Phi)| < \infty$. *Then* (1.1) *holds, where all expectations exist and the series converges absolutely.*

Proof. We perform a formal calculation using Fubini's theorem. This will be justified below. Denoting the right-hand side of (1.1) by I, and using (1.2), we have that

$$I = \sum_{n=0}^{\infty} \frac{1}{n!} \int \Biggl(\sum_{J\subset\{1,\dots,n\}} (-1)^{n-|J|}\mathbb{E}_\lambda f\Biggl(\Phi + \sum_{j\in J}\delta_{x_j}\Biggr)\Biggr)\mu^n\bigl(\mathrm{d}(x_1,\dots,x_n)\bigr).$$

By symmetry,

$$
\begin{aligned}
I &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} \mu(\mathbb{X})^{n-m} \int \mathbb{E}_\lambda f(\Phi + \delta_{x_1} + \cdots + \delta_{x_m}) \mu^m \big(\mathrm{d}(x_1, \ldots, x_m)\big) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=m}^{\infty} \frac{(-1)^{n-m}}{(n-m)!} \mu(\mathbb{X})^{n-m} \int \mathbb{E}_\lambda f(\Phi + \delta_{x_1} + \cdots + \delta_{x_m}) \mu^m \big(\mathrm{d}(x_1, \ldots, x_m)\big) \\
&= \mathrm{e}^{-\mu(\mathbb{X})} \sum_{m=0}^{\infty} \frac{1}{m!} \int \mathbb{E}_\lambda f(\Phi + \delta_{x_1} + \cdots + \delta_{x_m}) \mu^m \big(\mathrm{d}(x_1, \ldots, x_m)\big) \\
&= \int \mathbb{E}_\lambda f(\Phi + \varphi) \Pi_\mu(\mathrm{d}\varphi),
\end{aligned}
$$

where in the last step we have used the mixed sample representation of finite Poisson processes, see, for example, [12], Theorem 12.7. Noting that the distribution $\mathbb{P}_{\lambda+\mu}(\Phi \in \cdot)$ is that of a sum of two independent Poisson processes with intensity measures λ and μ, respectively, we obtain that $I = \mathbb{E}_{\lambda+\mu} f(\Phi)$, as desired.

To justify the use of Fubini's theorem, we need to show that

$$
c := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} \mu(\mathbb{X})^{n-m} \int \mathbb{E}_\lambda \big| f(\Phi + \delta_{x_1} + \cdots + \delta_{x_m}) \big| \mu^m \big(\mathrm{d}(x_1, \ldots, x_m)\big) < \infty.
$$

By a similar calculation as above,

$$
\begin{aligned}
c &= \mathrm{e}^{\mu(\mathbb{X})} \sum_{m=0}^{\infty} \frac{1}{m!} \int \mathbb{E}_\lambda \big| f(\Phi + \delta_{x_1} + \cdots + \delta_{x_m}) \big| \mu^m \big(\mathrm{d}(x_1, \ldots, x_m)\big) \\
&= \mathrm{e}^{2\mu(\mathbb{X})} \mathbb{E}_{\lambda+\mu} \big| f(\Phi) \big| < \infty.
\end{aligned}
$$

This proves the theorem. □

4. General perturbations

In this section, we allow also signed and infinite perturbations of the intensity measure of Φ. This requires more advanced techniques, as the Fock space representation (2.4) and Proposition 2.1.

We consider two σ-finite measures λ and ν on $\mathbb{X}$. We take a σ-finite measure ρ dominating λ and ν, that is, $\lambda + \nu \ll \rho$. Let $h_\lambda := \mathrm{d}\lambda/\mathrm{d}\rho$, $h_\nu := \mathrm{d}\nu/\mathrm{d}\rho$. The integral of a measurable function $g : \mathbb{X}^n \to \mathbb{R}$ with respect to $(\nu - \lambda)^n$ is defined by

$$
\int g \, \mathrm{d}(\nu - \lambda)^n := \int g(x_1, \ldots, x_n)(h_\nu - h_\lambda)^{\otimes n}(x_1, \ldots, x_n) \rho^n \big(\mathrm{d}(x_1, \ldots, x_n)\big), \tag{4.1}
$$

where, for any function $h:\mathbb{X}\to\mathbb{R}$, the function $h^{\otimes n}:\mathbb{X}^n\to\mathbb{R}$ is given by

$$h^{\otimes n}(x_1,\dots,x_n):=\prod_{j=1}^n h(x_j).$$

Note that our definition of $\int g\,\mathrm{d}(\nu-\lambda)^n$ does not depend on the choice of ρ. The following theorem is the main result of this paper.

Theorem 4.1. *Assume that*

$$\int(1-h_\lambda)^2\,\mathrm{d}\rho+\int(1-h_\nu)^2\,\mathrm{d}\rho<\infty. \tag{4.2}$$

Let $f:\mathbf{N}\to\mathbb{R}$ be a measurable function such that $\mathbb{E}_\rho f(\Phi)^2<\infty$. Then

$$\sum_{n=0}^\infty\frac{1}{n!}\int\big|\mathbb{E}_\lambda D^n_{x_1,\dots,x_n}f(\Phi)\big||h_\nu-h_\lambda|^{\otimes n}(x_1,\dots,x_n)\rho^n\big(\mathrm{d}(x_1,\dots,x_n)\big)<\infty \tag{4.3}$$

and (1.1) *holds.*

Remark 4.2. Let Φ_λ be a Poisson process with intensity measure λ defined on some abstract probability space. Then we can use independent thinning and superposition to generate a Poisson process Φ_ν with intensity measure ν. Let $A:=\{x\in\mathbb{X}:h_\lambda(x)>h_\nu(x)\}$ and define $p:\mathbb{X}\to[0,1]$ by $p(x):=h_\nu(x)/h_\lambda(x)$ for $x\in A$ and by $p(x):=1$, otherwise. Let Φ' be a p-thinning of Φ_λ, see [12], Chapter 12. Then Φ' is a Poisson process with intensity measure

$$p(x)\lambda(\mathrm{d}x)=\mathbf{1}_A(x)h_\nu(x)\rho(\mathrm{d}x)+\mathbf{1}_{\mathbb{X}\setminus A}h_\lambda(x)\rho(\mathrm{d}x).$$

Let Φ'' be a Poisson process with intensity measure $\mathbf{1}_{\mathbb{X}\setminus A}(x)(h_\nu(x)-h_\lambda(x))\rho(\mathrm{d}x)$, independent of Φ'. Then $\Phi'+\Phi''$ is a Poisson process with intensity measure $h_\nu(x)\rho(\mathrm{d}x)=\nu(\mathrm{d}x)$. In some applications, it might be convenient to couple Φ_λ and the perturbed process Φ_ν in a different way. For instance, Φ_λ could be an independent marking of a homogeneous Poisson process of arrival times and one might wish to keep the times and to change only the marks.

Proof of Theorem 4.1. By assumption (4.2), we can apply Proposition 2.1 to both λ and ν. It follows from (2.6) and (2.8) that $L_{\lambda,\rho}(\Phi+\delta_x)=h_\lambda(x)L_{\lambda,\rho}(\Phi)$ for all $x\in\mathbb{X}$. Therefore,

$$D^n_{x_1,\dots,x_n}L_{\lambda,\rho}(\Phi)=L_{\lambda,\rho}(\Phi)\prod_{i=1}^n\big(h_\lambda(x_i)-1\big).$$

Since $\mathbb{E}_\rho L_{\lambda,\rho}(\Phi)=1$ we obtain that

$$\mathbb{E}_\rho D^n_{x_1,\dots,x_n}L_{\lambda,\rho}(\Phi)=\prod_{i=1}^n\big(h_\lambda(x_i)-1\big),\qquad x_1,\dots,x_n\in\mathbb{X},n\in\mathbb{N}. \tag{4.4}$$

Denoting the right-hand side of (1.1) by I, we have

$$I=\sum_{n=0}^{\infty}\frac{1}{n!}\int\big(\mathbb{E}_\rho L_{\lambda,\rho}(\Phi)D^n_{x_1,\dots,x_n}f(\Phi)\big)(\nu-\lambda)^n\big(\mathrm{d}(x_1,\dots,x_n)\big). \tag{4.5}$$

In the following, we assume that f is bounded, an assumption that will be removed in the final part of the proof. Then $D^n_{x_1,\dots,x_n}f(\Phi)$ is for any fixed $(x_1,\dots,x_n)$ bounded and hence square-integrable. Hence, we can apply (2.4) to the expectations in (4.5) and use (4.4) to obtain that

$$\begin{aligned}I&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n!}\frac{1}{k!}\iint\big(\mathbb{E}_\rho D^{n+k}_{y_1,\dots,y_k,x_1,\dots,x_n}f(\Phi)\big)\prod_{j=1}^{k}\big(h_\lambda(y_j)-1\big)\\&\qquad\times\rho^k\big(\mathrm{d}(y_1,\dots,y_k)\big)(\nu-\lambda)^n\big(\mathrm{d}(x_1,\dots,x_n)\big)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n!}\frac{1}{k!}\int\big(\mathbb{E}_\rho D^{n+k}_{x_1,\dots,x_{n+k}}f(\Phi)\big)\prod_{j=1}^{k}\big(h_\lambda(x_j)-1\big)\\&\qquad\times\prod_{j=k+1}^{n+k}\big(h_\nu(x_j)-h_\lambda(x_j)\big)\rho^{n+k}\big(\mathrm{d}(x_1,\dots,x_{n+k})\big),\end{aligned}$$

where the use of Fubini's theorem will be justified below. Swapping the order of summation, we obtain

$$\begin{aligned}I&=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{1}{(n-k)!}\frac{1}{k!}\int\big(\mathbb{E}_\rho D^{n}_{x_1,\dots,x_n}f(\Phi)\big)\prod_{j=1}^{k}\big(h_\lambda(x_j)-1\big)\\&\qquad\times\prod_{j=k+1}^{n}\big(h_\nu(x_j)-h_\lambda(x_j)\big)\rho^{n}\big(\mathrm{d}(x_1,\dots,x_n)\big)\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\int\sum_{k=0}^{n}\binom{n}{k}\prod_{j=1}^{k}\big(h_\lambda(x_j)-1\big)\prod_{j=k+1}^{n}\big(h_\nu(x_j)-h_\lambda(x_j)\big)\\&\qquad\times\big(\mathbb{E}_\rho D^{n}_{x_1,\dots,x_n}f(\Phi)\big)\rho^{n}\big(\mathrm{d}(x_1,\dots,x_n)\big)\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{j=1}^{n}\big(h_\nu(x_j)-1\big)\big(\mathbb{E}_\rho D^{n}_{x_1,\dots,x_n}f(\Phi)\big)\rho^{n}\big(\mathrm{d}(x_1,\dots,x_n)\big),\end{aligned}$$

where we have used

$$\prod_{j=1}^{n}\big(h_\nu(x_j)-h_\lambda(x_j)+h_\lambda(x_j)-1\big)=\sum_{J\subset\{1,\dots,n\}}\prod_{j_1\in J}\big(h_\lambda(x_{j_1})-1\big)\prod_{j_2\notin J}\big(h_\nu(x_{j_2})-h_\lambda(x_{j_2})\big)$$

and the permutation invariance of $(\mathbb{E}_\rho D^n_{x_1,\dots,x_n} f(\Phi))\rho^n(\mathrm{d}(x_1,\dots,x_n))$ to obtain the last equality. We are now using Proposition 2.1, the identity (4.4) with λ replaced by ν, and (2.4) to obtain that

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_\rho D^n L_{\nu,\rho}(\Phi)\big)\big(\mathbb{E}_\rho D^n f(\Phi)\big)\, \mathrm{d}\rho^n \\ &= \mathbb{E}_\rho L_{\nu,\rho}(\Phi) f(\Phi) = \mathbb{E}_\nu f(\Phi), \end{aligned}$$

where $D^n f(\varphi)$ denotes for any $\varphi \in \mathbf{N}$ the mapping $(x_1,\dots,x_n) \mapsto D^n_{x_1,\dots,x_n} f(\varphi)$. This proves (1.1) for bounded f.

