

Efficient estimation of moments in linear mixed models

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In the linear random effects model, when distributional assumptions such as normality of the error variables cannot be justified, moments may serve as alternatives to describe relevant distributions in neighborhoods of their means. Generally, estimators may be obtained as solutions of estimating equations. It turns out that there may be several equations, each of them leading to consistent estimators, in which case finding the efficient estimator becomes a crucial problem. In this paper, we systematically study estimation of moments of the errors and random effects in linear mixed models.

Keywords: asymptotic normality; linear mixed model; moment estimator

1. Introduction

Normality or, more generally, the existence of a parametric structure on the distribution of random effects is a routine assumption for linear mixed models. In such a case, both the maximum likelihood estimator (MLE) and the restricted maximum likelihood estimator (RMLE) work well. Moreover, they are standard outputs in statistical software packages such as SAS and R. A comprehensive account of the methodology is contained in the monograph of Verbeke and Molenberghs [7]. In recent years, more efforts were devoted to relaxing this assumption and using semiparametric or nonparametric methods to estimate the parameters of interest. Zhang and Davidian [9] suggested using the seminonparametric representation of Gallant and Nychka [4] to approximate the random effect density in order to estimate parameters for linear mixed models. Cui, Ng and Zhu [3] used the estimation of moments in mixed effect models with errors in variables. Rank estimation was applied by Wang and Zhu [8] to estimate fixed effects.

However, the aforementioned papers do not consider the estimation of higher moments that are useful for hypothesis testing and interval estimation for the parameters in the models. To the best of our knowledge, Cox and Hall [2] is the only reference in the literature that defines and studies the estimators of the errors and random effects for higher than second moments. The authors of that work obtained the cumulants of the two components of variance based on homogeneous polynomials in a simple random effects location model, which is the sum of the one-level random effect and the error. For this model, Hall and Yao [5] studied nonparametric estimation of the distributions of the errors and the random effects via empirical cumulant generating functions. To the best of our knowledge, no paper has investigated this issue for the linear mixed model under consideration.

The contents of this paper are as follows:

- In Section 2.1 we introduce the linear mixed model and derive basic properties of the generalized least squares estimator under weak conditions on the group sizes and the design variables. The fundamental Lemma 2.1 yields representations of certain polynomial functions of the overall errors in terms of individual and group errors. This will be the basic tool to answer a question posed by Cox and Hall [2] in the context of the simple random effects location model, namely, how to properly weight and combine certain polynomial functions of the residuals.
- As a warmup, in Section 2.2, we consider the estimation of second moments. It turns out that by a proper combination of polynomial functions of the residuals, we can obtain second moment estimators which are asymptotically normal and have the same limit variance as if the unknown errors were known.
- For third and fourth moments, the situation is more complex. In Sections 2.3 and 2.4, we propose and study estimators yielding efficiency and asymptotic normality under weak conditions on the design and group sizes.
- As an alternative, in Sections 3.1 and 3.2, we study an extension of an estimator due to Cox and Hall [2] which may therefore be considered as a first step estimator. When the group sizes are all equal, our estimators have similar asymptotic properties to theirs. We show that for unequal group sizes, the obtained estimators may converge at slower rates unless some restrictive regularity assumptions are satisfied.
- Section 4 presents some simulation studies, while proofs are deferred to the Appendix.

2. Minimum variance estimation of moments

2.1. Motivation and first results

Assume that data are available from a linear mixed model, that is, we observe pairs (x_{ij}, y_{ij}) , $1 \leq i \leq n$, $1 \leq j \leq l_i$, satisfying

$$y_{ij} = \alpha + x'_{ij}\beta + b_i + \varepsilon_{ij}. \tag{2.1}$$

Here, i denotes the group index, while the measurements within this group are indexed by j . The integer l_i is the sample size within group i . The row vector x'_{ij} is a p -dimensional input vector corresponding to the j th observation in the i th group leading to the output y_{ij} . The relation between x_{ij} and y_{ij} described by (2.1) contains the intercept parameter α , the fixed effect regression parameter β and the one-level random effect b_i for group i , all unknown. Moreover, these quantities are disturbed by random errors ε_{ij} . It is assumed throughout that b_1, \dots, b_n are independent and identically distributed (i.i.d.) and also independent of all ε_{ij} , which are also i.i.d. Finally, we may assume without loss of generality that

$$\mathbb{E}b_i = 0 \quad \text{and} \quad \mathbb{E}\varepsilon_{ij} = 0 \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq l_i. \tag{2.2}$$

Otherwise, we may incorporate unknown nonzero expectations in the intercept α . Let γ_b^k and γ_ε^k denote the k th moments of the random effects and errors, respectively. In this paper, we

shall construct and analyze estimators of α , β , γ_b^k and γ_ε^k , $k = 2, 3, 4$, that are based on various estimating equations. These equations are obtained from proper nonlinear combinations of the residuals. For these, we first have to estimate β and α via a generalized least squares method. In the model (2.1), this leads to

$$\hat{\beta} = \hat{\Sigma}_n^{-1} \frac{\sum_{i=1}^n \sum_{j=1}^{l_i} (x_{ij} - \bar{x}_{i\cdot})(y_{ij} - \bar{y}_{i\cdot})}{\sum_{i=1}^n l_i} \tag{2.3}$$

and

$$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^n \bar{y}_{i\cdot} - \frac{1}{n} \sum_{i=1}^n \bar{x}'_{i\cdot} \hat{\beta}. \tag{2.4}$$

Here,

$$\hat{\Sigma}_n = \frac{1}{\sum_{i=1}^n l_i} \sum_{i=1}^n \sum_{j=1}^{l_i} (x_{ij} - \bar{x}_{i\cdot})(x_{ij} - \bar{x}_{i\cdot})', \tag{2.5}$$

while

$$\bar{x}_{i\cdot} = \frac{1}{l_i} \sum_{j=1}^{l_i} x_{ij} \quad \text{and} \quad \bar{y}_{i\cdot} = \frac{1}{l_i} \sum_{j=1}^{l_i} y_{ij}$$

denote the corresponding group averages. Furthermore, we let

$$N = \sum_{i=1}^n l_i,$$

the overall sample size.

Theorem 2.1. *Assume that the following conditions (2.6)–(2.8) are satisfied:*

$$\lim_{n \rightarrow \infty} \hat{\Sigma}_n = \Sigma \text{ for some positive definite } p \times p \text{ matrix } \Sigma; \tag{2.6}$$

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{l_i} \rightarrow 0 \quad \text{and} \quad N/n \rightarrow \infty \quad \text{as } n \rightarrow \infty; \tag{2.7}$$

$$\frac{\max_{i \leq i \leq n, 1 \leq j \leq l_i} \|x_{ij} - \bar{x}_{i\cdot}\|}{\sqrt{N}} \rightarrow 0 \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n \bar{x}_{i\cdot} \text{ is bounded.} \tag{2.8}$$

Then, in distribution, we have

$$N^{1/2}(\hat{\beta} - \beta) \rightarrow \mathcal{N}_p(0, \gamma_\varepsilon^2 \Sigma^{-1}) \tag{2.9}$$

and

$$n^{1/2}(\hat{\alpha} - \alpha) \rightarrow \mathcal{N}_1(0, \gamma_b^2). \tag{2.10}$$

The estimators $\hat{\beta}$ and $\hat{\alpha}$ and their distributional behavior play an important role for motivating the estimation of γ_b^k and γ_ε^k since this will be based on the residuals

$$\hat{e}_{ij} = y_{ij} - \hat{\alpha} - x'_{ij}\hat{\beta}.$$

Set

$$\bar{b} \equiv \bar{b}_n = \frac{1}{n} \sum_{i=1}^n b_i, \quad \bar{\varepsilon} \equiv \bar{\varepsilon}_n = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i} \sum_{j=1}^{l_i} \varepsilon_{ij}$$

and

$$\bar{x} \equiv \bar{x}_n = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i} \sum_{j=1}^{l_i} x_{ij}.$$

In view of (2.4), we have

$$\hat{\alpha} - \alpha = \bar{b} + \bar{\varepsilon} - \bar{x}'(\hat{\beta} - \beta),$$

from which it follows that

$$\begin{aligned} \hat{e}_{ij} &= (b_i - \bar{b}) + (\varepsilon_{ij} - \bar{\varepsilon}) + (x_{ij} - \bar{x})'(\beta - \hat{\beta}) \\ &\equiv (b_i + \varepsilon_{ij}) - (\bar{b} + \bar{\varepsilon}) + z'_{ij}(\beta - \hat{\beta}). \end{aligned} \tag{2.11}$$

Set

$$e_{ij} = b_i + \varepsilon_{ij},$$

a sum of two independent zero-mean random variables.

