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Invariance principles for linear processes with application to isotonic regression

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In this paper, we prove maximal inequalities and study the functional central limit theorem for the partial sums of linear processes generated by dependent innovations. Due to the general weights, these processes can exhibit long-range dependence and the limiting distribution is a fractional Brownian motion. The proofs are based on new approximations by a linear process with martingale difference innovations. The results are then applied to study an estimator of the isotonic regression when the error process is a (possibly long-range dependent) time series.

Keywords: fractional Brownian motion; generalizations of martingales; invariance principles; isotonic regression; linear processes; moment inequalities

1. Introduction and notation

Without loss of generality, we assume that all the strictly stationary sequences $(\xi_i)_{i\in\mathbf{Z}}$ considered in this paper are given by $\xi_i = \xi_0 \circ T^i$, where $T:\Omega \mapsto \Omega$ is a bijective bimeasurable transformation preserving the probability $\mathbf{P}$ on $(\Omega, \mathcal{A})$. We denote by $\mathcal{I}$ the σ-algebra of all T-invariant sets. For a subfield $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, let $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let $\mathcal{F}_{-\infty} = \bigcap_{n\geq 0} \mathcal{F}_{-n}$ and $\mathcal{F}_\infty = \bigvee_{k\in\mathbf{Z}} \mathcal{F}_k$. The sequence $(\mathcal{F}_i)_{i\in\mathbf{Z}}$ will be called a *stationary filtration*. We also assume that ξ_0 is *regular*, that is, $\mathbf{E}(\xi_0|\mathcal{F}_{-\infty}) = 0$ and ξ_0 is $\mathcal{F}_\infty$-measurable. On $\mathbf{L}^2$, we define the projection operator P_j by

$$P_j(Y) = \mathbf{E}(Y|\mathcal{F}_j) - \mathbf{E}(Y|\mathcal{F}_{j-1}).$$

For any random variable Y, $\|Y\|_p$ denotes the norm in $\mathbf{L}^p$.

Recall that the linear process $X_k = \sum_{i\in\mathbf{Z}} a_i \xi_{k-i}$ is well defined in $\mathbf{L}^2$ for any $(a_i)_{i\in\mathbf{Z}}$ in ℓ^2 (i.e., $\sum_{i\in\mathbf{Z}} a_i^2 < \infty$) if and only if the stationary sequence $(\xi_i)_{i\in\mathbf{Z}}$ has a bounded spectral density. Let $S_n = X_1 + \cdots + X_n$ and $c_{n,j} = a_{1-j} + \cdots + a_{n-j}$. In the case where ξ_0 is $\mathcal{F}_0$-measurable, Peligrad and Utev [19] have proven that if the sequence $(\xi_i)_{i\in\mathbf{Z}}$ satisfies an appropriate weak dependence condition, then

$$\left(\sum_{j\in\mathbf{Z}} c_{n,j}^2\right)^{-1/2} S_n$$

1350-7265

converges in distribution to $\sqrt{\eta}N$, where η is a non-negative $\mathcal{I}$-measurable random variable and N is a standard normal random variable independent of η. Their result extends the classical result of Ibragimov [12] from i.i.d. ξ_i's to the case of weakly dependent sequences. In particular, the result applies if

$$\sum_{i\in\mathbf{Z}} \|P_0(\xi_i)\|_2 < \infty. \tag{1}$$

Note that if this condition is satisfied, then the series $\sum_{k\in\mathbf{Z}} |\mathbf{E}(\xi_0\xi_k)|$ converges. Indeed, since $\xi_k = \sum_{i\in\mathbf{Z}} P_i(\xi_k)$ and since $\mathbf{E}(P_i(\xi_0)P_j(\xi_k)) = 0$ if $i \neq j$, it follows that for any $k \in \mathbf{Z}$,

$$|\mathbf{E}(\xi_0\xi_k)| \le \left|\sum_{i\in\mathbf{Z}} \mathbf{E}(P_i(\xi_0)P_i(\xi_k))\right| \le \sum_{i\in\mathbf{Z}} \|P_0(\xi_i)\|_2 \|P_0(\xi_{k+i})\|_2$$

so that $\sum_{k\in\mathbf{Z}} |\mathbf{E}(\xi_0\xi_k)| \le (\sum_{i\in\mathbf{Z}} \|P_0(\xi_i)\|_2)^2$. In addition, under condition (1), the non-negative random variable η satisfies $\eta = \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0\xi_k|\mathcal{I})$.