To justify the formal calculation above and to establish (4.3), we need to show that

$$\begin{aligned} c := \sum_{n=0}^{\infty} \frac{1}{n!} \int \sum_{k=0}^{n} \binom{n}{k} \prod_{j=1}^{k} \big|h_\lambda(x_j) - 1\big| \prod_{j=k+1}^{n} \big|h_\nu(x_j) - h_\lambda(x_j)\big| \\ \times \big|\mathbb{E}_\rho D^n_{x_1,\dots,x_n} f(\Phi)\big| \rho^n\big(\mathrm{d}(x_1,\dots,x_n)\big) \end{aligned}$$

is finite. By permutation invariance,

$$\begin{aligned} c &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \big(|h_\lambda - 1| + |h_\nu - h_\lambda|\big)^{\otimes n} \big|\mathbb{E}_\rho D^n f(\Phi)\big|\, \mathrm{d}\rho^n \\ &\le \sum_{n=0}^{\infty} \frac{1}{n!} \int \big(2|h_\lambda - 1| + |h_\nu - 1|\big)^{\otimes n} \big|\mathbb{E}_\rho D^n f(\Phi)\big|\, \mathrm{d}\rho^n. \end{aligned}$$

The Cauchy–Schwarz inequality yields,

$$\begin{aligned} c &\le \sum_{n=0}^{\infty} \frac{\sqrt{a_n}}{n!} \bigg(\int \Big(\big(2|h_\lambda - 1| + |h_\nu - 1|\big)^{\otimes n}\Big)^2 \mathrm{d}\rho^n\bigg)^{1/2} \\ &= \sum_{n=0}^{\infty} \frac{\sqrt{a_n}}{n!} \bigg(\int \big(2|h_\lambda - 1| + |h_\nu - 1|\big)^2 \mathrm{d}\rho\bigg)^{n/2}, \end{aligned}$$

where

$$a_n := \int \big(\mathbb{E}_\rho D^n f(\Phi)\big)^2 \mathrm{d}\rho^n, \qquad n \in \mathbb{N}_0.$$

Applying Cauchy–Schwarz again, yields

$$c^2 \le \bigg(\sum_{n=0}^{\infty} \frac{a_n}{n!}\bigg) \sum_{n=0}^{\infty} \frac{1}{n!} \bigg(\int \big(2|h_\lambda - 1| + |h_\nu - 1|\big)^2 \mathrm{d}\rho\bigg)^n.$$

The first series in the above product converges by (2.4) (we have $\mathbb{E}_\rho f(\Phi)^2 < \infty$). The second series converges, since the integral there is finite by (4.2) and the Minkowski inequality.

We now extend the result to general f satisfying $\mathbb{E}_\rho f(\Phi)^2 < \infty$. We take a sequence of bounded functions f_l, $l \in \mathbb{N}$, such that $\mathbb{E}_\rho(f(\Phi) - f_l(\Phi))^2 \to 0$ as $l \to \infty$. We know already that

$$\mathbb{E}_\nu f_l(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_\lambda D^n f_l(\Phi)\big)\, \mathrm{d}(\nu - \lambda)^n \tag{4.6}$$

holds for all $l \in \mathbb{N}$. By Cauchy–Schwarz,

$$\begin{aligned} \mathbb{E}_\nu \big| f(\Phi) - f_l(\Phi)\big| &= \mathbb{E}_\rho L_{\nu,\rho} \big| f(\Phi) - f_l(\Phi)\big| \\ &\le \big(\mathbb{E}_\rho L_{\nu,\rho}^2\big)^{1/2} \big(\mathbb{E}_\rho\big(f(\Phi) - f_l(\Phi)\big)^2\big)^{1/2} \to 0 \end{aligned} \tag{4.7}$$

as $l \to \infty$. Hence, the left-hand side of (4.6) tends to $\mathbb{E}_\nu f(\Phi)$ as $l \to \infty$. To deal with the right-hand side, we consider sequences $\mathbf{g} = (g_n)_{n \ge 0}$, where $g_0 \in \mathbb{R}$ and g_n, $n \ge 1$, is a measurable function on $\mathbb{X}^n$. Introduce the space $\mathbf{V}$ of all such sequences satisfying

$$\|\mathbf{g}\| := \sum_{n=0}^{\infty} \frac{1}{n!} \int |g_n| |h_\nu - h_\lambda|^{\otimes n}\, \mathrm{d}\rho^n < \infty.$$

Then $\mathbf{V}$ is a direct sum of Banach spaces and hence a Banach space as well. For $l \in \mathbb{N}$ define

$$g_{l,n} := \mathbb{E}_\lambda D^n f_l(\Phi), \qquad n \ge 0, \qquad \mathbf{g}_l := (g_{l,n})_{n \ge 0} \in \mathbf{V}.$$

Our next aim is to show that $(\mathbf{g}_l)$ is a Cauchy-sequence. We have for $l, m \in \mathbb{N}$ that

$$\begin{aligned} \|\mathbf{g}_l - \mathbf{g}_m\| &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \big|\mathbb{E}_\lambda D^n f_l(\Phi) - \mathbb{E}_\lambda D^n f_m(\Phi)\big| |h_\nu - h_\lambda|^{\otimes n}\, \mathrm{d}\rho^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \big|\mathbb{E}_\rho L_{\lambda,\rho}(\Phi) D^n f_{l,m}(\Phi)\big| |h_\nu - h_\lambda|^{\otimes n}\, \mathrm{d}\rho^n, \end{aligned}$$

where $f_{l,m} := f_l - f_m$. From the calculation in the first part of the proof, we obtain that

$$\|\mathbf{g}_l - \mathbf{g}_m\| \le \sum_{n=0}^{\infty} \frac{1}{n!} \int \big|\mathbb{E}_\rho D^n f_{l,m}(\Phi)\big| \big(2|h_\lambda - 1| + |h_\nu - 1|\big)^{\otimes n}\, \mathrm{d}\rho^n.$$

Applying the Cauchy–Schwarz inequality twice, as in the second part of the proof yields

$$\|\mathbf{g}_l - \mathbf{g}_m\|^2 \le a \sum_{n=0}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_\rho D^n f_{l,m}(\Phi)\big)^2\, \mathrm{d}\rho^n,$$

where

$$a := \sum_{n=0}^{\infty} \frac{1}{n!} \bigg(\int \big(2|h_\lambda - 1| + |h_\nu - 1|\big)^2\, \mathrm{d}\rho \bigg)^n.$$

By (2.4),

$$\|\mathbf{g}_l - \mathbf{g}_m\|^2 \le a\mathbb{E}_\rho f_{l,m}(\Phi)^2 = a\mathbb{E}_\rho\big(f_l(\Phi) - f_m(\Phi)\big)^2.$$

By the choice of f_l the sequence $(\mathbf{g}_l)$ has the Cauchy property. Because $\mathbf{V}$ is complete, there is a $\mathbf{g} = (g_n) \in \mathbf{V}$ such that $\|\mathbf{g}_l - \mathbf{g}\| \to 0$ as $l \to \infty$. Since,

$$\left|\sum_{n=0}^{\infty} \frac{1}{n!} \int g_n \mathrm{d}(\nu - \lambda)^n - \sum_{n=0}^{\infty} \frac{1}{n!} \int g_{l,n}\, \mathrm{d}(\nu - \lambda)^n\right| \le \sum_{n=0}^{\infty} \frac{1}{n!} \int |g_n - g_{l,n}||h_\nu - h_\lambda|^{\otimes n}\, \mathrm{d}\rho^n,$$

we obtain from (4.6) and (4.7) that

$$\mathbb{E}_\nu f(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int g_n \mathrm{d}(\nu - \lambda)^n.$$

It remains to show that, for any fixed $n \ge 0$,

$$|h_\nu - h_\lambda|^{\otimes n} g_n = |h_\nu - h_\lambda|^{\otimes n} \mathbb{E}_\lambda D^n f(\Phi), \qquad \rho^n\text{-a.e.} \tag{4.8}$$

We claim that

$$\lim_{l\to\infty} \int_{B^n} \mathbb{E}_\lambda \big|D^n f(\Phi) - D^n f_l(\Phi)\big|\, \mathrm{d}\rho^n = 0 \tag{4.9}$$

for all $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $\rho(B) < \infty$. As in the proof of [13], Lemma 2.3, it suffices to demonstrate that

$$\lim_{l\to\infty} \int_{B^m} \mathbb{E}_\lambda \left| f\left(\Phi + \sum_{i=1}^{m} \delta_{y_i}\right) - f_l\left(\Phi + \sum_{i=1}^{m} \delta_{y_i}\right)\right| \rho^m\big(\mathrm{d}(y_1, \dots, y_m)\big) = 0 \tag{4.10}$$

for all $m \in \{1, \dots, n\}$. By the (multivariate) Mecke equation (see, e.g., [17] or [13], (2.10)) the integral in (4.10) equals

$$\mathbb{E}_\rho \int_{B^m} L_{\lambda,\rho}(\Phi - \delta_{y_1} - \cdots - \delta_{y_m})\big|f(\Phi) - f_l(\Phi)\big|\Phi^{(m)}\big(\mathrm{d}(y_1, \dots, y_m)\big), \tag{4.11}$$

where, for $\varphi \in \mathbf{N}$, $\varphi^{(m)}$ is the measure on $\mathbb{X}^m$ defined by

$$\begin{aligned}\varphi^{(m)}(C) := \int \cdots \int \mathbf{1}_C(y_1, \dots, y_m)&\left(\varphi - \sum_{j=1}^{m-1} \delta_{y_j}\right)(\mathrm{d}y_m)\\ &\times \left(\varphi - \sum_{j=1}^{m-2} \delta_{y_j}\right)(\mathrm{d}y_{m-1}) \times \cdots \\ &\times (\varphi - \delta_{y_1})(\mathrm{d}y_2)\varphi(\mathrm{d}y_1), \qquad C \in \mathcal{X}^{\otimes m}.\end{aligned} \tag{4.12}$$

By Lemma 4.3 below and the Cauchy–Schwarz inequality, (4.11) tends to 0 as $l \to \infty$. Now (4.9) implies that $g_{l,n} = \mathbb{E}_\lambda D^n f_l(\Phi)$ tends to $\mathbb{E}_\lambda D^n f(\Phi)$ ρ^n-a.e. on B^n as $l \to \infty$ along a subsequence. Since

$$\lim_{l\to\infty} \int |g_n - g_{l,n}||h_\nu - h_\lambda|^{\otimes n} \, \mathrm{d}\rho^n = 0,$$

there is a further subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $|h_\nu - h_\lambda|^{\otimes n} g_{l,n}$ tends to $|h_\nu - h_\lambda|^{\otimes n} g_n$ ρ^n-a.e. on B^n as $l \to \infty$ along $\mathbb{N}'$. It follows that (4.8) holds for ρ^n restricted to B^n. Since ρ and λ are σ-finite we obtain (4.8). This completes the proof of the theorem. $\square$

In the final part of the above proof, we have used the following lemma. Recall the definition (4.12).