When the l_i 's are equal and $\beta = 0$, that is, in the simple random effects location model, Cox and Hall [2] used homogeneous polynomial functions to construct estimating equations. In the present paper, we consider more general situations in which new special nonlinear functions of the e_{ij} 's are important tools to derive estimating equations for

$$\gamma_b^k = \mathbb{E}b_i^k \quad \text{and} \quad \gamma_\varepsilon^k = \mathbb{E}\varepsilon_{ij}^k.$$

For this, define, for $1 \leq i \leq n$ and $1 \leq m \leq k$,

$$f_m^k(i) = \sum_{j=1}^{l_i} e_{ij}^m \left[\sum_{j=1}^{l_i} e_{ij} \right]^{k-m}.$$

The following lemma turns out to be crucial for our analysis.

Lemma 2.1. *We have*

$$f_m^k(i) = \sum_{t=0}^k \sum_{s=(t-k+m) \vee 0}^{t \wedge m} \binom{m}{s} \binom{k-m}{t-s} \left(\sum_{j=1}^{l_i} \varepsilon_{ij}^s \right) \left(\sum_{j=1}^{l_i} \varepsilon_{ij} \right)^{t-s} b_i^{k-t} l_i^{k-m-t+s}.$$

Here, $a \wedge b$ and $a \vee b$ denote the minimum and maximum, respectively, of two real numbers a and b .

The proof follows from simple arithmetic. When we take expectations, usually many of the terms in the expansion of $f_m^k(i)$ will vanish, mainly because the ε_{ij} 's and b_i 's are centered and independent; see (2.2). Moreover, by taking proper linear combinations of the $f_m^k(i)$'s, we shall be able to represent the γ_b^k 's and γ_ε^k 's in terms of the f 's. These so-called estimating equations will then lead to associated estimators.

For example, in the case of γ_ε^2 , we have

$$l_i f_2^2(i) - f_1^2(i) = l_i \sum_{j=1}^{l_i} \varepsilon_{ij}^2 - \left[\sum_{j=1}^{l_i} \varepsilon_{ij} \right]^2,$$

from which it follows that

$$\mathbb{E}[l_i f_2^2(i) - f_1^2(i)] = l_i(l_i - 1)\gamma_\varepsilon^2.$$

This equation does not incorporate any b -term, so it may serve as a basis for the estimation of γ_ε^2 . For moments γ_ε^k and γ_b^k , $k > 2$, things become more delicate. At first, it is not clear how to combine the $f_m^k(i)$'s in order to get efficient estimators. This issue is dealt with in Sections 2.2–2.4, for $k = 2, 3$ and 4, respectively. In Section 3, we briefly discuss the extension of Cox and Hall [2] to the regression case and show that it may cause some inefficiencies.

Remark 2.1. We only remark in passing that the results of this and the following sections may be extended to group sizes l_{ni} , $1 \leq i \leq n$, that is, when the l 's depend on the number n of groups and therefore form a triangular array.

2.2. Estimation of γ_ε^2 and γ_b^2

We start by estimating γ_ε^2 and γ_b^2 . As mentioned above,

$$\mathbb{E}[l_i f_2^2(i) - f_1^2(i)] = l_i(l_i - 1)\gamma_\varepsilon^2.$$

Averaging over $1 \leq i \leq n$ and replacing the unknown ε 's by the residuals leads to the estimator

$$\hat{\gamma}_\varepsilon^2 = \frac{\sum_{i=1}^n (1/(l_i - 1)) \{l_i \sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^2 - (\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij})^2\}}{N}.$$

Similarly, the equation

$$\mathbb{E}[f_1^2(i) - f_2^2(i)] = l_i(l_i - 1)\gamma_b^2$$

leads to the estimator

$$\hat{\gamma}_b^2 = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)} \left\{ \left(\sum_{j=1}^{l_i} \hat{\epsilon}_{ij} \right)^2 - \sum_{j=1}^{l_i} \hat{\epsilon}_{ij}^2 \right\}.$$

Theorem 2.2. *Under the conditions of Theorem 2.1, when γ_ϵ^4 and γ_b^4 are finite, we have that*

$$N^{1/2}[\hat{\gamma}_\epsilon^2 - \gamma_\epsilon^2] \rightarrow \mathcal{N}_1(0, \mu_\epsilon^2) \tag{2.12}$$

and

$$n^{1/2}[\hat{\gamma}_b^2 - \gamma_b^2] \rightarrow \mathcal{N}_1(0, \mu_b^2), \tag{2.13}$$

where

$$\mu_\epsilon^2 = \gamma_\epsilon^4 - (\gamma_\epsilon^2)^2 \quad \text{and} \quad \mu_b^2 = \gamma_b^4 - (\gamma_b^2)^2.$$

It is interesting to note that (2.12) and (2.13) will be shown by verifying

$$N^{1/2}[\hat{\gamma}_\epsilon^2 - \gamma_\epsilon^2] = N^{-1/2} \left[\sum_{i=1}^n \sum_{j=1}^{l_i} (\epsilon_{ij}^2 - \gamma_\epsilon^2) \right] + o_{\mathbb{P}}(1)$$

and

$$n^{1/2}[\hat{\gamma}_b^2 - \gamma_b^2] = \frac{1}{\sqrt{n}} \left[\sum_{i=1}^n (b_i^2 - \gamma_b^2) \right] + o_{\mathbb{P}}(1).$$

In other words, $\hat{\gamma}_\epsilon^2$ and $\hat{\gamma}_b^2$ are as efficient as the moment estimators based on the true (but unknown) ϵ_{ij} and b_i .

2.3. Estimation of γ_ϵ^3 and γ_b^3

In this section we show how to estimate γ_ϵ^3 and γ_b^3 with minimal variance. Again, this may be achieved by properly combining the $f_m^k(i)$'s. From Lemma 2.1, we obtain

$$\mathbb{E}f_3^3(i) = l_i\gamma_b^3 + l_i\gamma_\epsilon^3,$$

$$\mathbb{E}f_2^3(i) = l_i^2\gamma_b^3 + l_i\gamma_\epsilon^3$$

and

$$\mathbb{E}f_1^3(i) = l_i^3\gamma_b^3 + l_i\gamma_\epsilon^3.$$

We conclude that

$$\mathbb{E}[2f_1^3(i) + l_i^2 f_3^3(i) - 3l_i f_2^3(i)] = l_i(l_i - 1)(l_i - 2)\gamma_\varepsilon^3.$$

The corresponding estimator of γ_ε^3 becomes

$$\hat{\gamma}_\varepsilon^3 = N^{-1} \sum_{i=1}^n \frac{1}{(l_i - 1)(l_i - 2)} \left\{ 2 \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij} \right)^3 + l_i^2 \sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^3 - 3l_i \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^2 \right) \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij} \right) \right\}.$$

For γ_b^3 , the relevant equation is

$$\mathbb{E}[f_1^3(i) - 3f_2^3(i) + 2f_3^3(i)] = l_i(l_i - 1)(l_i - 2)\gamma_b^3,$$

leading to the estimator

$$\hat{\gamma}_b^3 = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)(l_i - 2)} \left\{ \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij} \right)^3 - 3 \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^2 \right) \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij} \right) + 2 \sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^3 \right\}.$$

Theorem 2.3. *Under the conditions of Theorem 2.1, when γ_ε^6 and γ_b^6 are finite, we have that*

$$N^{1/2}(\hat{\gamma}_\varepsilon^3 - \gamma_\varepsilon^3) \rightarrow \mathcal{N}_1(0, \mu_\varepsilon^3)$$

and

$$n^{1/2}(\hat{\gamma}_b^3 - \gamma_b^3) \rightarrow \mathcal{N}_1(0, \mu_b^3),$$

where

$$\begin{aligned} \mu_\varepsilon^3 &= \gamma_\varepsilon^6 - (\gamma_\varepsilon^3)^2 - 6\gamma_\varepsilon^2\gamma_\varepsilon^4 + 9(\gamma_\varepsilon^2)^3, \\ \mu_b^3 &= \gamma_b^6 - (\gamma_b^3)^2 - 6\gamma_b^2\gamma_b^4 + 9(\gamma_b^2)^3. \end{aligned}$$

As for second moments, these quantities denote the minimum variances, which may be achieved for empirical estimators based on the true ε_{ij} and b_i , respectively.