Condition (1) was introduced by Hannan [9], and by Heyde [10] in a slightly weaker form, and is well adapted to the analysis of time series (see, in particular, the application to time series regression given in the paper by Hannan [9]). As we shall see in our Remark 3.3, condition (1) is also satisfied if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \|\mathbf{E}(\xi_n|\mathcal{F}_0)\|_2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \|\xi_{-n} - \mathbf{E}(\xi_{-n}|\mathcal{F}_0)\|_2 < \infty, \tag{2}$$

which is weaker than the condition introduced by Gordin [7]. If ξ_0 is $\mathcal{F}_0$-measurable, then condition (2) leads to interesting new conditions for weakly dependent sequences and can be successfully applied to functions of dynamical systems (see [19], Section 3, and [4], Section 6, for more details).

A natural question is now: what can we say about the weak convergence of the partial sum process

$$\left\{\left(\sum_{j\in\mathbf{Z}} c_{n,j}^2\right)^{-1/2} S_{[nt]}, t \in [0,1]\right\} \tag{3}$$

in the space $D([0,1])$ of cadlag functions equipped with the uniform topology? Due to the results of Davydov [3] for i.i.d. ξ_i's, we know that the question is not as simple as for the central limit question and that the limiting process (when it exists) depends on the behavior of the normalizing sequence $v_n^2 = \sum_{j\in\mathbf{Z}} c_{n,j}^2$. More precisely, if (1) holds and if there exists $\beta \in]0,2]$ such that

$$\text{for any } t \in]0,1] \qquad \lim_{n\to\infty} \frac{v_{[nt]}^2}{v_n^2} = t^\beta, \tag{4}$$

then we show in Theorems 3.1 and 3.2 that the finite-dimensional marginals of the process (3) converge in distribution to those of $\sqrt{\eta}W_H$, where W_H is a fractional Brownian motion, independent of η, with Hurst index $H = \beta/2$. The question is now: under what conditions can we obtain the tightness in $D([0,1])$?

In Theorem 3.1 of Section 3.1, we show that if $\beta \in]1, 2]$, then condition (1) is sufficient for weak convergence in $D([0, 1])$. If $\beta \in]0, 1]$, we point out in Theorem 3.1 that the convergence in $D([0, 1])$ holds if (1) is replaced by the stronger condition

$$\sum_{i \in \mathbf{Z}} \|P_0(\xi_i)\|_q < \infty \qquad \text{for } q > 2/\beta. \tag{5}$$

As a matter of fact, for $\beta = 1$, it is known from counterexamples given in [29] and [16] that if the sequence $(\xi_i)_{i \in \mathbf{Z}}$ is i.i.d. with $\mathbf{E}(\xi_0^2) < \infty$, then the weak invariance principle may not be true for the partial sums of the linear process, so a reinforcement of (1) is necessary. The case $\beta = 1$, where $W_{1/2}$ is a standard Brownian motion, is of special interest and is known as the weakly dependent case. In that case, we point out in Section 3.2 that if we impose some additional assumptions on $(a_i)_{i \in \mathbf{Z}}$, then condition (1) is sufficient for the weak invariance principle (Comments 3.1 and 3.2) or may be reinforced in a weaker way than (5) (Theorem 3.3).

Note that, with the notation above, the sum S_n may be written as

$$S_n = \sum_{i \in \mathbf{Z}} c_{n,i} \xi_i. \tag{6}$$

Consequently, to prove our main theorems, in Section 2, we give two preliminary results for linear statistics of type (6): first, a moment inequality given in Proposition 2.1 and, next, a martingale approximation result given in Proposition 2.2, which enables us to go back to the standard case where the ξ_i's are martingale differences. Both results are given in terms of Orlicz norms.

Our results provide, besides the invariance principles, estimates of the maximums of partial sums that make them appealing to the study of statistics involving linear processes. In Section 4, we apply our results to the so-called isotonic regression problem

$$y_k = \phi\left(\frac{k}{n}\right) + X_k, \qquad k = 1, 2, \ldots, n, \tag{7}$$

where ϕ is non-decreasing and the error X_k is a linear process. We follow the general scheme given in [1], who showed that in the context of dependent errors, the main tools to obtain the asymptotic distribution of the isotonic estimator $\hat{\phi}$ are the convergence in $D([0, 1])$ of the partial sum process defined in (3) and a suitable maximal inequality for the rescaled stochastic term (see their condition (14)). Zhao and Woodroofe [30] shed light on the fact that, in addition to the weak invariance principle, it is, in fact, enough to prove a suitable maximal inequality directly on the partial sums of the error process. As in [1], the rate of convergence of $\hat{\phi}$ is determined by the asymptotic behavior of the normalizing sequence $v_n^2 = \sum_{j \in \mathbf{Z}} c_{n,j}^2$ and the limiting distribution depends on the limiting process W_H.