Lemma 4.3. *Assume that* (4.2) *holds and let $B \in \mathcal{X}$ satisfy $\lambda(B) < \infty$ and $\rho(B) < \infty$. Then we have for all $m \geq 1$ that*

$$\mathbb{E}_\rho \left(\int_{B^m} L_{\lambda,\rho}(\Phi - \delta_{x_1} - \cdots - \delta_{x_m}) \Phi^{(m)}\big(\mathrm{d}(x_1, \ldots, x_m)\big) \right)^2 < \infty.$$

Proof. Writing the square of the inner integral as a double integral and using a combinatorial argument, we see that it suffices to prove that

$$\begin{aligned}\mathbb{E}_\rho \int_{B^{m-k}} \int_{B^m} & L_{\lambda,\rho}(\Phi - \delta_{x_1} - \cdots - \delta_{x_m}) L_{\lambda,\rho}(\Phi - \delta_{x_1} - \cdots - \delta_{x_k} - \delta_{y_1} - \cdots - \delta_{y_{m-k}}) \\ & \times (\Phi - \delta_{y_1} - \cdots - \delta_{y_{m-k}})^{(m)}\big(\mathrm{d}(x_1, \ldots, x_m)\big) \Phi^{(m-k)}\big(\mathrm{d}(y_1, \ldots, y_{m-k})\big) < \infty\end{aligned}$$

for all $k \in \{0, \ldots, m\}$ (with the obvious convention for $k = m$). Applying the Mecke equation twice, we obtain that this expression equals

$$\begin{aligned}\mathbb{E}_\rho \int_{B^{m-k}} \int_{B^m} & L_{\lambda,\rho}(\Phi + \delta_{y_1} + \cdots + \delta_{y_{m-k}}) L_{\lambda,\rho}(\Phi + \delta_{x_{k+1}} + \cdots + \delta_{x_m}) \\ & \times \rho^m\big(\mathrm{d}(x_1, \ldots, x_m)\big) \rho^{m-k}\big(\mathrm{d}(y_1, \ldots, y_{m-k})\big).\end{aligned}$$

Since

$$L_{\lambda,\rho}(\Phi + \delta_{y_1} + \cdots + \delta_{y_{m-k}}) = L_{\lambda,\rho}(\Phi) h_\lambda(y_1) \times \cdots \times h_\lambda(y_{m-k}),$$

we obtain that the above expectation equals $\rho(B)^{m-k} \lambda(B)^m \mathbb{E}_\rho L_{\lambda,\rho}(\Phi)^2$ which is finite by Proposition 2.1. $\square$

Remark 4.4. In the case $\rho = \lambda$ (this requires $\nu \ll \lambda$) the proof of Theorem 4.1 becomes considerably simpler. Another simplification is possible if $\mathbb{E}_\rho f(\Phi)^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$. Then $\mathbb{E}_\rho (D^n_{x_1,\ldots,x_n} f(\Phi))^2 < \infty$ for all $n \geq 1$ and ρ^n-a.e. $(x_1, \ldots, x_n)$. Indeed, by the proof of Lemma 2.3 in [13] it is enough to show that $\mathbb{E}_\rho f(\Phi)^2 \Phi(B)^k < \infty$ for all $k \in \mathbb{N}$ and any $B \in \mathcal{X}$ with $\rho(B) < \infty$. Since $\Phi(B)$ has finite moments of any order, this is a direct consequence of

Hölder's inequality. We can then apply (2.4) to the expectations in (4.5) and proceed exactly as in the proof of Theorem 4.1. This makes the final (and somewhat tricky) part of this proof superfluous.

We continue with providing special cases of Theorem 4.1. We let $\nu = \nu_1 + \nu_2$ (resp., $\lambda = \lambda_1 + \lambda_2$) be the Lebesgue decomposition of ν (resp., λ) with respect to λ (resp., ν). Hence $\nu_1 \ll \lambda$ and $\nu_2 \perp \lambda$, where the latter means that ν_2 and λ are *singular*, that is concentrated on disjoint measurable subsets of $\mathbb{X}$.

Theorem 4.5. *Let $f : \mathbf{N} \to \mathbb{R}$ be measurable. Assume that either*

$$\int \left(1 - \frac{\mathrm{d}\nu_1}{\mathrm{d}\lambda}\right)^2 \mathrm{d}\lambda + \nu_2(\mathbb{X}) < \infty \tag{4.13}$$

and $\mathbb{E}_{\lambda+\nu_2} f(\Phi)^2 < \infty$, or that

$$\int \left(1 - \frac{\mathrm{d}\lambda_1}{\mathrm{d}\nu}\right)^2 \mathrm{d}\nu + \lambda_2(\mathbb{X}) < \infty \tag{4.14}$$

and $\mathbb{E}_{\nu+\lambda_2} f(\Phi)^2 < \infty$. Then (1.1) holds.

Proof. We prove only the first assertion. There are disjoint measurable subsets B_1 and B_2 of $\mathbb{X}$ such that

$$\lambda(\mathbb{X} \setminus B_1) = \nu_2(\mathbb{X} \setminus B_2) = 0. \tag{4.15}$$

In particular, $\nu_1(\mathbb{X} \setminus B_1) = 0$. Let $\rho := \lambda + \nu_2$. It is easy to check that

$$h_\lambda = \mathbf{1}_{B_1}, \qquad h_\nu = \mathbf{1}_{B_1} h_1 + \mathbf{1}_{B_2},$$

where $h_1 := \mathrm{d}\nu_1/\mathrm{d}\lambda$. We have

$$\int (h_\nu - 1)^2 \,\mathrm{d}\rho = \int (\mathbf{1}_{B_1} h_1 - \mathbf{1}_{\mathbb{X}\setminus B_2})^2 \,\mathrm{d}\rho = \int (\mathbf{1}_{B_1} h_1 - \mathbf{1}_{\mathbb{X}\setminus B_2})^2 \,\mathrm{d}\lambda = \int (h_1 - 1)^2 \,\mathrm{d}\lambda$$

and $\int (h_\lambda - 1)^2 \,\mathrm{d}\rho = \rho(\mathbb{X} \setminus B_1) = \nu_2(\mathbb{X})$. Therefore, (4.2) holds and the result follows from Theorem 4.1. □

The next corollary deals with a monotone perturbation of λ.

Corollary 4.6. *Let μ be a σ-finite measure on $\mathbb{X}$ and assume that $h := \mathrm{d}\lambda/\mathrm{d}(\lambda + \mu)$ satisfies*

$$\int (1 - h)^2 \,\mathrm{d}(\lambda + \mu) < \infty. \tag{4.16}$$

Then we have for all measurable f with $\mathbb{E}_{\lambda+\mu} f(\Phi)^2 < \infty$ that

$$\mathbb{E}_{\lambda+\mu} f(\Phi) = \mathbb{E}_\lambda f(\Phi) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_\lambda D^n_{x_1,\dots,x_n} f(\Phi)\big) \mu^n\big(\mathrm{d}(x_1,\dots,x_n)\big). \tag{4.17}$$

Proof. Apply the second part of Theorem 4.5 with $\nu = \lambda + \mu$. Then $\lambda_2 = 0$ and $\mathrm{d}\lambda/\mathrm{d}\nu = h$. $\square$

Remark 4.7. In the situation of Corollary 4.6, we may assume that $h \le 1$. Then $1-h$ is a density of μ with respect to $\lambda + \mu$, so that $\int (1-h)^2 \,\mathrm{d}(\lambda+\mu) = \int (1-h)\,\mathrm{d}\mu$. In particular, $\mu(\mathbb{X}) < \infty$ implies (4.16), cf. Theorem 3.1.

The results of this section can be extended so as to cover additional randomization.

Remark 4.8. Let $(\mathbb{Y}, \mathcal{Y})$ be a measurable space and $\eta : \Omega \to \mathbb{Y}$ be a measurable mapping such that $\mathbb{P}_\lambda((\eta,\Phi) \in \cdot) = \mathbb{V} \otimes \Pi_\lambda$ for all σ-finite measures λ, where $\mathbb{V}$ is a probability measure on $(\mathbb{Y}, \mathcal{Y})$, not depending on λ. The definition of the difference operator can be extended to measurable functions $f : \mathbb{Y} \times \mathbf{N} \to \mathbb{R}$ in the following natural way. If $n \in \mathbb{N}$ and $x_1,\dots,x_n \in \mathbb{X}$ then $D^n_{x_1,\dots,x_n} f : \mathbb{Y} \times \mathbf{N} \to \mathbb{R}$ is defined by $D^n_{x_1,\dots,x_n} f(y,\varphi) := D^n_{x_1,\dots,x_n} f_y(\varphi)$, where $f_y := f(y,\cdot)$, $y \in \mathbb{Y}$. Assume now that λ, ν, ρ satisfy the assumptions of Theorem 4.1 and that $\mathbb{E}_\rho f(\eta,\Phi)^2 < \infty$. We claim that (4.3) and (1.1) hold when replacing Φ by (η,Φ). This implies that all results of this section (as well as those of Section 6) remain valid with the obvious changes.

To verify the above claim we define, for any $\varphi \in \mathbf{N}$, $\tilde{f}(\varphi) := \int f(y,\varphi)\mathbb{V}(\mathrm{d}y)$ and conclude from Jensen's inequality that $\mathbb{E}_\rho \tilde{f}(\Phi)^2 \le \mathbb{E}_\rho f(\eta,\Phi)^2 < \infty$. Hence, Theorem 4.1 applies and we need to show for all $n \in \mathbb{N}$ that

$$\mathbb{E}_\lambda D^n_{x_1,\dots,x_n} \tilde{f}(\Phi) = \mathbb{E}_\lambda D^n_{x_1,\dots,x_n} f(\eta,\Phi), \qquad \rho^n\text{-a.e. } (x_1,\dots,x_n).$$

In view of (1.2) and Fubini's theorem it is sufficient to show for all $m \ge 0$ that

$$\mathbb{E}_\lambda \int \big|f(y, \Phi + \delta_{x_1} + \cdots + \delta_{x_m})\big| \mathbb{V}(\mathrm{d}y) = \mathbb{E}_\lambda \big|f(\eta, \Phi + \delta_{x_1} + \cdots + \delta_{x_m})\big| < \infty$$

for ρ^m-a.e. $(x_1,\dots,x_m)$ (with the obvious convention for $m = 0$). To this end, we take $B_1,\dots,B_m \in \mathcal{X}$ with finite measure with respect to both λ and ρ, let $B := B_1 \times \cdots \times B_m$, and obtain from the Mecke equation that

$$\begin{aligned}
&\int_B \mathbb{E}_\lambda \big|f(\eta, \Phi + \delta_{x_1} + \cdots + \delta_{x_m})\big| \rho^m\big(\mathrm{d}(x_1,\dots,x_m)\big) \\
&\quad = \mathbb{E}_\rho \int_B L_{\lambda,\rho}(\Phi) \big|f(\eta, \Phi + \delta_{x_1} + \cdots + \delta_{x_m})\big| \rho^m\big(\mathrm{d}(x_1,\dots,x_m)\big) \\
&\quad = \mathbb{E}_\rho \big|f(\eta,\Phi)\big| \int_B L_{\lambda,\rho}(\Phi - \delta_{x_1} - \cdots - \delta_{x_m}) \Phi^{(m)}\big(\mathrm{d}(x_1,\dots,x_m)\big),
\end{aligned}$$

which is finite by Cauchy–Schwarz, Lemma 4.3 and our assumption $\mathbb{E}_\rho f(\eta,\Phi)^2 < \infty$.

5. Necessary conditions for the variational formulas

Again we consider two σ-finite measures λ, ν on $\mathbb{X}$. The squared *Hellinger distance* between these two measures is defined as

$$H(\lambda, \nu) := \frac{1}{2} \int (\sqrt{h_\lambda} - \sqrt{h_\nu})^2 \, \mathrm{d}\rho, \tag{5.1}$$

where (as before) ρ is a σ-finite measure dominating λ and ν and h_λ, respectively, h_ν are the corresponding densities.