2.4. Estimation of γ_ε^4 and γ_b^4

For γ_ε^4 , we are also looking for a combination of f_m^4 's such that the expectations include γ_ε^4 but no other moments. First, from Lemma 2.1, we have

$$\begin{aligned} \mathbb{E}f_4^4(i) &= l_i\gamma_b^4 + 6l_i\gamma_b^2\gamma_\varepsilon^2 + l_i\gamma_\varepsilon^4, \\ \mathbb{E}f_3^4(i) &= l_i^2\gamma_b^4 + 3l_i(l_i + 1)\gamma_b^2\gamma_\varepsilon^2 + l_i\gamma_\varepsilon^4, \\ \mathbb{E}f_2^4(i) &= l_i^3\gamma_b^4 + (l_i^3 + 5l_i^2)\gamma_b^2\gamma_\varepsilon^2 + l_i(l_i - 1)(\gamma_\varepsilon^2)^2 + l_i\gamma_\varepsilon^4 \end{aligned}$$

and

$$\mathbb{E}f_1^4(i) = l_i^4 \gamma_b^4 + 6l_i^3 \gamma_b^2 \gamma_\varepsilon^2 + \mathbb{E}\left[\left(\sum_j \varepsilon_{ij}\right)^4\right].$$

Finally, we put

$$f_5^4(i) = \left[\sum_{j=1}^{l_i} e_{ij}^2\right]^2.$$

Clearly,

$$\begin{aligned} \mathbb{E}f_5^4(i) &= \sum_{j=1}^{l_i} \sum_{k=1}^{l_i} \mathbb{E}[e_{ij}^2 e_{ik}^2] = \sum_{j=1}^{l_i} \sum_{k=1}^{l_i} \mathbb{E}[(b_i + \varepsilon_{ij})^2 (b_i + \varepsilon_{ik})^2] \\ &= l_i^2 \gamma_b^4 + l_i \gamma_\varepsilon^4 + (2l_i^2 + 4l_i) \gamma_b^2 \gamma_\varepsilon^2 + (l_i^2 - l_i) (\gamma_\varepsilon^2)^2. \end{aligned} \tag{2.14}$$

We now combine these expressions in a proper way. In particular, we check that

$$\begin{aligned} &\mathbb{E}[(l_i^2 - 2l_i + 3)(l_i f_4^4(i) - 4f_3^4(i)) + 6l_i f_2^4(i) - 3f_1^4(i) - 3(2l_i - 3)f_5^4(i)] \\ &= l_i(l_i - 1)(l_i - 2)(l_i - 3)\gamma_\varepsilon^4. \end{aligned}$$

At first sight, the coefficients may look a little strange, but they appear as solutions of linear equations incorporating $\mathbb{E}f_1^4, \dots, \mathbb{E}f_5^4$ such that all terms involving moments other than γ_ε^4 vanish. Our minimum variance estimator of γ_ε^4 thus becomes

$$\begin{aligned} \hat{\gamma}_\varepsilon^4 &= N^{-1} \sum_{i=1}^n \frac{1}{(l_i - 1)(l_i - 2)(l_i - 3)} \\ &\times \left\{ (l_i^2 - 2l_i + 3) \left[l_i \sum_{j=1}^{l_i} \hat{e}_{ij}^4 - 4 \sum_{j=1}^{l_i} \hat{e}_{ij}^3 \sum_{j=1}^{l_i} \hat{e}_{ij} \right] \right. \\ &\quad \left. + 6l_i \left(\sum_{j=1}^{l_i} \hat{e}_{ij}^2 \right) \left(\sum_{j=1}^{l_i} \hat{e}_{ij} \right)^2 - 3 \left(\sum_{j=1}^{l_i} \hat{e}_{ij} \right)^4 - 3(2l_i - 3) \left[\sum_{j=1}^{l_i} \hat{e}_{ij}^2 \right]^2 \right\}. \end{aligned}$$

For γ_b^4 , the relevant equation is

$$\mathbb{E}[f_1^4(i) - 6f_2^4(i) + 8f_3^4(i) - 6f_4^4(i) + 3f_5^4(i)] = l_i(l_i - 1)(l_i - 2)(l_i - 3)\gamma_b^4,$$

giving us

$$\hat{\gamma}_b^4 = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i-1)(l_i-2)(l_i-3)} \left\{ \left(\sum_{j=1}^{l_i} \hat{e}_{ij} \right)^4 - 6 \left(\sum_{j=1}^{l_i} \hat{e}_{ij}^2 \right) \left(\sum_{j=1}^{l_i} \hat{e}_{ij} \right)^2 + 8 \left(\sum_{j=1}^{l_i} \hat{e}_{ij}^3 \right) \left(\sum_{j=1}^{l_i} \hat{e}_{ij} \right) - 6 \sum_{j=1}^{l_i} \hat{e}_{ij}^4 + 3 \left(\sum_{j=1}^{l_i} \hat{e}_{ij}^2 \right)^2 \right\}.$$

Theorem 2.4. *Under the conditions of Theorem 2.1, when γ_ε^8 and γ_b^8 are finite, we have that*

$$N^{1/2}(\hat{\gamma}_\varepsilon^4 - \gamma_\varepsilon^4) \rightarrow \mathcal{N}_1(0, \mu_\varepsilon^4)$$

and

$$n^{1/2}(\hat{\gamma}_b^4 - \gamma_b^4) \rightarrow \mathcal{N}_1(0, \mu_b^4),$$

where

$$\mu_\varepsilon^4 = \gamma_\varepsilon^8 - (\gamma_\varepsilon^4)^2 - 8\gamma_\varepsilon^3\gamma_\varepsilon^5 + 16\gamma_\varepsilon^2(\gamma_\varepsilon^3)^2$$

and

$$\mu_b^4 = \gamma_b^8 - (\gamma_b^4)^2 - 8\gamma_b^3\gamma_b^5 + 16\gamma_b^2(\gamma_b^3)^2.$$

As in previous cases, μ_ε^4 and μ_b^4 are minimal variances.

3. First step estimation

3.1. Estimation of γ_ε^3 and γ_b^3

In this section, we briefly discuss the fact that different choices of estimating equations may lead to inefficiencies. These observations eventually lead us to the efficient estimators discussed in the previous section. For the third moments, recall that

$$\mathbb{E}f_3^3(i) = l_i\gamma_b^3 + l_i\gamma_\varepsilon^3 \quad \text{and} \quad \mathbb{E}f_2^3(i) = l_i^2\gamma_b^3 + l_i\gamma_\varepsilon^3,$$

from which

$$l_i\gamma_\varepsilon^3 = \frac{1}{l_i-1} [l_i\mathbb{E}f_3^3(i) - \mathbb{E}f_2^3(i)].$$

Summation over $1 \leq i \leq n$ yields

$$\gamma_\varepsilon^3 = \frac{\sum_{i=1}^n (1/(l_i-1)) [l_i\mathbb{E}f_3^3(i) - \mathbb{E}f_2^3(i)]}{N}.$$

If we replace the expectations by their sample analogs and the true e 's by the residuals, then we come up with an estimator of γ_ε^3 similar to that of Cox and Hall [2], where all l_i 's are equal and there are no covariate effects:

$$\hat{\gamma}_\varepsilon^{*3} = \frac{\sum_{i=1}^n (1/(l_i - 1)) [l_i \sum_{j=1}^{l_i} \hat{e}_{ij}^3 - (\sum_{j=1}^{l_i} \hat{e}_{ij}^2)(\sum_{j=1}^{l_i} \hat{e}_{ij})]}{N}.$$

In the same way, we obtain

$$\hat{\gamma}_b^{*3} = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)} \left\{ \sum_{j=1}^{l_i} \hat{e}_{ij}^2 \sum_{j=1}^{l_i} \hat{e}_{ij} - \sum_{j=1}^{l_i} \hat{e}_{ij}^3 \right\}.$$

To formulate limit results for $\hat{\gamma}_\varepsilon^{*3}$ and $\hat{\gamma}_b^{*3}$, we recall that

$$\begin{aligned} \mu_\varepsilon^3 &= \gamma_\varepsilon^6 - (\gamma_\varepsilon^3)^2 - 6\gamma_\varepsilon^2\gamma_\varepsilon^4 + 9(\gamma_\varepsilon^2)^3, \\ \mu_b^3 &= \gamma_b^6 - (\gamma_b^3)^2 - 6\gamma_b^2\gamma_b^4 + 9(\gamma_b^2)^3 \end{aligned}$$

and put

$$\mu_\varepsilon^{*3} = \gamma_\varepsilon^6 - (\gamma_\varepsilon^3)^2 - 6\gamma_\varepsilon^2\gamma_\varepsilon^4 + (4c + 5)(\gamma_\varepsilon^2)^3 + 4(\gamma_\varepsilon^2)^3 x_0' \Sigma^{-1} x_0.$$

Here,

$$\bar{x}_n^* = N^{-1} \sum_{i=1}^n \sum_{j=1}^{l_i} x_{ij}$$

and (as before)

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i} \sum_{j=1}^{l_i} x_{ij}.$$

The vector x_0 in μ_ε^{*3} equals

$$x_0 = \lim_{n \rightarrow \infty} (\bar{x}_n^* - \bar{x}_n),$$

while

$$c = \lim_{n \rightarrow \infty} \frac{N}{n^2} \sum_{i=1}^n l_i^{-1},$$

assuming that both limits exist.