2. Moment inequalities and martingale approximation for Orlicz norms

For $\Psi : \mathbf{R}_+ \to \mathbf{R}_+$ a Young function (convex, increasing, $\Psi(0) = 0$ and $\lim_{x \to \infty} \Psi(x) = \infty$), we denote by $\mathbf{L}_\Psi$ the Orlicz space defined as the space of all random variables X such that

$\mathbf{E}\Psi(|X|/c) < \infty$ for some $c > 0$. It is a Banach space for the norm

$$\|X\|_\Psi = \inf\{c > 0, \mathbf{E}\Psi(|X|/c) \le 1\}.$$

Note that if $\Psi(x) = x^q$, $1 \le q < \infty$, then $\mathbf{L}_\Psi = \mathbf{L}^q$.

Let us also introduce the following class of functions (see [5], page 60). For $\alpha > 0$, the class $\mathcal{A}_\alpha$ consists of functions $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$, where $\Phi(0) = 0$, Φ is non-decreasing continuous and such that

$$\Phi(cx) \le c^\alpha \Phi(x) \qquad \text{for all } c \ge 2, x \ge 0.$$

We also denote by $\mathcal{C}(\mathcal{A}_\alpha)$ the class of functions Ψ such that Ψ is a Young function in $\mathcal{A}_\alpha$ and $x \mapsto \Psi(\sqrt{x})$ is a convex function.

Proposition 2.1. *Let $\{Y_k\}_{k\in\mathbf{Z}}$ be a sequence of random variables such that for all k, $\mathbf{E}(Y_k | \mathcal{F}_{-\infty}) = 0$ almost surely and Y_k is $\mathcal{F}_\infty$-measurable. Let Ψ be a function in $\mathcal{C}(\mathcal{A}_\alpha)$. Assume that*

$$\|P_{k-j}(Y_k)\|_\Psi \le p_j \quad \text{and} \quad D_\Psi := \sum_{j=-\infty}^{\infty} p_j < \infty.$$

For any positive integer m, let $\{c_{m,j}\}_{j\in\mathbf{Z}}$ be a sequence in ℓ^2. Define $S_m = \sum_{j\in\mathbf{Z}} c_{m,j} Y_j$. Then, for all $m \ge 1$, there exists a positive constant C_α, depending only on α, such that

$$\|S_m\|_\Psi \le C_\alpha D_\Psi \left(\sum_{j\in\mathbf{Z}} c_{m,j}^2\right)^{1/2}. \tag{8}$$

Remark 2.1. Using the notation of the above proposition, we get, for the special function $\Psi(x) = x^q$ with $q \in [2, \infty[$, the following moment inequality. Assume that

$$\|P_{k-j}(Y_k)\|_q \le p_j \quad \text{and} \quad D_q := \sum_{j=-\infty}^{\infty} p_j < \infty.$$

Then, for any $m \ge 1$,

$$\|S_m\|_q \le C_q \left(\sum_{j\in\mathbf{Z}} c_{m,j}^2\right)^{1/2} D_q,$$

where $C_q^q = 18 q^{3/2}/(q-1)^{1/2}$.

For all $j \in \mathbf{Z}$, let $d_j = \sum_{\ell\in\mathbf{Z}} P_j(\xi_\ell)$. Clearly, $(d_j)_{j\in\mathbf{Z}}$ is a stationary sequence of martingale differences with respect to the filtration $(\mathcal{F}_j)_{j\in\mathbf{Z}}$.

Proposition 2.2. *For any positive integer n, let $\{c_{n,i}\}_{i\in\mathbf{Z}}$ be a sequence in ℓ^2. Let Ψ be a function in $\mathcal{C}(\mathcal{A}_\alpha)$. If $\sum_{j\in\mathbf{Z}}\|P_0(\xi_j)\|_\Psi<\infty$, then we have the following martingale-difference approximation: for any positive integer m, there exists a positive constant C_α, depending only on α, such that*

$$\Big\|\sum_{i\in\mathbf{Z}}c_{n,i}(\xi_i-d_i)\Big\|_\Psi\le 2C_\alpha\Big(\sum_{i\in\mathbf{Z}}c_{n,i}^2\Big)^{1/2}\sum_{|k|\ge m}\|P_0(\xi_k)\|_\Psi \\ +3C_\alpha m\Big(\sum_{j\in\mathbf{Z}}(c_{n,j}-c_{n,j-1})^2\Big)^{1/2}\sum_{j\in\mathbf{Z}}\|P_0(\xi_j)\|_\Psi.$$