Theorem 5.1. *Assume that* (1.1) *holds for all bounded measurable* $f : \mathbf{N} \to \mathbb{R}$. *Then* Π_λ *and* Π_ν *are not singular and*

$$H(\lambda, \nu) < \infty. \tag{5.2}$$

Proof. Assume on the contrary that Π_λ and Π_ν are singular. Then we find disjoint sets $F, G \in \mathcal{N}$ such that $\Pi_\lambda(F) = \Pi_\nu(G) = 1$. We now proceed as in the proof of Theorem 9.1.13 in [16]. Let $C_n \in \mathcal{X}$, $n \in \mathbb{N}$, be such that $\lambda(C_n) + \nu(C_n) < \infty$, and $C_n \uparrow \mathbb{X}$ as $n \to \infty$. Recall that the restriction of $\varphi \in \mathbf{N}$ to $B \in \mathcal{X}$ is denoted by φ_B. We have for any $n \in \mathbb{N}$ that

$$\begin{aligned}
\exp\bigl[-\lambda(C_n)\bigr] &= \mathbb{P}_\lambda\bigl(\Phi(C_n) = 0, \Phi \in F\bigr) = \mathbb{P}_\lambda\bigl(\Phi(C_n) = 0, \Phi_{\mathbb{X}\setminus C_n} \in F\bigr) \\
&= \mathbb{P}_\lambda\bigl(\Phi(C_n) = 0\bigr)\mathbb{P}_\lambda(\Phi_{\mathbb{X}\setminus C_n} \in F) = \exp\bigl[-\lambda(C_n)\bigr]\mathbb{P}_\lambda(\Phi_{\mathbb{X}\setminus C_n} \in F).
\end{aligned}$$

A similar calculation applies to $\mathbb{P}_\nu$. It follows that the sets

$$F_n := \bigcap_{m=n}^{\infty} \{\varphi \in \mathbf{N} : \varphi_{\mathbb{X}\setminus C_n} \in F\}, \qquad G_n := \bigcap_{m=n}^{\infty} \{\varphi \in \mathbf{N} : \varphi_{\mathbb{X}\setminus C_n} \in G\}, \qquad n \in \mathbb{N}, \tag{5.3}$$

have the properties

$$\mathbb{P}_\lambda(\Phi \in F_n) = \mathbb{P}_\nu(\Phi \in G_n) = 1, \qquad n \in \mathbb{N}.$$

This implies

$$\mathbb{P}_\lambda\bigl(\Phi \in F'\bigr) = \mathbb{P}_\nu\bigl(\Phi \in G'\bigr) = 1, \tag{5.4}$$

where $F' := \bigcup_{n\in\mathbb{N}} F_n$ and $G' := \bigcup_{n\in\mathbb{N}} G_n$. Since $F \cap G = \varnothing$ we have $F_n \cap G_n = \varnothing$ for any $n \in \mathbb{N}$. Since F_n and G_n are increasing, we obtain that $F' \cap G' = \varnothing$. Since $C_n \uparrow \mathbb{X}$ we have for any $(\varphi, x) \in \mathbf{N} \times \mathbb{X}$ that $\varphi \in F'$ if and only if $\varphi + \delta_x \in F'$. Therefore, for $f := \mathbf{1}_{F'}$, $D^n_{x_1,\dots,x_n} f \equiv 0$ for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in \mathbb{X}$. Using this fact as well as (5.4) (together with $F' \cap G' = \varnothing$), we see that (1.1) fails.

A classical result by Liese [14] (see also [15], Theorem (3.30)) says that

$$H(\Pi_\lambda, \Pi_\nu) = 1 - \mathrm{e}^{-H(\lambda,\nu)} \tag{5.5}$$

so that singularity of Π_λ and Π_ν is equivalent to $H(\lambda, \nu) = \infty$ (see [14], (3.2)). $\square$

Recall the Lebesgue decompositions $\nu = \nu_1 + \nu_2$ of ν with respect to λ and $\lambda = \lambda_1 + \lambda_2$ of λ with respect to ν.

Corollary 5.2. *Assume that* (1.1) *holds for all bounded measurable* $f : \mathbf{N} \to \mathbb{R}$. *Then*

$$\int \left(1 - \sqrt{\frac{d\nu_1}{d\lambda}}\right)^2 d\lambda + \nu_2(\mathbb{X}) + \int \left(1 - \sqrt{\frac{d\lambda_1}{d\nu}}\right)^2 d\nu + \lambda_2(\mathbb{X}) < \infty. \tag{5.6}$$

Moreover, we have that $\Pi_{\nu_1} \ll \Pi_\lambda$ *and* $\Pi_{\lambda_1} \ll \Pi_\nu$ *and in particular* $\Pi_\nu \ll \Pi_\lambda$ (*resp.*, $\Pi_\lambda \ll \Pi_\nu$) *provided that* $\lambda \ll \nu$ (*resp.*, $\nu \ll \lambda$). *If, in addition, the density* $d\nu_1/d\lambda$ (*resp.*, $d\lambda_1/d\nu$) *may be chosen bounded, then* (4.13) (*resp.*, (4.14)) *holds.*

Proof. Since the definition (5.1) of $H(\lambda, \nu)$ is independent of the dominating measure ρ, we have (see also the proof of Theorem 4.5)

$$H(\lambda, \nu) = \int \left(1 - \sqrt{\frac{d\nu_1}{d\lambda}}\right)^2 d\lambda + \nu_2(\mathbb{X}) = \int \left(1 - \sqrt{\frac{d\lambda_1}{d\nu}}\right)^2 d\nu + \lambda_2(\mathbb{X}). \tag{5.7}$$

Hence, (5.6) follows from (5.2) while the asserted absolute continuity relations follow from (5.6) and [14], Satz (3.3) (see [16], Theorem 1.5.12). If $d\nu_1/d\lambda$ may be chosen bounded, then (4.14) follows from (5.6) and the identity $(1 - x) = (1 - \sqrt{x})(1 + \sqrt{x})$, $x \geq 0$. $\square$

For monotone perturbations, Corollary 5.2 yields the following characterization of the variational formula.

Corollary 5.3. *Let* μ *be a* σ*-finite measure on* $\mathbb{X}$.

(i) *The variational formula* (4.17) *holds for all bounded and measurable* $f : \mathbf{N} \to \mathbb{R}$ *if and only if* $h := d\lambda/d(\lambda + \mu)$ *satisfies* (4.16).

(ii) *Assume that* $\mu \leq \lambda$ *and let* $\nu := \lambda - \mu$. *Then* (1.1) *holds for all bounded and measurable* $f : \mathbf{N} \to \mathbb{R}$ *if and only if* $h_\mu := d\mu/d\lambda$ *satisfies* $\int h_\mu^2 \, d\lambda < \infty$.

Remark 5.4. In general, inequality (5.6) is weaker than both (4.13) and (4.14). We do not know whether (5.6) is sufficient for (1.1) to hold for all bounded measurable f.

Example 5.5. Assume that λ is Lebesgue measure on $\mathbb{X} := \mathbb{R}^d$ for some $d \geq 1$. Let $\mu := c\lambda$ for some $c > 0$. Then $d\lambda/d(\lambda + \mu) = (1 + c)^{-1}$, so that (4.16) fails. Let B_n be a ball with centre at the origin and radius $n \in \mathbb{N}$ and let f be the measurable function on $\mathbf{N}$ defined by

$$f(\varphi) := \mathbf{1}\left\{\lim_{n\to\infty} \lambda(B_n)^{-1}\varphi(B_n) = 1\right\}.$$

Then $\mathbb{E}_\lambda f(\Phi) = 1$ while $\mathbb{E}_{\lambda+\mu} f(\Phi) = 0$. On the other hand we have $D^n_{x_1,\dots,x_n} f \equiv 0$ for all $n \geq 1$ and all $x_1, \dots, x_n \in \mathbb{R}^d$. Hence (4.17) fails.

Remark 5.6. Theorem 5.1 and (5.7) show that (1.1) can only hold for all bounded functions f if the non-absolutely continuous part of the perturbation of λ has finite mass while the absolutely continuous part of the perturbation leads to a distribution Π_ν that is absolutely continuous with respect to the original distribution Π_λ. Example 5.5 shows what can go wrong with (1.1) if this second condition fails. If one condition is violated, then this does not mean that (1.1) does not hold for *some* bounded measurable f. In fact, Theorem 4.1 shows that the formula holds whenever f depends on the restriction of Φ to a set $B \in \mathcal{X}$ with $\lambda(B) < \infty$ and $\nu(B) < \infty$.

6. Derivatives and Russo-type formulas

In this section, we consider σ-finite measures λ, ρ on $\mathbb{X}$ and assume that λ is absolutely continuous with respect to ρ with density h_λ. We also consider a measurable function $h : \mathbb{X} \to \mathbb{R}$ and assume that

$$\int (1 - h_\lambda)^2 \, \mathrm{d}\rho + \int h^2 \, \mathrm{d}\rho < \infty. \tag{6.1}$$

Theorem 6.1. *Assume that* (6.1) *holds. Let $\theta_0 \in \mathbb{R}$ and assume that $I \subset \mathbb{R}$ is an interval with non-empty interior such that $\theta_0 \in I$ and $h_\theta := h_\lambda + (\theta - \theta_0)h \ge 0$ ρ-a.e. for $\theta \in I$. For $\theta \in I$ let λ_θ denote the measure with density h_θ with respect to ρ. Let $f : \mathbf{N} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}_\rho f(\Phi)^2 < \infty$. Then,*

$$\mathbb{E}_{\lambda_\theta} f(\Phi) = \mathbb{E}_\lambda f(\Phi) + \sum_{n=1}^{\infty} \frac{(\theta - \theta_0)^n}{n!} \int \big(\mathbb{E}_\lambda D^n f(\Phi)\big) h^{\otimes n} \, \mathrm{d}\rho^n, \qquad \theta \in I, \tag{6.2}$$

where $\mathbb{E}_\lambda D^n(\Phi)$ denotes the function $(x_1, \dots, x_n) \mapsto \mathbb{E}_\lambda D^n_{x_1,\dots,x_n} f(\Phi)$ and the series converges absolutely. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{E}_{\lambda_\theta} f(\Phi) = \int \big(\mathbb{E}_{\lambda_\theta} D_x f(\Phi)\big) h(x) \rho(\mathrm{d}x), \qquad \theta \in I. \tag{6.3}$$

Proof. Let $\theta \in I$. By our assumptions $1 - h_\theta = (1 - h_\lambda) - (\theta - \theta_0)h$ is square-integrable with respect to ρ. Hence we can apply Theorem 4.1 with $\nu = \lambda_\theta$ to obtain (6.2). In particular we get (6.3) for $\theta = \theta_0$.

To derive (6.3) for general $\theta \in I$ we apply the above with $(\lambda_\theta, h_\theta)$ instead of (λ, h_λ) and with θ instead of θ_0. Since

$$h_{\tilde\theta} = h_\lambda + (\theta - \theta_0)h + (\tilde\theta - \theta)h = h_\theta + (\tilde\theta - \theta)h, \qquad \tilde\theta \in I,$$

we obtain the desired result from (6.3) using the same function h as before. □

Corollary 6.2. *Let ν be another σ-finite measure with density h_ν with respect to ρ. Assume that* (4.2) *holds. Then*

$$\mathbb{E}_{\lambda+\theta(\nu-\lambda)} f(\Phi) = \mathbb{E}_\lambda f(\Phi) + \sum_{n=1}^{\infty} \frac{\theta^n}{n!} \int \big(\mathbb{E}_\lambda D^n f(\Phi)\big) \, \mathrm{d}(\nu - \lambda)^n, \qquad \theta \in [0, 1], \tag{6.4}$$

provided that $\mathbb{E}_\rho f(\Phi)^2 < \infty$.