A detailed qualitative interpretation of these quantities will be deferred to the end of this section.

Theorem 3.1. *Under the conditions of Theorem 2.1, when γ_ε^6 and γ_b^6 are finite, we have that*

$$N^{1/2}(\hat{\gamma}_\varepsilon^{*3} - \gamma_\varepsilon^3) \rightarrow \mathcal{N}_1(0, \mu_\varepsilon^{*3} + 4\gamma_b^2(\gamma_\varepsilon^4 - (1-d)(\gamma_\varepsilon^2)^2)), \quad (3.1)$$

where

$$d = \lim_{n \rightarrow \infty} \left[\frac{\sum_{i=1}^n l_i^2}{N} - \frac{\sum_{i=1}^n l_i}{n} \right].$$

As to $\hat{\gamma}_b^{*3}$, we have that

$$n^{1/2}[\hat{\gamma}_b^{*3} - \gamma_b^3] \rightarrow \mathcal{N}_1(0, \mu_b^3) \quad \text{as } n \rightarrow \infty. \tag{3.2}$$

Remark 3.1. As in Section 2, the estimator in the b -case achieves the minimum variance. It equals the variance of the moment estimator based on the true but unknown b_i . In the ε -case, things are less transparent. For example, assume that $l_{ni} \equiv l_n^0$ are all equal for $1 \leq i \leq n$, a situation studied by Cox and Hall [2]. If $l_n^0 \rightarrow \infty$, then $c = 1, x_0 = 0$ and $d = 0$. Hence,

$$\mu_\varepsilon^{*3} = \gamma_\varepsilon^6 - (\gamma_\varepsilon^3)^2 - 6\gamma_\varepsilon^2\gamma_\varepsilon^4 + 9(\gamma_\varepsilon^2)^3 = \mu_\varepsilon^3,$$

the variance of the (central) moment estimator based on the true ε_{ij} . The total variance therefore becomes

$$\mu_\varepsilon^3 + 4\gamma_b^3(\gamma_\varepsilon^4 - (\gamma_\varepsilon^2)^2),$$

which, by the Cauchy–Schwarz inequality, exceeds μ_ε^3 . Hence, in this situation, $\hat{\gamma}_\varepsilon^{*3}$ is inefficient.

Remark 3.2. If $l_i = i^a$ with $0 < a < 1$, then $c = 1/(1 - a^2) > 1$ becomes large as $a \rightarrow 1$. Hence, the quality of the Cox–Hall-type estimator deteriorates in such situations. Worse than that, as our proofs reveal, asymptotic normality may fail in situations where the limit d is not finite.

3.2. Estimation of γ_ε^4 and γ_b^4

For fourth moments, taking expectations of $f_4^4(i)$ and $f_3^4(i)$, we again obtain

$$\mathbb{E}f_4^4(i) = l_i\gamma_b^4 + 6l_i\gamma_b^2\gamma_\varepsilon^2 + l_i\gamma_\varepsilon^4 \tag{3.3}$$

and

$$\mathbb{E}f_3^4(i) = l_i^2\gamma_b^4 + 3l_i(l_i + 1)\gamma_b^2\gamma_\varepsilon^2 + l_i\gamma_\varepsilon^4, \tag{3.4}$$

from which it follows that

$$\gamma_b^4 = \frac{\mathbb{E}f_3^4(i) - \mathbb{E}f_4^4(i)}{l_i(l_i - 1)} - 3\gamma_b^2\gamma_\varepsilon^2.$$

Averaging over $1 \leq i \leq n$ leads to the estimator

$$\hat{\gamma}_b^{*4} = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)} \left\{ \sum_{j=1}^{l_i} \hat{e}_{ij}^3 \sum_{j=1}^{l_i} \hat{e}_{ij} - \sum_{j=1}^{l_i} \hat{e}_{ij}^4 \right\} - 3\hat{\gamma}_b^2\hat{\gamma}_\varepsilon^2,$$

where $\hat{\gamma}_b^2$ and $\hat{\gamma}_\varepsilon^2$ were studied in Section 2.2. From (3.3) and (3.4) we immediately obtain

$$l_i \gamma_\varepsilon^4 = \frac{l_i \mathbb{E} f_4^4(i) - \mathbb{E} f_3^4(i)}{l_i - 1} - 3l_i \gamma_b^2 \gamma_\varepsilon^2$$

and therefore to

$$\hat{\gamma}_\varepsilon^{*4} = N^{-1} \sum_{i=1}^n \frac{1}{l_i - 1} \left\{ l_i \sum_{j=1}^{l_i} \hat{e}_{ij}^4 - \sum_{j=1}^{l_i} \hat{e}_{ij}^3 \sum_{j=1}^{l_i} \hat{e}_{ij} \right\} - 3\hat{\gamma}_b^2 \hat{\gamma}_\varepsilon^2.$$

Cox and Hall [2] also considered these estimators; however, we have discovered that the limit variances are larger than those given in their paper. Therefore, we propose the following modification. First, recall that

$$\mathbb{E} f_2^4(i) = l_i^3 \gamma_b^4 + (l_i^3 + 5l_i^2) \gamma_b^2 \gamma_\varepsilon^2 + l_i(l_i - 1)(\gamma_\varepsilon^2)^2 + l_i \gamma_\varepsilon^4. \quad (3.5)$$

In addition to the $f_m^k(i)$ with $m \leq k$, we again need

$$f_5^4(i) = \left[\sum_{j=1}^{l_i} e_{ij}^2 \right]^2.$$

It follows from (3.5) and (2.14) that

$$\mathbb{E}[f_2^4(i) - f_5^4(i)] = (l_i^3 - l_i^2) \gamma_b^4 + (l_i^3 + 3l_i^2 - 4l_i) \gamma_\varepsilon^2 \gamma_b^2. \quad (3.6)$$

To estimate γ_ε^4 , we are looking for a linear combination of (3.3), (3.4) and (3.6) so that the terms γ_b^4 and $\gamma_\varepsilon^2 \gamma_b^2$ cancel out. In fact, it is easily seen that

$$\mathbb{E}[(2l_i^2 - l_i) f_4^4(i) - (5l_i - 4) f_3^4(i) + 3f_2^4(i) - 3f_5^4(i)] = 2l_i(l_i - 1)(l_i - 2) \gamma_\varepsilon^4.$$

The corresponding estimator of γ_ε^4 becomes

$$\begin{aligned} \hat{\gamma}_\varepsilon^{*4} = N^{-1} \sum_{i=1}^n \frac{1}{2(l_i - 1)(l_i - 2)} & \left\{ (2l_i^2 - l_i) \sum_{j=1}^{l_i} \hat{e}_{ij}^4 - (5l_i - 4) \sum_{j=1}^{l_i} \hat{e}_{ij}^3 \sum_{j=1}^{l_i} \hat{e}_{ij} \right. \\ & \left. + 3 \left(\sum_{j=1}^{l_i} \hat{e}_{ij}^2 \right) \left(\sum_{j=1}^{l_i} \hat{e}_{ij} \right)^2 - 3 \left[\sum_{j=1}^{l_i} \hat{e}_{ij}^2 \right]^2 \right\}. \end{aligned}$$

Following this idea, we also get an estimator of γ_b^4 . Subtracting (3.3) from (3.4), we obtain

$$\mathbb{E}[f_3^4(i) - f_4^4(i)] = (l_i^2 - l_i) \gamma_b^4 + 3(l_i^2 - l_i) \gamma_b^2 \gamma_\varepsilon^2.$$

Together with (3.6), this yields

$$3\mathbb{E}[f_2^4(i) - f_5^4(i)] - (l_i + 4)\mathbb{E}[f_3^4(i) - f_4^4(i)] = 2l_i(l_i - 1)(l_i - 2) \gamma_b^4$$

and therefore

$$\hat{\gamma}_b^{*4} = \frac{1}{n} \sum_{i=1}^n \frac{1}{2l_i(l_i-1)(l_i-2)} \left\{ 3 \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^2 \right) \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij} \right)^2 - 3 \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^2 \right)^2 - (l_i+4) \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^3 \right) \left(\sum_{j=1}^{l_i} \hat{\varepsilon}_{ij} \right) + (l_i+4) \sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^4 \right\}.$$

In the following theorem, we summarize the main results on the limit distributions of $\hat{\gamma}_\varepsilon^{*4}$ and $\hat{\gamma}_b^{*4}$.