Corollary 2.1. *Let $(a_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2. Let Ψ be a function in $\mathcal{C}(\mathcal{A}_\alpha)$. Assume that $\xi_0\in L_\Psi$ and $\sum_j\|P_0(\xi_j)\|_\Psi<\infty$. Let $X_k=\sum_{j\in\mathbf{Z}}a_j\xi_{k-j}$ and $Y_k=\sum_{j\in\mathbf{Z}}a_jd_{k-j}$. Set $S_n=\sum_{k=1}^n X_k$ and $T_n=\sum_{k=1}^n Y_k$. Then, for any positive m, there exist positive constants C_1 and C_2 such that*

$$\|S_n-T_n\|_\Psi\le C_1 v_n\sum_{|k|\ge m}\|P_0(\xi_k)\|_\Psi+C_2 m, \tag{9}$$

where $v_n^2=\sum_{j\in\mathbf{Z}}c_{n,j}^2$ and $c_{n,j}=a_{1-j}+\cdots+a_{n-j}$.

Proof. We apply Proposition 2.2 by noting that $S_n-T_n=\sum_{j\in\mathbf{Z}}c_{n,j}(\xi_j-d_j)$ and that

$$\sum_{j\in\mathbf{Z}}(c_{n,j}-c_{n,j-1})^2\le 4\sum_{j\in\mathbf{Z}}a_j^2.$$

□

Using the Orlicz norms, we give the following maximal inequality, which is a refinement of inequality (6) in [27], Proposition 1.

Lemma 2.1. *Let Ψ be a Young function. Let $p\ge 1$ and write $\Psi_p(x)$ for $\Psi(x^p)$. Let $(Y_i)_{1\le i\le 2^N}$ be a strictly stationary sequence of random variables such that $\|Y_1\|_{\Psi_p}<\infty$. Let $S_n=Y_1+\cdots+Y_n$. Then*

$$\Big\|\max_{1\le m\le 2^N}|S_m|\Big\|_p\le\sum_{L=0}^N\|S_{2^L}\|_{\Psi_p}(\Psi^{-1}(2^{N-L}))^{1/p}.$$

Remark 2.2. Clearly, we can take $\Psi(x)=x$ in Lemma 2.1. Hence, in the stationary case, we recover the inequality (6) in [27].

3. Invariance principle for linear processes

In this section, we shall focus on the weak invariance principle for linear processes. Let $(a_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2. Let

$$X_k = \sum_{i\in\mathbf{Z}} a_i \xi_{k-i} \quad \text{and} \quad S_{[nt]} = \sum_{k=1}^{[nt]} X_k, \tag{10}$$

and

$$v_n^2 = \sum_{j\in\mathbf{Z}} c_{n,j}^2, \qquad \text{where } c_{n,j} = a_{1-j} + \cdots + a_{n-j}. \tag{11}$$

The behavior of the process $\{S_{[nt]}, t \in [0,1]\}$, properly normalized, strongly depends on the behavior of the sequence $(a_i)_{i\in\mathbf{Z}}$.

In the next two sections, we treat separately the case where the limit process is a mixture of fractional Brownian motions and the case where it is a mixture of standard Brownian motions.

3.1. Convergence to a mixture of fractional Brownian motions

Definition 3.1. *We say that a positive sequence $(v_n^2)_{n\geq 1}$ is regularly varying with exponent $\beta > 0$ if, for any $t \in]0,1]$,*

$$\frac{v_{[nt]}^2}{v_n^2} \to t^\beta \qquad \text{as } n \to \infty. \tag{12}$$

We shall separate the case $\beta \in]1,2]$ from the case $\beta \in]0,1]$.