Proof. We take in Theorem 6.1 $h := h_\nu - h_\lambda$, $I := [0, 1]$ and $\theta_0 := 0$. The result follows upon noting that square-integrability of h is implied by the Minkowski inequality. □

Remark 6.3. Fix a measurable function $f : \mathbf{N} \to \mathbb{R}$ such that $\mathbb{E}_\rho f(\Phi)^2 < \infty$. Let h_λ satisfy $\int (1 - h_\lambda)^2 \, \mathrm{d}\rho < \infty$ and let H_λ be the set of all measurable functions $h : \mathbb{X} \to \mathbb{R}$ such that $\int h^2 \, \mathrm{d}\rho < \infty$ and $h_\lambda + \theta h \geq 0$ ρ-a.e. for all θ in some (possibly one-sided) neighborhood I_h of 0. For $h \in H_\lambda$ and $\theta \in I_h$ we let λ_θ denote the measure with density $h_\theta := h_\lambda + \theta h$ with respect to ρ. Then Theorem 6.1 states that

$$\lim_{\theta \to 0} \theta^{-1} \big(\mathbb{E}_{\lambda_\theta} f(\Phi) - \mathbb{E}_\lambda f(\Phi) \big) = G_{\lambda, f}(h), \qquad h \in H_\lambda, \tag{6.5}$$

where

$$G_{\lambda, f}(h) := \int \big(\mathbb{E}_\lambda D_x f(\Phi) \big) h(x) \rho(\mathrm{d}x). \tag{6.6}$$

Hence $G_{\lambda, f}(h)$ is the *Gâteaux derivative* of the mapping $\nu \mapsto \mathbb{E}_\nu f(\Phi)$ at λ in the *direction* h.

If the perturbation is absolutely continuous with respect to the original measure λ, then we can strengthen (6.5) to *Fréchet* differentiability as follows. Let H_λ^* be the set of all measurable functions $h : \mathbb{X} \to \mathbb{R}$ such that $\int h^2 \, \mathrm{d}\lambda < \infty$ and $1 + h \geq 0$ λ-a.e.

Proposition 6.4. *Let $f : \mathbf{N} \to \mathbb{R}$ be measurable and such that $\mathbb{E}_\lambda f(\Phi)^2 < \infty$. For $h \in H_\lambda^*$ let λ_h denote the measure with density $1 + h$ with respect to λ. Then*

$$\mathbb{E}_{\lambda_h} f(\Phi) = \mathbb{E}_\lambda f(\Phi) + G_{\lambda, f}(h) + \mathrm{o}\big(\|h\|\big), \qquad h \in H_\lambda^*, \tag{6.7}$$

where $G_{\lambda, f}(h)$ is defined by (6.6), $\|h\| := \sqrt{\int h^2 \, \mathrm{d}\lambda}$ *and* $\lim_{t \to 0} t^{-1} \mathrm{o}(t) = 0$.

Proof. We apply Theorem 4.1 with $\rho = \lambda$ (so that $h_\lambda \equiv 1$) and $\nu = \lambda_h$ to obtain that

$$\mathbb{E}_{\lambda_h} f(\Phi) = \mathbb{E}_\lambda f(\Phi) + G_{\lambda, f}(h) + c_h,$$

where

$$c_h := \sum_{n=2}^{\infty} \frac{1}{n!} \int \mathbb{E}_\lambda D^n f(\Phi) h^{\otimes n} \, \mathrm{d}\lambda^n.$$

Applying the triangle inequality and then the Cauchy–Schwarz inequality to each summand gives

$$|c_h| \leq \sum_{n=2}^{\infty} \frac{1}{n!} \left(\int \big(\mathbb{E}_\lambda D^n f(\Phi) \big)^2 \, \mathrm{d}\lambda^n \right)^{1/2} \left(\int h^2 \, \mathrm{d}\lambda \right)^{n/2}.$$

Applying the Cauchy–Schwarz inequality again yields

$$|c_h| \le \left(\sum_{n=2}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_\lambda D^n f(\Phi)\big)^2 \, \mathrm{d}\lambda^n\right)^{1/2} \left(\sum_{n=2}^{\infty} \frac{1}{n!}\left(\int h^2 \, \mathrm{d}\lambda\right)^n\right)^{1/2}.$$

The first factor is finite by (2.4) and the second equals $\tilde{o}(\|h\|)$, where $\tilde{o}(t) := \sqrt{e^{t^2} - 1 - t^2}$. □

Next, we generalize (6.3) to possibly non-linear perturbations of λ.

Theorem 6.5. *Assume that* (6.1) *holds. Let $\theta_0 \in \mathbb{R}$ and assume that $I \subset \mathbb{R}$ is an interval with non-empty interior such that $\theta_0 \in I$. For any $\theta \in I$ let $R_\theta : \mathbb{X} \to \mathbb{R}$ be a measurable function such that the following assumptions are satisfied*:

(i) *For all $\theta \in I$, $h_\lambda + (\theta - \theta_0)(h + R_\theta) \ge 0$ ρ-a.e.*
(ii) $\lim_{\theta \to \theta_0} R_\theta = 0$ *ρ-a.e.*
(iii) *There is a measurable function $R : \mathbb{X} \to [0, \infty)$ such that $|R_\theta| \le R$ ρ-a.e. for all $\theta \in I$ and $\int R^2 \, \mathrm{d}\rho < \infty$.*

For $\theta \in I$, let λ_θ denote the measure with density $h_\lambda + (\theta - \theta_0)(h + R_\theta)$ with respect to ρ. Let $f : \mathbf{N} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}_\rho f(\Phi)^2 < \infty$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{E}_{\lambda_\theta} f(\Phi)\bigg|_{\theta=\theta_0} = \int \big(\mathbb{E}_\lambda D_x f(\Phi)\big) h(x) \rho(\mathrm{d}x). \tag{6.8}$$

Proof. In view of $\int h^2 \, \mathrm{d}\rho < \infty$ and assumption (iii), it is possible to apply Theorem 4.1 to the measure $\nu = \lambda_\theta$. This gives for $\theta \in I \setminus \{\theta_0\}$

$$\begin{aligned}(\theta - \theta_0)^{-1}&\big(\mathbb{E}_{\lambda_\theta} f(\Phi) - \mathbb{E}_\lambda f(\Phi)\big) \\ &= \int \big(\mathbb{E}_\lambda D f(\Phi)\big)(h + R_\theta) \, \mathrm{d}\rho \\ &\quad + \sum_{n=2}^{\infty} \frac{(\theta - \theta_0)^{n-1}}{n!} \int \big(\mathbb{E}_\lambda D^n f(\Phi)\big)(h + R_\theta)^{\otimes n} \, \mathrm{d}\rho^n.\end{aligned} \tag{6.9}$$

Applying Theorem 4.1 to the measure ν with density $h_\lambda + |h| + R$ with respect to ρ yields

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int \big|\mathbb{E}_\lambda D^n f(\Phi)\big| \big(|h| + R\big)^{\otimes n} \, \mathrm{d}\rho^n < \infty.$$

Hence the result follows from assumption (ii) and bounded convergence. □

The case where the perturbed measure λ_θ is absolutely continuous with respect to λ is of special interest. Then the assumptions (ii) and (iii) in Theorem 6.5 can be simplified.

Theorem 6.6. *Assume that* $\int h^2 \, d\lambda < \infty$. *Let* $\theta_0 \in \mathbb{R}$ *and assume that* $I \subset \mathbb{R}$ *is an interval with non-empty interior such that* $\theta_0 \in I$. *For any* $\theta \in I$ *let* $R_\theta : \mathbb{X} \to \mathbb{R}$ *be a measurable function such that the following assumptions are satisfied*:

(i) *For all* $\theta \in I$, $1 + (\theta - \theta_0)(h + R_\theta) \geq 0$ λ*-a.e.*
(ii) $\lim_{\theta \to \theta_0} \int R_\theta^2 \, d\lambda = 0$.

For $\theta \in I$, *let* λ_θ *denote the measure with density* $1 + (\theta - \theta_0)(h + R_\theta)$ *with respect to* λ. *Let* $f : \mathbf{N} \to \mathbb{R}$ *be a measurable function such that* $\mathbb{E}_\lambda f(\Phi)^2 < \infty$. *Then*

$$\frac{d}{d\theta} \mathbb{E}_{\lambda_\theta} f(\Phi) \bigg|_{\theta = \theta_0} = \int \big(\mathbb{E}_\lambda D_x f(\Phi)\big) h(x) \lambda(dx). \tag{6.10}$$

Proof. This time we apply Theorem 4.1 with $\rho = \lambda$ (so that $h_\lambda \equiv 1$) and $\nu = \lambda_\theta$. To treat the right-hand side of (6.9), we first note that

$$\int \big|\mathbb{E}_\lambda D f(\Phi)\big| |R_\theta| \, d\lambda \leq \left(\int \big(\mathbb{E}_\lambda D f(\Phi)\big)^2 \, d\lambda \right)^{1/2} \left(\int R_\theta^2 \, d\lambda \right)^{1/2}.$$

By assumption (ii), this tends to zero as $\theta \to \theta_0$. It remains to show that

$$c_\theta := \sum_{n=2}^{\infty} \frac{1}{n!} \int \big|\mathbb{E}_\lambda D^n f(\Phi)\big| |h + R_\theta|^{\otimes n} \, d\lambda^n$$

is bounded in θ. As in the proof of Proposition 6.4, it follows that

$$c_\theta^2 \leq \left(\sum_{n=2}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_\lambda D^n f(\Phi)\big)^2 \, d\lambda^n \right) \left(\sum_{n=2}^{\infty} \frac{1}{n!} \left(\int (h + R_\theta)^2 \, d\lambda \right)^n \right).$$

Here the first factor is finite by Theorem 4.1 while the second remains bounded by assumption (ii). □

Corollary 6.7. *Let the assumptions of Theorem 6.6 be satisfied. Then*

$$\frac{d}{d\theta} \mathbb{E}_{\lambda_\theta} f(\Phi) \bigg|_{\theta = \theta_0} = \mathbb{E}_\lambda \int \big(f(\Phi) - f(\Phi - \delta_x)\big) h(x) \Phi(dx). \tag{6.11}$$

Proof. The result follows from (6.10) and the Mecke equation from [17]. □

Remark 6.8. The results of this section generalize the Poisson cases of the derivative formulas in [2] and [9], where one can also find some earlier predecessors. We note that [2] and [9] study more general point processes.

Finally in this section, we deal with the case, where λ_θ is a multiple of a finite measure.

Corollary 6.9. *Assume that* λ *is a finite measure and let* $f : \mathbf{N} \to \mathbb{R}$ *be a measurable function such that* $\mathbb{E}_{\theta_0\lambda} f(\Phi)^2 < \infty$ *for some* $\theta_0 > 0$. *Then* $\theta \mapsto \mathbb{E}_{\theta\lambda} f(\Phi)$ *is analytic on* $[0, \infty)$. *Moreover,*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_{\theta\lambda} f(\Phi) = \int \mathbb{E}_{\theta\lambda} D_x f(\Phi)\lambda(\mathrm{d}x), \qquad \theta \geq 0, \tag{6.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_{\theta\lambda} f(\Phi) = \theta^{-1}\mathbb{E}_{\theta\lambda} \int \big(f(\Phi) - f(\Phi - \delta_x)\big)\Phi(\mathrm{d}x), \qquad \theta > 0. \tag{6.13}$$

Proof. Apply Theorem 6.1 with $\rho := \theta_0\lambda$, $\lambda := 0$, $h := \theta_0^{-1}$, $\theta_0 := 0$ and $I := [0, \infty)$. This yields the first two assertions. As before, formula (6.13) is a consequence of (6.12) and the Mecke formula. □

Remark 6.10. Consider in Corollary 6.9 a general σ-finite measure λ but assume that the function f does only depend on the restriction of Φ to some set $B \in \mathcal{X}$ with $\lambda(B) < \infty$. Applying the corollary to $\lambda(B \cap \cdot)$ gives (6.12). This is extended in [6] to functions that depend measurably on the σ-field associated with a *stopping set* satisfying suitable integrability assumptions.