Theorem 3.2. *Under the conditions of Theorem 3.1, when γ_ε^8 and γ_b^8 are finite, we have that*

$$N^{1/2}(\hat{\gamma}_\varepsilon^{*4} - \gamma_\varepsilon^4) \rightarrow \mathcal{N}_1(0, \mu_\varepsilon^{*4} + \frac{9}{4}\gamma_b^2[\gamma_\varepsilon^6 - (1-d)(\gamma_\varepsilon^3)^2 - 6\gamma_\varepsilon^2\gamma_\varepsilon^4 + 9(\gamma_\varepsilon^2)^3])$$

and

$$n^{1/2}[\hat{\gamma}_b^{*4} - \gamma_b^4] \rightarrow \mathcal{N}_1(0, \mu_b^4) \quad \text{as } n \rightarrow \infty,$$

where, again,

$$\mu_b^4 = \gamma_b^8 - (\gamma_b^4)^2 - 8\gamma_b^3\gamma_b^5 + 16\gamma_b^2(\gamma_b^3)^2$$

and

$$\mu_\varepsilon^{*4} = \gamma_\varepsilon^8 - (\gamma_\varepsilon^4)^2 - 8\gamma_\varepsilon^3\gamma_\varepsilon^5 + \left(\frac{9}{4}c + \frac{55}{4}\right)\gamma_\varepsilon^2(\gamma_\varepsilon^3)^2 + \frac{9}{4}\gamma_\varepsilon^2(\gamma_\varepsilon^3)^2 x_0' \Sigma^{-1} x_0.$$

Remark 3.3. Our earlier Remarks 3.1 and 3.2 also apply to fourth moments. This more or less led us to look for the new estimators studied in Section 2.

Remark 3.4. We will now discuss the results of this paper in a qualitative way. Suppose that all the b_i 's and ε_{ij} 's are known to the observer. Then, rather than computing residuals, they could be used directly to nonparametrically estimate the (central) moments of interest. Simple computations then show that the variances of these estimators equal μ_b^k and μ_ε^k , respectively. In the case where only residuals are available, the improper weighting in $\hat{\gamma}_\varepsilon^{*k}$ yields variances which heavily depend on the design (via $x_0' \Sigma^{-1} x_0$), the group sizes (via the constants c and d) and the noise variables b_i (via γ_b^2). In such a situation, Theorems 2.3 and 2.4 provide new estimators which also attain the minimum variance in the ε -case and are less vulnerable to the model design.

4. Simulation study

To demonstrate the usefulness of our estimation procedures, a small simulation study will be carried out. The data sets are generated from the model (2.1) with $\alpha = 1$ and $\beta = (1, 2)'$. To estimate the model parameters and the third and fourth moments using the methods developed in this paper, the group values are randomly drawn from a Poisson distribution with mean 5.

Table 1. The true values of the 2nd–4th moments of the random and group errors

c.d.f	2nd	3rd	4th
$0.5N(0, 1)$	0.25	0	0.1875
$0.5t(8)$	0.333	0	0.5
$0.5\Gamma(1, 1) - 0.5$	0.25	0.25	0.5625

The design matrices are generated from a zero-mean normal distribution with covariance matrix $\begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}$. For the random effects b_i and the errors ε_{ij} , we consider the following five cases:

- (a) $\varepsilon_{ij} \sim_{\text{i.i.d.}} 0.5\mathcal{N}_1(0, 1)$ and $b_i \sim_{\text{i.i.d.}} 0.5\mathcal{N}_1(0, 1)$;
- (b) $\varepsilon_{ij} \sim_{\text{i.i.d.}} 0.5\mathcal{N}_1(0, 1)$ and $b_i \sim_{\text{i.i.d.}} 0.5t(8)$;
- (c) $\varepsilon_{ij} \sim_{\text{i.i.d.}} 0.5\mathcal{N}_1(0, 1)$ and $b_i \sim_{\text{i.i.d.}} 0.5\Gamma(1, 1) - 0.5$;
- (d) $\varepsilon_{ij} \sim_{\text{i.i.d.}} 0.5t(8)$ and $b_i \sim_{\text{i.i.d.}} 0.5t(8)$;
- (e) $\varepsilon_{ij} \sim_{\text{i.i.d.}} 0.5t(8)$ and $b_i \sim_{\text{i.i.d.}} 0.5\Gamma(1, 1) - 0.5$.

The true values of the 2nd–4th moments of the errors and random effects are given in Table 1. \mathcal{N}_1 , Γ and t correspond to the normal, gamma and t distributions, respectively.

The following simulation results are based on 1000 samples of data $\{(x_{ij}, y_{ij}) : i = 1, \dots, n, j = 1, \dots, l_i\}$ with $n = 50, 100$. The estimated mean, standard deviation and root mean squared error of the estimators suggested above are reported in Tables 2 and 3. Table 2 presents the results for the model parameters and second moments. For the purposes of comparison, we also include the results for the MLE. Table 3 presents the results for the minimum variance estimators of the third and fourth moments.

In Table 2, the comparison with the MLE shows that our estimators are very competitive, although such a comparison is actually in favor of the MLE when we assume that the distribution is parametric. In fact, empirical studies in the literature also show that the assumption concerning the distribution of the random effects hardly influences the parameter estimates; see Butler and Louis [1] and Verbeke and Lesaffre [6] for details. This indicates that the estimation of moments for the model parameters performs very well.

Appendix

Proof of Theorem 2.1. We first study $\hat{\beta}$. It follows from (2.1) and (2.3) that

$$\hat{\beta} - \beta = \frac{1}{\sum_{i=1}^n l_i} \sum_{i=1}^n \sum_{j=1}^{l_i} \hat{\Sigma}_n^{-1} (x_{ij} - \bar{x}_i) \varepsilon_{ij}, \tag{5.1}$$

Table 2. The results for $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}_\varepsilon^2$ and $\hat{\gamma}_b^2$ in cases (a)–(e) (the numbers in brackets correspond to the MLE)

Case	n	Result	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\gamma}_\varepsilon^2$	$\hat{\gamma}_b^2$
(a)	50	mean	1.0004 (1.0005)	0.9986 (0.9984)	2.0009 (2.0012)	0.2497 (0.2480)	0.2451 (0.2449)
		std	0.0748 (0.0749)	0.0428 (0.0424)	0.0430 (0.0427)	0.0177 (0.0177)	0.0550 (0.0549)
		rmse	0.0748 (0.0749)	0.0428 (0.0424)	0.0430 (0.0427)	0.0178 (0.0178)	0.0552 (0.0552)
	100	mean	0.9965 (0.9964)	1.0000 (0.9999)	1.9995 (1.9996)	0.2496 (0.2496)	0.2476 (0.2473)
		std	0.0544 (0.0544)	0.0307 (0.0304)	0.0302 (0.0300)	0.0128 (0.0128)	0.0392 (0.0393)
		rmse	0.0545 (0.0546)	0.0307 (0.0304)	0.0302 (0.0300)	0.0128 (0.0128)	0.0393 (0.0394)
(b)	50	mean	1.0022 (1.0023)	1.0004 (1.0002)	2.0010 (2.0014)	0.2490 (0.2490)	0.3301 (0.3296)
		std	0.0860 (0.0860)	0.0438 (0.0438)	0.0427 (0.0427)	0.0184 (0.0184)	0.0978 (0.0978)
		rmse	0.0861 (0.0860)	0.0438 (0.0438)	0.0427 (0.0427)	0.0185 (0.0184)	0.0979 (0.0979)
	100	mean	1.0026 (1.0026)	0.9991 (0.9991)	2.0005 (2.0004)	0.2489 (0.2489)	0.3322 (0.3319)
		std	0.0598 (0.0598)	0.0299 (0.0296)	0.0299 (0.0296)	0.0128 (0.0127)	0.0652 (0.0651)
		rmse	0.0598 (0.0598)	0.0299 (0.0296)	0.0299 (0.0296)	0.0128 (0.0128)	0.0652 (0.0651)
(c)	50	mean	0.9988 (0.9988)	1.0004 (1.0003)	1.9962 (1.9960)	0.2485 (0.2485)	0.2421 (0.2418)
		std	0.0754 (0.0754)	0.0430 (0.0426)	0.0418 (0.0415)	0.0178 (0.0178)	0.0981 (0.0981)
		rmse	0.0754 (0.0754)	0.0430 (0.0426)	0.0420 (0.0417)	0.0179 (0.0179)	0.0984 (0.0984)
	100	mean	0.9974 (0.9974)	0.9993 (0.9993)	2.0005 (2.0007)	0.2481 (0.2481)	0.2437 (0.2440)
		std	0.0527 (0.0528)	0.0298 (0.0294)	0.0302 (0.0298)	0.0121 (0.0121)	0.0731 (0.0730)
		rmse	0.0527 (0.0528)	0.0298 (0.0294)	0.0302 (0.0298)	0.0123 (0.0123)	0.0733 (0.0733)
(d)	50	mean	1.0008 (1.0007)	0.9992 (0.9992)	2.0001 (2.0011)	0.3298 (0.3297)	0.3239 (0.3232)
		std	0.0843 (0.0842)	0.0479 (0.0474)	0.0484 (0.0479)	0.0294 (0.0294)	0.0896 (0.0894)
		rmse	0.0843 (0.0842)	0.0479 (0.0474)	0.0484 (0.0480)	0.0296 (0.0296)	0.0900 (0.0900)
	100	mean	1.0004 (1.0003)	1.0006 (1.0004)	1.9996 (1.9996)	0.3315 (0.3315)	0.3305 (0.3303)
		std	0.0618 (0.0618)	0.0336 (0.0334)	0.0351 (0.0350)	0.0213 (0.0213)	0.0697 (0.0697)
		rmse	0.0618 (0.0618)	0.0336 (0.0334)	0.0351 (0.0350)	0.0214 (0.0214)	0.0697 (0.0697)
(e)	50	mean	1.0025 (1.0023)	1.0016 (1.0012)	1.9977 (1.9977)	0.3311 (0.3313)	0.2433 (0.2425)
		std	0.0670 (0.0670)	0.0488 (0.0488)	0.0494 (0.0492)	0.0297 (0.0298)	0.1010 (0.1014)
		rmse	0.0670 (0.0671)	0.0489 (0.0488)	0.0495 (0.0493)	0.0297 (0.0298)	0.1012 (0.1017)
	100	mean	1.0007 (1.0006)	0.9994 (0.9993)	2.0011 (2.0010)	0.3323 (0.3323)	0.2446 (0.2442)
		std	0.0537 (0.0536)	0.0348 (0.0344)	0.0335 (0.0330)	0.0220 (0.0220)	0.0708 (0.0707)
		rmse	0.0537 (0.0536)	0.0348 (0.0344)	0.0335 (0.0330)	0.0220 (0.0220)	0.0710 (0.0709)