Remark 6.11. Let $f := \mathbf{1}_A$, where $A \in \mathcal{N}$ is *increasing*, that is, whenever $\varphi \in A$ then $\varphi + \delta_x \in A$ for all $x \in \mathbb{X}$. Then

$$\int \big(f(\Phi) - f(\Phi - \delta_x)\big)\Phi(\mathrm{d}x) = \int \mathbf{1}\{\Phi \in A, \Phi - \delta_x \notin A\}\Phi(\mathrm{d}x)$$

is the number of points of Φ that are *pivotal* for A. Hence (6.13) expresses the derivative of $\mathbb{P}_{\theta\lambda}(\Phi \in A)$ in terms of the expected number of pivotal elements. This Poisson counterpart of the *Margulis–Russo formula* for Bernoulli fields was first proved in [19]. In the more general setting of Corollary 6.7, the pivotal elements have to be counted in a weighted way.

7. Perturbation analysis of Lévy processes

In this section, we apply our results to $\mathbb{R}^d$-valued *Lévy processes*, that is, to processes $X = (X_t)_{t\geq 0}$ with homogeneous and independent increments and $X_0 = 0$. We assume that X is continuous in probability. By Proposition II.3.36 in [11] and Theorem 15.4 in [12], we can then assume that a.s.

$$X_t = bt + W_t + \int_{|x|\leq 1}\int_0^t x\big(\Phi(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\,\nu(\mathrm{d}x)\big) + \int_{|x|>1}\int_0^t x\Phi(\mathrm{d}s, \mathrm{d}x), \qquad t \geq 0, \tag{7.1}$$

where $b \in \mathbb{R}^d$, $W = (W)_{t\geq 0}$ is a d-dimensional Wiener process with covariance matrix Σ and Φ is an independent Poisson process on $[0, \infty) \times \mathbb{R}^d$ with intensity measure $\lambda_1 \otimes \nu$. Here λ_1 is Lebesgue measure on $[0, \infty)$ and ν is a *Lévy measure* on $\mathbb{R}^d$, that is, a measure on $\mathbb{R}^d$ having $\nu(\{0\}) = 0$, and $\int (|x|^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. The integrals in (7.1) have to be interpreted as limits in probability. Let $\mathbf{D}$ denote the space of all $\mathbb{R}^d$-valued right-continuous functions on $\mathbb{R}_+$ with left-hand limits on $(0, \infty)$. By [12], Theorem 15.1, we can and will interpret X as a random element in $\mathbf{D}$ equipped with the Kolmogorov product σ-field. The *characteristic triplet* (Σ, b, ν) determines the distribution of X. In this section, we fix Σ and let $\mathbb{P}_{b,\nu}$ denote a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}_{b,\nu}(X \in \cdot)$ is the distribution of a Lévy process with characteristic triplet (Σ, b, ν). The expectation with respect to this measure is denoted by $\mathbb{E}_{b,\nu}$. As before, we let $\mathbb{P}_{\lambda_1 \otimes \nu}$ denote a probability measure such that $\mathbb{P}_{\lambda_1 \otimes \nu}(\Phi \in \cdot) = \Pi_{\lambda_1 \otimes \nu}$. Similarly as in Remark 4.8, we assume that under $\mathbb{P}_{\lambda_1 \otimes \nu}$ the (fixed) process $W = (W)_{t \geq 0}$ is a Wiener process as above, independent of Φ.

Let $\mathbf{F}$ denote the space of all $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$ equipped with the Kolmogorov product σ-field. For $w \in \mathbf{F}$ and $(t_1, x_1) \in [0, \infty) \times \mathbb{R}^d$ we define $w^{t_1,x_1} \in \mathbf{F}$ by $w_t^{t_1,x_1} := w_t + \mathbf{1}\{t \geq t_1\} x_1$. Clearly the mapping $(w, t_1, x_1) \mapsto w^{t_1,x_1}$ is measurable. Moreover, if $w \in \mathbf{D}$ then also $w^{t_1,x_1} \in \mathbf{D}$. For any measurable $f : \mathbf{F} \to \mathbb{R}$, the measurable function $\Delta_{t_1,x_1} f : \mathbf{F} \to \mathbb{R}$ is defined by

$$\Delta_{t_1,x_1} f(w) := f\big(w^{t_1,x_1}\big) - f(w), \qquad w \in \mathbf{F}. \tag{7.2}$$

Similarly as at (2.2), we can iterate this definition to obtain, for $(t_1, x_1, \ldots, t_n, x_n) \in ([0, \infty) \times \mathbb{R}^d)^n$ a function $\Delta^n_{t_1,x_1,\ldots,t_n,x_n} f : \mathbf{F} \to \mathbb{R}$. Further, we define $\Delta^0 f := f$. For $s > 0$ and $w \in \mathbf{F}$ let $w^{(s)} \in \mathbf{F}$ be defined by $w^{(s)}(t) := w(t \wedge s)$ and let $\tilde{\mathcal{A}}_s$ denote the σ-field generated by the mapping $w \mapsto w^{(s)}$. An $\tilde{\mathcal{A}}_s$-measurable function $f : \mathbf{F} \to \mathbb{R}$ has the property that $\Delta_{t,x} f \equiv 0$ whenever $t > s$. Define $\mathcal{A}_s := \tilde{\mathcal{A}}_s \cap \mathbf{D}$.

In the next theorem, we consider three Lévy measures ν, ν', ν^*. We assume that ν and ν' are absolutely continuous with respect to ν^* with densities g_ν and $g_{\nu'}$, respectively, that satisfy

$$\int (1 - g_\nu)^2 \, \mathrm{d}\nu^* + \int (1 - g_{\nu'})^2 \, \mathrm{d}\nu^* < \infty, \tag{7.3}$$

$$\int_{|x| \leq 1} |x| \big|1 - g_\nu(x)\big| \nu^*(\mathrm{d}x) + \int_{|x| \leq 1} |x| \big|1 - g_{\nu'}(x)\big| \nu^*(\mathrm{d}x) < \infty. \tag{7.4}$$

We also consider $b, b', b^* \in \mathbb{R}^d$ such that

$$b = b^* + \int_{|x| \leq 1} x\big(g_\nu(x) - 1\big)\nu^*(\mathrm{d}x), \qquad b' = b^* + \int_{|x| \leq 1} x\big(g_{\nu'}(x) - 1\big)\nu^*(\mathrm{d}x). \tag{7.5}$$

In the following theorem and also later, we abuse our notation by interpreting for a function $g : \mathbb{R}^d \to \mathbb{R}$ and $n \in \mathbb{N}$, $g^{\otimes n}$ as a function on $([0, \infty) \times \mathbb{R}^d)^n$.

Theorem 7.1. *Assume that* (7.3), (7.4) *and* (7.5) *hold. Let* $f : \mathbf{D} \to \mathbb{R}$ *be* $\mathcal{A}_{t_0}$*-measurable for some* $t_0 > 0$ *and assume that* $\mathbb{E}_{b^*,\nu^*} f(X)^2 < \infty$. *Then*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int \big|\mathbb{E}_{b,\nu} \Delta^n f(X)\big| |g_{\nu'} - g_\nu|^{\otimes n} \, \mathrm{d}\big(\lambda_1 \otimes \nu^*\big)^n < \infty, \tag{7.6}$$

where $\mathbb{E}_{b,\nu}\Delta^n f(X)$ *denotes the function* $(t_1, x_1, \dots, t_n, x_n) \mapsto \mathbb{E}_{b,\nu}\Delta^n_{t_1,x_1,\dots,t_n,x_n} f(X)$. *Furthermore,*

$$\mathbb{E}_{b',\nu'} f(X) = \mathbb{E}_{b,\nu} f(X) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \big(\mathbb{E}_{b,\nu}\Delta^n f(X)\big)(g_{\nu'} - g_\nu)^{\otimes n} \,\mathrm{d}\big(\lambda_1 \otimes \nu^*\big)^n. \tag{7.7}$$

Proof. Let $\mathbb{X} := [0, \infty) \times \mathbb{R}^d$ and define $\mathbf{N}$ as before. Let $\mathbf{N}_0$ be the measurable set of all $\varphi \in \mathbf{N}$ such that $\varphi([0, s] \times \{x : 1/n \le |x| \le n\}) < \infty$ for all $s > 0$ and $n \in \mathbb{N}$. Since ν is a Lévy measure we have $\mathbb{P}_{\lambda_1\otimes\nu}(\Phi \in \mathbf{N}_0) = 1$. For $\varphi \in \mathbf{N}_0$ and $n \in \mathbb{N}$, we define $T^n(\varphi) \in \mathbf{F}$ by the pathwise integrals

$$T^n(\varphi)_t := bt + \int_{1/n\le|x|\le1} \int_0^t x\big(\varphi(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\,\nu(\mathrm{d}x)\big) + \int_{n\ge|x|>1} \int_0^t x\varphi(\mathrm{d}s, \mathrm{d}x).$$

Define $T_{b,\nu}(\varphi) \in \mathbf{F}$ by

$$T_{b,\nu}(\varphi)_t := \liminf_{n\to\infty} T^n(\varphi), \qquad t \ge 0,$$

whenever this is finite, and by $T_{b,\nu}(\varphi)_t := 0$, otherwise. For $\varphi \notin \mathbf{N}_0$ we let $T_{b,\nu}(\varphi) \equiv 0$. Then $T_{b,\nu}$ is a measurable mapping from $\mathbf{N}$ to $\mathbf{F}$. It is a basic property of Poisson and Lévy processes ([12], Chapter 15) that $T^n(\Phi)_t$ converges in $\mathbb{P}_{\lambda_1\otimes\nu}$-probability and that

$$\mathbb{P}_{\lambda_1\otimes\nu}\big(W + T_{b,\nu}(\Phi) \in \cdot\big) = \mathbb{P}_{b,\nu}(X \in \cdot) \qquad \text{on } \mathbf{F}, \tag{7.8}$$

where here and later we interpret X also as a random element in $\mathbf{F}$. Assumptions (7.4) and (7.5) imply that $T_{b,\nu} = T_{b',\nu'} = T_{b^*,\nu^*} =: T$, so that the following holds on $\mathbf{F}$:

$$\mathbb{P}_{\lambda_1\otimes\nu'}\big(W + T(\Phi) \in \cdot\big) = \mathbb{P}_{b',\nu'}(X \in \cdot), \qquad \mathbb{P}_{\lambda_1\otimes\nu^*}\big(W + T(\Phi) \in \cdot\big) = \mathbb{P}_{b^*,\nu^*}(X \in \cdot). \tag{7.9}$$

Let $\lambda_1^{t_0}$ be the restriction of λ_1 to the interval $[0, t_0]$. Let $\tilde{f} : \mathbf{F} \to \mathbb{R}$ be an $\tilde{\mathcal{A}}_{t_0}$-measurable function satisfying $\mathbb{E}_{b^*,\nu^*}\tilde{f}(X)^2 < \infty$. We apply Theorem 4.1 and Remark 4.8 with (λ, ν, ρ) replaced with $(\lambda_1^{t_0} \otimes \nu, \lambda_1^{t_0} \otimes \nu', \lambda_1^{t_0} \otimes \nu^*)$, with $\eta = W$ and with the function $(w, \varphi) \mapsto \tilde{f}(w + T(\varphi))$. Assumption (4.2) is implied by (7.3), while $\mathbb{E}_{\lambda_1^{t_0}\otimes\nu^*}(W + \tilde{f}(T(\Phi)))^2 < \infty$ follows from (7.9) and assumption on $\tilde{f}$. (By $\tilde{\mathcal{A}}_{t_0}$-measurability of $\tilde{f}$ we have $\tilde{f}(T(\varphi)) = \tilde{f}(T(\varphi_{t_0}))$ for any $\varphi \in \mathbf{N}_0$, where φ_{t_0} is the restriction of φ to $[0, t_0] \times \mathbb{R}^d$.) Using that for $\varphi \in \mathbf{N}_0$,

$$D^n_{(t_1,x_1),\dots,(t_n,x_n)}(\tilde{f} \circ T)(\varphi) = \big(\Delta^n_{t_1,x_1,\dots,t_n,x_n}\tilde{f}\big)\big(T(\varphi)\big), \qquad (t_1, x_1, \dots, t_n, x_n) \in \big([0, \infty) \times \mathbb{R}^d\big)^n,$$

we obtain (7.6) and (7.7) with $\tilde{f}$ instead of f.