where $\hat{\Sigma}_n$ is given in (2.5). To show (2.9), we fix $a \in \mathbb{R}^p$. It suffices to prove that

$$N^{1/2}a'(\hat{\beta} - \beta) \rightarrow \mathcal{N}_1(0, \gamma_\varepsilon^2 a' \Sigma^{-1} a) \quad \text{in distribution.}$$

Since, according to (5.1), $\hat{\beta} - \beta$ is a sum of zero-mean independent random vectors, it remains to check the variance and verify Lindeberg's condition. The variance of $N^{1/2}a'(\hat{\beta} - \beta)$ equals

$$\frac{\gamma_\varepsilon^2}{N} \sum_{i=1}^n \sum_{j=1}^{l_i} [a' \hat{\Sigma}_n^{-1} (x_{ij} - \bar{x}_{i\cdot})]^2 = \gamma_\varepsilon^2 a' \hat{\Sigma}_n^{-1} a \rightarrow \gamma_\varepsilon^2 a' \Sigma^{-1} a,$$

Table 3. The results for $\hat{\gamma}_\varepsilon^k$ and $\hat{\gamma}_b^k$ ($k = 3, 4$) in cases (a)–(e)

Case	n	Result	$\hat{\gamma}_\varepsilon^3$	$\hat{\gamma}_\varepsilon^4$	$\hat{\gamma}_b^3$	$\hat{\gamma}_b^4$
(a)	50	mean	−0.0004	0.1852	−0.0003	0.1813
		std	0.0170	0.0325	0.0519	0.0974
		rmse	0.0170	0.0326	0.0519	0.0976
	100	mean	−0.0002	0.1867	0.0005	0.1835
		std	0.0125	0.0234	0.0362	0.0688
		rmse	0.0125	0.0234	0.0362	0.0689
(b)	50	mean	0.0002	0.1866	−0.0003	0.5054
		std	0.0171	0.0331	0.1948	0.8577
		rmse	0.0171	0.0331	0.1948	0.8578
	100	mean	0.0002	0.1858	0.0076	0.5003
		std	0.0121	0.0239	0.1276	0.4835
		rmse	0.0121	0.0240	0.1278	0.4835
(c)	50	mean	0.0007	0.1864	0.2290	0.4962
		std	0.0174	0.0335	0.2235	0.7649
		rmse	0.0174	0.0335	0.2244	0.7678
	100	mean	0.0002	0.1848	0.2380	0.5317
		std	0.0123	0.0216	0.1778	0.6445
		rmse	0.0123	0.0216	0.1782	0.6445
(d)	50	mean	0.0001	0.4754	0.0033	0.4477
		std	0.0557	0.1832	0.1520	0.4510
		rmse	0.0557	0.1848	0.1520	0.4542
	100	mean	0.0007	0.4862	0.0014	0.4796
		std	0.0403	0.1611	0.1098	0.4100
		rmse	0.0403	0.1617	0.1098	0.4102
(e)	50	mean	−0.0011	0.4832	0.2278	0.5007
		std	0.0578	0.2068	0.2311	0.8052
		rmse	0.0578	0.2075	0.2322	0.8075
	100	mean	0.0001	0.4979	0.2385	0.5355
		std	0.0427	0.1726	0.1753	0.6569
		rmse	0.0427	0.1726	0.1757	0.6575

by (2.6). To verify Lindeberg’s condition, we first fix $\delta > 0$. The Lindeberg function then equals

$$L_n(\delta) = N^{-1} \sum_{i=1}^n \sum_{j=1}^{l_i} [a' \hat{\Sigma}_n^{-1} (x_{ij} - \bar{x}_i)]^2 \int_{\{|a' \hat{\Sigma}_n^{-1} (x_{ij} - \bar{x}_i) | |\varepsilon_{ij}| \geq \delta \sqrt{\Sigma l_i}\}} \varepsilon_{ij}^2 d\mathbb{P}.$$

Recall that, by (2.8),

$$C_n = \frac{\max_{i,j} \|x_{ij} - \bar{x}_i\|}{N^{1/2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We conclude, by the Cauchy–Schwarz inequality and (2.6), that

$$L_n(\delta) \leq a' \hat{\Sigma}_n^{-1} a \int_{\{\|a' \hat{\Sigma}_n^{-1} \|\|\varepsilon\| \geq \delta C_n^{-1}\}} \varepsilon^2 d\mathbb{P} \rightarrow 0,$$

as required. This proves (2.9). For $\hat{\alpha}$, we immediately get from (2.4) that

$$\hat{\alpha} - \alpha = \frac{1}{n} \sum_{i=1}^n b_i + \bar{\varepsilon} - \frac{1}{n} \sum_{i=1}^n \bar{x}'_i (\hat{\beta} - \beta).$$

From (2.7), it follows that $n^{1/2} \bar{\varepsilon} \rightarrow 0$ in squared mean and hence in probability.

Furthermore, by (2.7)–(2.9),

$$n^{-1/2} \sum_{i=1}^n \bar{x}'_i (\hat{\beta} - \beta) \rightarrow 0 \quad \text{in probability.}$$

Summarizing,

$$n^{1/2}(\hat{\alpha} - \alpha) = n^{-1/2} \sum_{i=1}^n b_i + o_{\mathbb{P}}(1) \rightarrow \mathcal{N}_1(0, \gamma_b^2).$$

This shows (2.10) and thereby completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. From the definition of $\hat{\gamma}_\varepsilon^2$ and (2.11), we readily get

$$\begin{aligned} N\hat{\gamma}_\varepsilon^2 &= \sum_{i=1}^n \frac{1}{l_i - 1} \left\{ l_i \sum_{j=1}^{l_i} [\varepsilon_{ij}^2 + (z'_{ij}(\beta - \hat{\beta}))^2 + 2\varepsilon_{ij}z'_{ij}(\beta - \hat{\beta})] \right. \\ &\quad \left. - \sum_{j=1}^{l_i} \sum_{k=1}^{l_i} [\varepsilon_{ij} + z'_{ij}(\beta - \hat{\beta})][\varepsilon_{ik} + z'_{ik}(\beta - \hat{\beta})] \right\} \\ &= \sum_{i=1}^n \sum_{j=1}^{l_i} \varepsilon_{ij}^2 - \sum_{i=1}^n \frac{1}{l_i - 1} \sum_{j \neq k} \varepsilon_{ij} \varepsilon_{ik} + 2 \sum_{i=1}^n \sum_{j=1}^{l_i} \varepsilon_{ij} z'_{ij}(\beta - \hat{\beta}) \\ &\quad + \sum_{i=1}^n \frac{l_i}{l_i - 1} \sum_{j=1}^{l_i} (z'_{ij}(\beta - \hat{\beta}))^2 - \sum_{i=1}^n \frac{1}{l_i - 1} \sum_{j=1}^{l_i} \sum_{k=1}^{l_i} [z'_{ij}(\beta - \hat{\beta})][z'_{ik}(\beta - \hat{\beta})] \\ &\quad - \sum_{i=1}^n \frac{2}{l_i - 1} \sum_{j \neq k} \varepsilon_{ij} z'_{ik}(\beta - \hat{\beta}) \\ &\equiv I - II + III + IV - V - VI. \end{aligned}$$

Of these six terms, only the first will be a leading term, while the others are remainders. For example, II is a sum of centered independent random variables with variance

$$2(\gamma_\varepsilon^2)^2 \sum_{i=1}^n \frac{l_i}{l_i - 1} \leq 4n(\gamma_\varepsilon^2)^2.$$

We conclude, in view of (2.7), that

$$N^{-1/2}II \rightarrow 0 \quad \text{in probability.}$$

To show the same for III, it suffices to prove, because of (2.9), that

$$N^{-1} \sum_{i=1}^n \sum_{j=1}^{l_i} \varepsilon_{ij} z_{ij} \rightarrow 0 \quad \text{in probability.}$$

Again, this is a sum of centered random vectors with covariance

$$\gamma_\varepsilon^2 \left[\frac{\sum_{i=1}^n l_i (\bar{x}_i - \bar{x})(\bar{x}_i - \bar{x})'}{N^2} + \frac{\hat{\Sigma}_n}{N} \right] \rightarrow 0.$$

Similarly, the convergence of $N^{-1/2}IV$, $N^{-1/2}V$ and $N^{-1/2}VI$ to zero follows from (2.6)–(2.9). All together, this shows that

$$N^{1/2}[\hat{\gamma}_\varepsilon^2 - \gamma_\varepsilon^2] = N^{-1/2} \left[\sum_{i=1}^n \sum_{j=1}^{l_i} (\varepsilon_{ij}^2 - \gamma_\varepsilon^2) \right] + o_{\mathbb{P}}(1)$$

and hence (2.12), by a simple application of the central limit theorem. To show (2.13), we note that

$$\hat{\gamma}_b^2 = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)} \sum_{j \neq k} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{ik}.$$

Again using (2.11) and applying similar arguments to those used before, we obtain

$$\hat{\gamma}_b^2 = \frac{1}{n} \sum_{i=1}^n b_i^2 + o_{\mathbb{P}}(n^{-1/2}),$$

from which it follows that (2.13) holds. □

Proof of Theorem 2.3. We first deal with $\hat{\gamma}_b^3$. Simple algebraic manipulations yield

$$\hat{\gamma}_b^3 = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)(l_i - 2)} \sum_{j \neq k \neq l} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{ik} \hat{\varepsilon}_{il}.$$

Expanding $\hat{\varepsilon}_{ij}$ into $\hat{\varepsilon}_{ij} = \varepsilon_{ij} + b_i - (\bar{b} + \bar{\varepsilon}) + z'_{ij}(\beta - \hat{\beta})$, we may again neglect all contributions involving the $z'_{ij}(\beta - \hat{\beta})$. Hence, up to an $\mathfrak{o}_{\mathbb{P}}(n^{-1/2})$ term,

$$\begin{aligned} \hat{\gamma}_b^3 &= \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i-1)(l_i-2)} \sum_{j \neq k \neq l} \varepsilon_{ij} \varepsilon_{ik} \varepsilon_{il} \\ &+ \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i-1)} \sum_{j \neq k} [3b_i \varepsilon_{ij} \varepsilon_{ik} - 3(\bar{b} + \bar{\varepsilon}) \varepsilon_{ij} \varepsilon_{ik}] \\ &+ \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i} \sum_j [3b_i^2 \varepsilon_{ij} - 6b_i(\bar{b} + \bar{\varepsilon}) \varepsilon_{ij} + 3(\bar{b} + \bar{\varepsilon})^2 \varepsilon_{ij}] \\ &+ \frac{1}{n} \sum_{i=1}^n [b_i^3 - 3(\bar{b} + \bar{\varepsilon})b_i^2] + 3\bar{b}(\bar{b} + \bar{\varepsilon})^2 - (\bar{b} + \bar{\varepsilon})^3. \end{aligned}$$

Under the assumptions of the theorem, the first three sums are negligible, as are the last two terms. Hence,

$$\begin{aligned} n^{1/2}[\hat{\gamma}_b^3 - \gamma_b^3] &= n^{-1/2} \sum_{i=1}^n [b_i^3 - \gamma_b^3 - 3(\bar{b} + \bar{\varepsilon})b_i^2] + \mathfrak{o}_{\mathbb{P}}(1) \\ &= n^{-1/2} \sum_{i=1}^n [b_i^3 - \gamma_b^3 - 3\gamma_b^3 b_i] + \mathfrak{o}_{\mathbb{P}}(1). \end{aligned}$$

The conclusion for $\hat{\gamma}_b^3$ now readily follows from the central limit theorem. For $\hat{\gamma}_\varepsilon^3$, we may write

$$\begin{aligned} N\hat{\gamma}_\varepsilon^3 &= \sum_i \frac{2}{(l_i-1)(l_i-2)} \sum_{j \neq k \neq l} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{ik} \hat{\varepsilon}_{il} \\ &- \sum_i \frac{3}{l_i-1} \sum_{j \neq k} \hat{\varepsilon}_{ij}^2 \hat{\varepsilon}_{ik} + \sum_i \sum_j \hat{\varepsilon}_{ij}^3. \end{aligned}$$

If we once again ignore the higher order terms of $z'_{ij}(\beta - \hat{\beta})$, we find that in the expansion of $\hat{\gamma}_\varepsilon^3$, we have

$$\begin{aligned} N\hat{\gamma}_\varepsilon^3 &= \sum_i \frac{2}{(l_i-1)(l_i-2)} \sum_{j \neq k \neq l} \varepsilon_{ij} \varepsilon_{ik} \varepsilon_{il} \\ &- \sum_i \frac{3}{l_i-1} \sum_{j \neq k} \varepsilon_{ij}^2 \varepsilon_{ik} + \sum_i \sum_j \varepsilon_{ij}^3 + \mathfrak{o}_{\mathbb{P}}([\sum l_i]^{1/2}) \\ &= \sum_i \sum_j [\varepsilon_{ij}^3 - 3\gamma_\varepsilon^2 \varepsilon_{ij}] + \mathfrak{o}_{\mathbb{P}}(N^{1/2}). \end{aligned}$$

The conclusion for $\hat{\gamma}_\varepsilon^3$ now follows easily from the central limit theorem after centering the ε_{ij}^3 . \square

Proof of Theorem 2.4. For $\hat{\gamma}_b^4$, we check that

$$n\hat{\gamma}_b^4 = \sum_{i=1}^n \frac{1}{l_i(l_i-1)(l_i-2)(l_i-3)} \sum_{j \neq k \neq l \neq m} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{ik} \hat{\varepsilon}_{il} \hat{\varepsilon}_{im}.$$

Again neglecting all terms that contain $z'_{ij}(\beta - \hat{\beta})$, we get, with $v_i = b_i - (\bar{b} + \bar{\varepsilon})$,

$$\begin{aligned} n\hat{\gamma}_b^4 &= \sum_{i=1}^n \frac{1}{l_i(l_i-1)(l_i-2)(l_i-3)} \sum_{j \neq k \neq l \neq m} \varepsilon_{ij} \varepsilon_{ik} \varepsilon_{il} \varepsilon_{im} \\ &+ 4 \sum_{i=1}^n \frac{v_i}{l_i(l_i-1)(l_i-2)} \sum_{j \neq k \neq l} \varepsilon_{ij} \varepsilon_{ik} \varepsilon_{il} \\ &+ 6 \sum_{i=1}^n \frac{v_i^2}{l_i(l_i-1)} \sum_{j \neq k} \varepsilon_{ij} \varepsilon_{ik} + 4 \sum_{i=1}^n \frac{v_i^3}{l_i} \sum_j \varepsilon_{ij} \\ &+ \sum_{i=1}^n v_i^4. \end{aligned}$$

Since the first four sums are all $o_{\mathbb{P}}(n^{1/2})$, we obtain

$$n^{1/2}[\hat{\gamma}_b^4 - \gamma_b^4] = n^{-1/2} \sum_{i=1}^n [v_i^4 - \gamma_b^4] + o_{\mathbb{P}}(1).$$

The distributional convergence of $\hat{\gamma}_b^4$ now readily follows from the central limit theorem after an expansion of the last sum into

$$n^{-1/2} \sum_{i=1}^n [b_i^4 - \gamma_b^4 - 4\gamma_b^3 b_i] + o_{\mathbb{P}}(1).$$