To conclude the proof, we need a $\tilde{\mathcal{A}}_{t_0}$-measurable function $\tilde{f} : \mathbf{F} \to \mathbb{R}$ such that $f = \tilde{f}$ on $\mathbf{D}$. Such a function trivially exists if $f(w) = g(w_{t_1}, \dots, w_{t_n})$, where $0 \le t_1 \le \cdots \le t_n \le t_0$ and $g : \mathbb{R}^n \to \mathbb{R}$ is Borel-measurable. Therefore, the existence follows by a monotone class argument. □

Remark 7.2. In the above proof, we cannot apply Theorem 4.1 with (λ, ν, ρ) replaced with $(\lambda_1 \otimes \nu, \lambda_1 \otimes \nu', \lambda_1 \otimes \nu^*)$. For instance, the first integral in (4.2) would diverge as soon as $\nu \neq \nu^*$. Therefore, we have assumed the function f to depend only on the restriction of X to a finite time interval.

Remark 7.3. If

$$\int \big(|x| \wedge 1\big)\nu(\mathrm{d}x) < \infty, \tag{7.10}$$

it is common, to rewrite (7.1) as

$$X_t = at + W_t + \int_{\mathbb{R}^d} \int_0^t x\,\Phi(\mathrm{d}s, \mathrm{d}x), \qquad t \geq 0, \tag{7.11}$$

where $a := b - \int_{|x|\leq 1} x\,\nu(\mathrm{d}x)$. If all three measures ν, ν', ν^* satisfy (7.10), then we might replace (b, b', b^*) by (a, a', a^*) (with a' and a^* defined similarly as a) and simplify (7.5) to $a = a' = a^*$.

Remark 7.4. By [11], Theorem IV.4.39, the finiteness of the first integrals in (7.3) and (7.4) together with the first identity in (7.5) imply that $\mathbb{P}_{b,\nu}(X^{(t)} \in \cdot)$ is, for every $t \geq 0$, absolutely continuous with respect to $\mathbb{P}_{b^*,\nu^*}(X^{(t)} \in \cdot)$. (Recall that $X_s^{(t)} := X_{t\wedge s}$.) In fact, this conclusion remains true under the weaker assumption $\int (1 - \sqrt{g_\nu})^2\,\mathrm{d}\nu^* < \infty$. We do not know whether the assumption (7.3) in Theorem 7.1 can be weakend to $\int (1 - \sqrt{g_\nu})^2\,\mathrm{d}\nu^* + \int (1 - \sqrt{g_{\nu'}})^2\,\mathrm{d}\nu^* < \infty$, see also Remark 5.4.

Our next theorem is the Lévy version of Theorem 6.1. We consider a Lévy measure ν with density g_ν with respect to some other Lévy measure ν^* and a measurable function $g : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int (1 - g_\nu)^2\,\mathrm{d}\nu^* < \infty, \qquad \int g^2\,\mathrm{d}\nu^* < \infty, \qquad \int \big(\|x\| \wedge 1\big)\big|g(x)\big|\nu^*(\mathrm{d}x) < \infty. \tag{7.12}$$

Theorem 7.5. *Assume that (7.12) holds and the first integral in (7.4) is finite. Let b and b^* satisfy the first identity in (7.5). Let $\theta_0 \in \mathbb{R}$ and assume that $I \subset \mathbb{R}$ is an interval with non-empty interior such that $\theta_0 \in I$ and $g_\theta := g_\nu + (\theta - \theta_0)g \geq 0$ ν^*-a.e. for $\theta \in I$. For $\theta \in I$ let*

$$b_\theta := b + (\theta - \theta_0) \int_{|x|\leq 1} x\,g(x)\nu^*(\mathrm{d}x) \tag{7.13}$$

and let ν_θ denote the measure with density g_θ with respect to ν^*. Let $f : \mathbf{D} \to \mathbb{R}$ be $\mathcal{A}_{t_0}$-measurable for some $t_0 > 0$ and assume that $\mathbb{E}_{b^*,\nu^*} f(X)^2 < \infty$. Then

$$\begin{aligned} \mathbb{E}_{b_\theta,\nu_\theta} f(X) &= \mathbb{E}_{b,\nu} f(X) \\ &\quad + \sum_{n=1}^{\infty} \frac{(\theta - \theta_0)^n}{n!} \int \big(\mathbb{E}_{b,\nu} \Delta^n f(X)\big) g^{\otimes n}\,\mathrm{d}\big(\lambda_1 \otimes \nu^*\big)^n, \qquad \theta \in I, \end{aligned} \tag{7.14}$$

where the series converges absolutely. In particular,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_{b_\theta,\nu_\theta} f(X)\bigg|_{\theta=\theta_0} = \iint \big(\mathbb{E}_{b,\nu}\Delta_{t,x} f(X)\big) g(x)\,\mathrm{d}t\,\nu^*(\mathrm{d}x). \tag{7.15}$$

Proof. Noting that

$$b_\theta = b^* + \int_{|x|\le 1} x\big(g_\theta(x) - 1\big)\nu^*(\mathrm{d}x), \tag{7.16}$$

and using the mapping T defined in the proof of Theorem 7.1, the result follows from Theorem 6.1 and Remark 4.8. □

Remark 7.6. Consider ν and ν^* such that the first integrals in (7.3), respectively, in (7.4) are finite. Let b and b^* satisfy the first identity in (7.5). Let $f : \mathbf{D} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}_{b^*,\nu^*} f(X)^2 < \infty$. By Theorem 7.5

$$G_{b,\nu,f}(g) := \iint \big(\mathbb{E}_{b,\nu}\Delta_{t,x} f(X)\big) g(x)\,\mathrm{d}t\,\nu^*(\mathrm{d}x) \tag{7.17}$$

can be interpreted as the Gâteaux derivative of the mapping $\nu' \mapsto \mathbb{E}_{\nu',b'} f(X)$ at ν in the direction g, where b' is determined by b, and ν' and the function g satisfies the second and third equality in (7.12) as well as $g_\lambda + \theta g \ge 0$ ν^*-a.e. for all θ in some open neighborhood of 0. Proposition 6.4 on Fréchet derivatives can be adapted in a similar way. Details are left to the reader.

The next result deals with non-linear perturbations and is a consequence of Theorem 6.5.

Theorem 7.7. *Assume that* (7.12) *holds. Let* $\theta_0 \in \mathbb{R}$ *and assume that* $I \subset \mathbb{R}$ *is an interval with non-empty interior such that* $\theta_0 \in I$. *For any* $\theta \in I$ *let* $R_\theta : \mathbb{R}^d \to [0,\infty)$ *be a measurable function such that the following assumptions are satisfied*:

(i) *For all* $\theta \in I$, $g_\nu + (\theta - \theta_0)(g + R_\theta) \ge 0$ ν^**-a.e.*
(ii) $\int (|x| \wedge 1)|R_\theta(x)|\nu^*(\mathrm{d}x) < \infty$.
(iii) $\lim_{\theta\to\theta_0} R_\theta = 0$ ν^**-a.e.*
(iv) *There is a measurable function* $R : \mathbb{R}^d \to [0,\infty)$ *such that* $|R_\theta| \le R$ ν^**-a.e. and* $\int R(x)^2 \nu^*(\mathrm{d}x) < \infty$.

For $\theta \in I$, *let* ν_θ *denote the measure with density* $g_\nu + (\theta - \theta_0)(g + R_\theta)$ *with respect to* ν^*. *Let* $b, b^* \in \mathbb{R}$ *satisfy the first identity in* (7.5) *and define*

$$b_\theta := b + (\theta - \theta_0)\int_{|x|\le 1} x\big(g(x) + R_\theta(x)\big)\nu^*(\mathrm{d}x). \tag{7.18}$$

Let $f : \mathbf{D} \to \mathbb{R}$ *be* $\mathcal{A}_{t_0}$*-measurable for some* $t_0 > 0$ *and such that* $\mathbb{E}_{b^*,\nu^*} f(X)^2 < \infty$. *Then* (7.15) *holds.*

Remark 7.8. Assume that we can take $\nu^* = \nu$ in Theorem 7.7 (yielding that $\nu_\theta \ll \nu$). By Theorem 6.6, assumptions (iii) and (iv) can then be replaced with $\lim_{\theta\to\theta_0} \int R_\theta^2 \, \mathrm{d}\nu = 0$.

We finish this section with some examples.

Example 7.9. Let $\alpha \in (0,2)$ and let $\mathbb{Q}$ be a finite measure on the unit sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| \le 1\}$. Then

$$\nu := \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{ru \in \cdot\} r^{-\alpha-1} \, \mathrm{d}r \mathbb{Q}(\mathrm{d}u)$$

is the Lévy measure of an α-*stable* Lévy process, see, for example, [3]. Consider the Lévy measure

$$\mu := \int_{\mathbb{S}^{d-1}} \int_0^1 \mathbf{1}\{ru \in \cdot\} r^{-\alpha'-1} \, \mathrm{d}r \mathbb{Q}'(\mathrm{d}u),$$

where $0 < \alpha' < \alpha/2$ and $\mathbb{Q}'$ is a finite measure on $\mathbb{S}^{d-1}$. Assume that $\mathbb{Q}' \ll \mathbb{Q}$ with a density that is square-integrable with respect to $\mathbb{Q}$. It is not difficult to check that the density $g := \mathrm{d}\mu/\mathrm{d}\nu$ satisfies the assumptions of Theorem 7.5 with $\nu^* = \nu$, $I = [0,\infty)$ and $\theta_0 = 0$.

Example 7.10. Let $d = 1$, $\alpha \in (0,2)$ and $\nu(\mathrm{d}x) := \mathbf{1}\{x \neq 0\} x^{-\alpha-1} \, \mathrm{d}x$ be the Lévy measure of a (symmetric) α-stable process. It is again easy to check that, for $\beta > 0$, the density g of the measure $\mu_\beta(\mathrm{d}x) := \mathbf{1}\{x > 0\} x^{-1} \mathrm{e}^{-\beta x} \, \mathrm{d}x$ with respect to ν satisfies the assumptions of Theorem 7.5 with $\nu^* = \nu$, $I = [0,\infty)$ and $\theta_0 = 0$. For $\theta \ge 0$, the measure $\theta\mu$ is the Lévy measure of a *Gamma process* with shape parameter θ and scale parameter β, see, for example, [3]. (Under $\mathbb{P}_{0,\theta\mu_\beta}$ and for $W \equiv 0$, the random variable X_t has a Gamma distribution with shape parameter θt and scale parameter β.)