For $\hat{\gamma}_\varepsilon^4$, we check that

$$\begin{aligned} N\hat{\gamma}_\varepsilon^4 &= -3 \sum_{i=1}^n \frac{1}{(l_i-1)(l_i-2)(l_i-3)} \sum_{j \neq k \neq l \neq m} \hat{\varepsilon}_{ij} \hat{\varepsilon}_{ik} \hat{\varepsilon}_{il} \hat{\varepsilon}_{im} \\ &- 4 \sum_{i=1}^n \sum_{i=1}^n \frac{1}{l_i-1} \sum_{j \neq k} \hat{\varepsilon}_{ij}^3 \hat{\varepsilon}_{ik} + 6 \sum_{i=1}^n \frac{1}{(l_i-1)} \frac{1}{(l_i-2)} \sum_{j \neq k \neq l} \hat{\varepsilon}_{ij}^2 \hat{\varepsilon}_{ik} \hat{\varepsilon}_{il} \\ &+ \sum_{i=1}^n \sum_j \hat{\varepsilon}_{ij}^4. \end{aligned}$$

Similarly to the proof of Theorem 2.3, it can be shown that

$$N(\hat{\gamma}_\varepsilon^4 - \gamma_\varepsilon^4) = \sum_i \sum_j [\varepsilon_{ij}^4 - \gamma_\varepsilon^4 - 4\gamma_\varepsilon^3 \varepsilon_{ij}] + o_{\mathbb{P}}(N^{1/2}),$$

from which the conclusion follows. □

Proof of Theorem 3.1. We first deal with $\hat{\gamma}_b^{*3}$. By definition,

$$\hat{\gamma}_b^{*3} = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)} \sum_{j \neq k} \hat{\varepsilon}_{ij}^2 \hat{\varepsilon}_{ik}.$$

Our goal will be to use (2.11) in order to express the last double sum in terms of b_i , ε_{ij} and negligible remainders. Actually, in view of (2.7)–(2.9), since the standardizing factor of $\hat{\gamma}_b^{*3}$ is $n^{1/2} = o(N^{1/2})$, all terms in the expansion of $\hat{\gamma}_b^{*3}$ containing $z'_{ij}(\beta - \hat{\beta})$ are negligible. In other words,

$$\hat{\gamma}_b^{*3} = \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)} \sum_{j \neq k} (b_i + \varepsilon_{ij} - (\bar{b} + \bar{\varepsilon}))^2 (b_i + \varepsilon_{ik} - (\bar{b} + \bar{\varepsilon})) + o_{\mathbb{P}}(n^{-1/2}).$$

After some simple but tedious rearrangements, this becomes

$$\begin{aligned} \hat{\gamma}_b^{*3} &= \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i(l_i - 1)} \left[\sum_{j \neq k} (\varepsilon_{ij}^2 - \gamma_\varepsilon^2) \varepsilon_{ik} + 2b_i \sum_{j \neq k} \varepsilon_{ij} \varepsilon_{ik} - 2(\bar{b} + \bar{\varepsilon}) \sum_{j \neq k} \varepsilon_{ij} \varepsilon_{ik} \right] \\ &\quad + \frac{1}{n} \sum_{i=1}^n \frac{1}{l_i} \left[b_i \sum_j (\varepsilon_{ij}^2 - \gamma_\varepsilon^2) + 3(b_i^3 - \gamma_b^2) \sum_j \varepsilon_{ij} \right. \\ &\quad \quad \left. - (\bar{b} + \bar{\varepsilon}) \sum_j (\varepsilon_{ij}^2 - \gamma_\varepsilon^2) - (\bar{b} + \bar{\varepsilon}) 6b_i \sum_j \varepsilon_{ij} \right] \\ &\quad - 3(\bar{b} + \bar{\varepsilon}) \frac{1}{n} \sum_{i=1}^n (b_i^2 - \gamma_b^2) + 2(\bar{b} + \bar{\varepsilon})^3 + \frac{1}{n} \sum_{i=1}^n b_i^3 - 3\bar{b} \gamma_b^2 + o_{\mathbb{P}}(n^{-1/2}). \end{aligned}$$

To identify remainders, we note that $n^{1/2}(\bar{b} + \bar{\varepsilon}) = O_{\mathbb{P}}(1)$. Also, all summands in the double and triple sums are centered and independent. Computation of variances shows that they are all negligible. In summary, we get

$$n^{1/2}[\hat{\gamma}_b^{*3} - \gamma_b^3] = n^{-1/2} \sum_{i=1}^n [b_i^3 - \gamma_b^3 - 3\gamma_b^2 b_i] + o_{\mathbb{P}}(1).$$

This i.i.d. representation of $\hat{\gamma}_b^{*3}$ is the key tool for (3.2) – just apply the central limit theorem to the leading sum.

We will only study $\hat{\gamma}_\varepsilon^{*3}$ briefly. First, by definition,

$$N\hat{\gamma}_\varepsilon^{*3} = \sum_{i=1}^n \sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^3 - \sum_{i=1}^n \frac{\sum_{j \neq k} \hat{\varepsilon}_{ij}^2 \hat{\varepsilon}_{ik}}{l_i - 1}.$$

To expand the two expressions into leading terms and remainders, recall that the final standardizing factor in (3.1) will be $N^{1/2}$, which is the same as in (2.9). We conclude that, under the conditions of the theorem, terms containing higher orders of $z'_{ij}(\beta - \hat{\beta})$ are negligible. Hence, up to remainders,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^{l_i} \hat{\varepsilon}_{ij}^3 &= \sum_{i=1}^n \sum_{j=1}^{l_i} (\varepsilon_{ij} - \bar{\varepsilon} + b_i - \bar{b} + z'_{ij}(\beta - \hat{\beta}))^3 \\ &= \sum_{i=1}^n \sum_{j=1}^{l_i} \varepsilon_{ij}^3 + \sum_{i=1}^n \sum_{j=1}^{l_i} 3\varepsilon_{ij}^2(-\bar{\varepsilon} + b_i - \bar{b}) + \sum_{i=1}^n \sum_{j=1}^{l_i} 3\varepsilon_{ij}(-\bar{\varepsilon} + b_i - \bar{b})^2 \\ &\quad + \sum_{i=1}^n \sum_{j=1}^{l_i} (-\bar{\varepsilon} + b_i - \bar{b})^3 + \sum_{i=1}^n \sum_{j=1}^{l_i} 3\varepsilon_{ij}^2 z'_{ij}(\beta - \hat{\beta}) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^{l_i} 3(-\bar{\varepsilon} + b_i - \bar{b})^2 z'_{ij}(\beta - \hat{\beta}) + \sum_{i=1}^n \sum_{j=1}^{l_i} 6\varepsilon_{ij}(-\bar{\varepsilon} + b_i - \bar{b}) z'_{ij}(\beta - \hat{\beta}). \end{aligned}$$

A detailed study of these sums yields

$$\begin{aligned} N^{1/2}(\hat{\gamma}_\varepsilon^{*3} - \gamma_\varepsilon^3) &= N^{-1/2} \sum_{i=1}^n \left\{ 2\gamma_\varepsilon^2(l_i - \bar{l}_n)b_i + \sum_{j=1}^{l_i} \left[\varepsilon_{ij}^3 - \gamma_\varepsilon^3 + 2b_i(\varepsilon_{ij}^2 - \gamma_\varepsilon^2) \right. \right. \\ &\quad \left. \left. - \gamma_\varepsilon^2 \left(1 + \frac{2\bar{l}_n}{l_i} + 2x'_0 \hat{\Sigma}_n^{-1}(x_{ij} - \bar{x}_{i\cdot}) \right) \varepsilon_{ij} \right] \right\} + o_{\mathbb{P}}(1). \end{aligned}$$

The leading part is a sum of centered independent random variables to which the central limit theorem may be applied. Its variance satisfies

$$\begin{aligned} &\mathbb{E}[\varepsilon^3 - \gamma_\varepsilon^3]^2 + 4\gamma_b^2 \mathbb{E}[\varepsilon^2 - \gamma_\varepsilon^2]^2 + 4(\gamma_\varepsilon^2)^2 \gamma_b^2 d \\ &\quad - 6\gamma_\varepsilon^2 \mathbb{E}\varepsilon^4 + (\gamma_\varepsilon^2)^3 [5 + 4c + 4x'_0 \Sigma^{-1} x_0] + o(1) \\ &\rightarrow \mu_\varepsilon^{*3} + 4\gamma_b^2 (\gamma_\varepsilon^4 - (1-d)(\gamma_\varepsilon^2)^2), \end{aligned}$$

as desired. This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. The necessary arguments are similar to those used before and are therefore omitted. Details may be obtained from the authors. \square

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