Example 7.11. We let ν and μ_β be as in Example 7.10. This time we are interested in derivatives with respect to the scale parameter β. We fix $\beta_0 > 0$ and $\theta \ge 0$. Our aim is to apply Theorem 7.7 with $I = (\beta_0/2, 3\beta_0/2)$, $\theta_0 = \beta_0$, $\nu_\beta := \nu + \theta\mu_\beta$, and $\nu^* := \nu$. The measure ν in Theorem 7.7 is being replaced with ν_{β_0}. We have noted in Example 7.10 that $\int (1 - g_{\beta_0})^2 \, \mathrm{d}\nu < \infty$, where g_{β_0} is the Radon–Nikodym derivative of ν_{β_0} with respect to ν. Next, we note that

$$\begin{aligned} \nu_\beta(\mathrm{d}x) &= \nu_{\beta_0}(\mathrm{d}x) + \mathbf{1}\{x > 0\}\theta\big(\mathrm{e}^{-(\beta-\beta_0)x} - 1\big) x^\alpha \mathrm{e}^{-\beta_0 x} \nu(\mathrm{d}x) \\ &= \big[g_{\beta_0}(x) + (\beta - \beta_0)\big(g(x) + R_\beta(x)\big)\big]\nu(\mathrm{d}x), \end{aligned} \tag{7.19}$$

where

$$g(x) := -\mathbf{1}\{x > 0\}\theta x^{\alpha+1} \mathrm{e}^{-\beta_0 x}, \qquad R_\beta(x) := -\mathbf{1}\{x > 0\}\theta x^\alpha \mathrm{e}^{-\beta_0 x} \sum_{n=2}^\infty \frac{(\beta_0 - \beta)^{n-1}}{n!} x^n.$$

Clearly, $g(x)^2$ and $(|x| \wedge 1)|g(x)|$ are integrable with respect to ν. We need to check the assumptions (i)–(iv) of Theorem 7.7. Assumption (i) follows from (7.19) while (iii) is obvious. Further, we have for $\beta \in I$ that $|R_\beta| \le R$, where

$$R(x) := \mathbf{1}\{x > 0\}\theta x^\alpha \mathrm{e}^{-\beta_0 x} \sum_{n=0}^{\infty} \frac{(\beta_0/2)^{n-1}}{n!} x^n = \mathbf{1}\{x > 0\}\theta c^{-1} x^\alpha \mathrm{e}^{-cx},$$

where $c := \beta_0/2$. Since $\int (x \wedge 1) R(x) \nu(\mathrm{d}x) < \infty$ and $\int R(x)^2 \nu(\mathrm{d}x) < \infty$ we obtain (ii) and (iv). Let $b \in \mathbb{R}$. In view of (7.16), we define

$$b_\beta := b + \int_0^1 x\big(g_\beta(x) - 1\big)\nu(\mathrm{d}x) = b + \theta \int_0^1 \mathrm{e}^{-\beta x}\,\mathrm{d}x = b + \frac{\theta}{\beta}\big(1 - \mathrm{e}^{-\beta}\big).$$

Under the assumption $\mathbb{E}_{b,\nu} f(X)^2 < \infty$, we then obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \mathbb{E}_{b_\beta, \nu+\theta\mu_\beta} f(X)\bigg|_{\beta=\beta_0} = -\theta \iint \big(\mathbb{E}_{b_{\beta_0}, \nu+\theta\mu_{\beta_0}} \Delta_{t,x} f(X)\big)\mathrm{e}^{-\beta_0 x}\,\mathrm{d}t\,\mathrm{d}x. \tag{7.20}$$

Remark 7.12. It is a common feature of Examples 7.9 and 7.10 that the perturbation $\nu' - \nu$ is infinite. Theorem 3.1 would not be enough to treat these cases. In Example 7.11 however, $\nu_{\beta_0} - \nu_\beta$ is finite so that one might use Theorem 3.1 in the case $\beta < \beta_0$.

Finally in this section we assume $d = 1$ and apply our results to the *running supremum*

$$S_t := \sup\{X_s : 0 \le s \le t\}, \qquad t \ge 0,$$

of X. We fix $t_0 > 0$ and define

$$Z_t := \sup\{X_s : t \le s \le t_0\}, \qquad Y_t := S_t - Z_t, \qquad 0 \le t \le t_0.$$

Proposition 7.13. *Let $\nu, \nu^*, g, g_\nu, b, b^*, I, \theta_0$ be as in Theorem 7.5 and define b_θ, g_θ as in that theorem. Assume moreover that*

$$\int_{x>1} x^2 \nu^*(\mathrm{d}x) < \infty. \tag{7.21}$$

Then $\theta \mapsto \mathbb{E}_{b_\theta,\nu_\theta} S_{t_0}$ is analytic on I. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{E}_{b_\theta,\nu_\theta} S_{t_0}\bigg|_{\theta=\theta_0} = \iint \big((x-y)^+ - y^-\big) g(x) \mathbb{Q}(\mathrm{d}y)\nu^*(\mathrm{d}x), \tag{7.22}$$

where, for $x \in \mathbb{R}$, $x^+ := \max\{x, 0\}$, $x^- := -\min\{x, 0\}$, and

$$\mathbb{Q} := \int_0^{t_0} \mathbb{P}_{b,\nu}(Y_t \in \cdot)\,\mathrm{d}t. \tag{7.23}$$

Proof. We define a measurable function $f : \mathbf{D} \to \mathbb{R}$ by

$$f(w) := \sup\{w_s : 0 \le s \le t_0\}, \qquad w \in \mathbf{D}.$$

It follows from the Lévy–Khintchine representation (7.1), Doob's inequality and moment properties of Poisson integrals that (7.21) is sufficient (and actually also necessary) for $\mathbb{E}_{b^*,\nu^*} f(X)^2 < \infty$. (This argument is quite standard.) Hence, we can apply Theorem 7.5.

It remains to compute the right-hand side of (7.15). Let $t \in (0, t_0]$. For $x > 0$ we have

$$f\big(X^{t,x}\big) = \begin{cases} S_{t_0} + x, & \text{if } S_{t-} \le Z_t, \\ Z_t + x, & \text{if } Z_t < S_{t-} \le Z_t + x, \\ S_{t_0}, & \text{if } Z_t + x < S_{t-}, \end{cases}$$

so that

$$f\big(X^{t,x}\big) - f(X) = \mathbf{1}\{Y_t \le 0\}x + \mathbf{1}\{0 < Y_t \le x\}(x - Y_t) = (x - Y_t)^+ - (Y_t)^-,$$

provided that $S_{t-} = S_t$. Note that the latter equality holds for λ_1-a.e. $t > 0$. Similarly we obtain for $x < 0$ that

$$f\big(X^{t,x}\big) - f(X) = \mathbf{1}\{x < Y_t \le 0\}Y_t + \mathbf{1}\{Y_t \le x\}x = (x - Y_t)^+ - (Y_t)^-,$$

whenever $S_{t-} = S_t$. Hence, (7.22) follows from (7.15). Note that the integrability required for (7.22) is part of the assertion of Theorem 7.5. But it does also follow more directly from $|f(X^{t,x}) - f(X)| \le 2|x|$ and the fact that $\int |g(x)||x|\nu^*(\mathrm{d}x)$ is finite by assumption (7.12) on g, (7.21), and the Cauchy–Schwarz inequality. □

Remark 7.14. We consider the situation of Proposition 7.13 but do not assume (7.21). For $u \in \mathbb{R}$ wen can then apply Theorem 7.5 to the real and the imaginary part of the complex-valued and bounded function $f(X) := \mathrm{e}^{iuS_{t_0}}$. This shows that $\theta \mapsto \mathbb{E}_{b_\theta,\nu_\theta} f(X)$ is analytic. The derivative at θ_0 can be expressed in terms of the measure

$$\mathbb{Q}' := \int_0^{t_0} \mathbb{P}_{b,\nu}\big((S_t, Z_t) \in \cdot\big)\,\mathrm{d}t \tag{7.24}$$

that contains more information than the measure (7.23). The same remark applies to the bounded function $f(X) := \mathbf{1}\{S_{t_0} > u\}$. The details are left to the reader.

For a general Lévy process the distribution of S_t is not known. The measures (7.23) and (7.24) are not known either. This hints at the fact that perturbation analysis cannot help in finding explicit distributions. What equation (7.22) does, however, is to identify the Gâteaux derivative of $\mathbb{E}_{b,\nu} S_{t_0}$ in the direction g, see Remark 7.6. The measure (7.23), controlling all these derivatives, is completely determined by the distribution of the process $(X_t)_{t \le t_0}$ under $\mathbb{P}_{b,\nu}$. We do not make any attempt to review the vast literature on the running supremum of Lévy processes but just refer to [8] for some recent progress.

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References

[1] Asmussen, S. and Glynn, P.W. (2007). *Stochastic Simulation*: *Algorithms and Analysis*. *Stochastic Modelling and Applied Probability* **57**. New York: Springer. MR2331321
[2] Baccelli, F., Klein, M. and Zuyev, S. (1995). Perturbation analysis of functionals of random measures. *Adv. in Appl*. *Probab.* **27** 306–325. MR1334815
[3] Bertoin, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge: Cambridge Univ. Press. MR1406564
[4] Błaszczyszyn, B. (1995). Factorial moment expansion for stochastic systems. *Stochastic Process*. *Appl*. **56** 321–335. MR1325226
[5] Bollobás, B. and Riordan, O. (2006). *Percolation*. New York: Cambridge Univ. Press. MR2283880
[6] Bordenave, C. and Torrisi, G.L. (2008). Monte Carlo methods for sensitivity analysis of Poisson-driven stochastic systems, and applications. *Adv. in Appl*. *Probab.* **40** 293–320. MR2431298
[7] Brown, M. (1971). Discrimination of Poisson processes. *Ann*. *Math*. *Statist*. **42** 773–776.
[8] Chaumont, L. (2013). On the law of the supremum of Lévy processes. *Ann*. *Probab.* **41** 1191–1217.
[9] Decreusefond, L. (1998). Perturbation analysis and Malliavin calculus. *Ann*. *Appl*. *Probab.* **8** 496–523. MR1624953
[10] Ho, Y.C. and Cao, X.R. (1991). *Perturbation Analysis of Discrete-Event Dynamic Systems*. Boston, MA: Kluwer.
[11] Jacod, J. and Shiryaev, A.N. (1987). *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Berlin: Springer. MR0959133
[12] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). New York: Springer. MR1876169
[13] Last, G. and Penrose, M.D. (2011). Poisson process Fock space representation, chaos expansion and covariance inequalities. *Probab*. *Theory Related Fields* **150** 663–690. MR2824870
[14] Liese, F. (1975). Eine informationstheoretische Bedingung für die Äquivalenz unbegrenzt teilbarer Punktprozesse. *Math*. *Nachr.* **70** 183–196. Collection of articles dedicated to the memory of Wolfgang Richter. MR0478321
[15] Liese, F. and Vajda, I. (1987). *Convex Statistical Distances*. *Teubner-Texte zur Mathematik* [*Teubner Texts in Mathematics*] **95**. Leipzig: Teubner. MR0926905
[16] Matthes, K., Kerstan, J. and Mecke, J. (1982). *Bezgranichno Delimye Tochechnye Protsessy*. Moscow: Nauka. Translated from the English by G. V. Martynov and V. I. Piterbarg. MR0673561
[17] Mecke, J. (1967). Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen. *Z. Wahrsch*. *Verw*. *Gebiete* **9** 36–58. MR0228027
[18] Molchanov, I. and Zuyev, S. (2000). Variational analysis of functionals of Poisson processes. *Math*. *Oper*. *Res*. **25** 485–508. MR1855179
[19] Zuev, S.A. (1992). Russo's formula for Poisson point fields and its applications. *Diskret*. *Mat*. **4** 149–160. MR1220977

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