Theorem 3.1. *Let $(a_i)_{i\in\mathbf{Z}}$ be in ℓ^2. Let $\beta \in]1,2]$ and assume that v_n^2 defined by (11) is regularly varying with exponent β. Let ξ_0 be a regular random variable such that $\|\xi_0\|_2 < \infty$ and let $\xi_i = \xi_0 \circ T^i$. Assume that condition (1) is satisfied. The process $\{v_n^{-1} S_{[nt]}, t \in [0,1]\}$ then converges in $D([0,1])$ to $\sqrt{\eta} W_H$, where W_H is a standard fractional Brownian motion independent of η with Hurst index $H = \beta/2$, $\eta = \sum_{k\in\mathbf{Z}} \mathbb{E}(\xi_0 \xi_k | \mathcal{I})$ and there exists a positive constant C (not depending on n) such that*

$$\mathbf{E}\Big(\max_{1\leq k\leq n} S_k^2\Big) \leq C v_n^2. \tag{13}$$

Theorem 3.2. *Let $\beta \in]0,1]$ and assume that v_n^2 defined by (11) is regularly varying with exponent β. Let ξ_0 be a regular random variable such that $\|\xi_0\|_2 < \infty$ and let $\xi_i = \xi_0 \circ T^i$. Assume that condition (1) is satisfied. The finite-dimensional distributions of $\{v_n^{-1} S_{[nt]}, t \in [0,1]\}$ then converge to the corresponding ones of $\sqrt{\eta} W_H$, where W_H is a standard fractional Brownian motion, independent of η, with Hurst index $H = \beta/2$ and $\eta = \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0 \xi_k | \mathcal{I})$. Assume, in addition,* *that for a $q > 2/\beta$, we have $\|\xi_0\|_q < \infty$ and*

$$\sum_{j\in\mathbf{Z}} \|P_0(\xi_j)\|_q < \infty. \tag{14}$$

Then the process $\{v_n^{-1}S_{[nt]}, t\in[0,1]\}$ converges in $D([0,1])$ to $\sqrt{\eta}W_H$ and (13) holds.

Remark 3.1. According to Peligrad and Utev [19], Corollary 2, we have

$$\lim_{n\to\infty}\frac{\mathrm{Var}(S_n)}{v_n^2} = \lim_{n\to\infty}\frac{\mathrm{Var}(\xi_1+\cdots+\xi_n)}{n} = v^2 = \Big\|\sum_{j\in\mathbf{Z}} P_0(\xi_j)\Big\|_2^2.$$

Remark 3.2. In the context of Theorem 3.1, condition (12) is necessary for the conclusion of this theorem (see [14]). This condition has also been imposed by Davydov [3] to study the weak invariance principle of linear processes with i.i.d. innovations. To be more precise, Davydov proved that if (12) holds and if $\xi_0\in\mathbb{L}^q$ with $q\ge 4$ and $q > 4(1/\beta-1)$, then $\{v_n^{-1}S_{[nt]}, t\in[0,1]\}$ converges in $D([0,1])$ to $\sqrt{\mathbf{E}(\xi_0^2)}W_{\beta/2}$. Later, in the case $\beta>1$, Konstantopoulos and Sakhanenko [13] sharpened Davydov's result, showing that the weak invariance principle holds if the ξ_i's are i.i.d. and in $\mathbf{L}^2$.

Example 1. For $0<d<1/2$, let us consider the linear process X_k defined by

$$X_k = (1-B)^{-d}\xi_k = \sum_{i\ge 0} a_i\xi_{k-i}, \tag{15}$$

where B is the lag operator, $a_0=1$, $a_i=\frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}$ for $i\ge 1$ and $(\xi_i)_{i\in\mathbf{Z}}$ is a strictly stationary sequence satisfying the condition of Theorem 3.1. In this case, Theorem 3.1 applies with $\beta = 2d+1$ since $a_k\sim(\Gamma(d))^{-1}k^{d-1}$.

Example 2. Now, consider the following choice of $(a_k)_{k\ge 0}$: $a_0=1$ and $a_i=(i+1)^{-\alpha}-i^{-\alpha}$ for $i\ge 1$ with $\alpha\in\,]0,1/2[$. Theorem 3.2 then applies. Indeed, for this choice, $v_n^2\sim\kappa_\alpha n^{1-2\alpha}$, where κ_α is a positive constant depending on α.

Example 3. For the choice $a_i\sim i^{-\alpha}\ell(i)$, where ℓ is a slowly varying function at infinity and $1/2<\alpha<1$, we have $v_n^2\sim\kappa_\alpha n^{3-2\alpha}\ell^2(n)$ (see, e.g., [26], relations (12)), where κ_α is a positive constant depending on α.

Example 4. Finally, if $a_i\sim i^{-1/2}(\log i)^{-\alpha}$ for some $\alpha>1/2$, then $v_n^2\sim n^2(\log n)^{1-2\alpha}/(2\alpha-1)$ (see [26], relations (12)). Hence, (12) is satisfied with $\beta=2$.

For the sake of applications, we now give a sufficient condition for (14) to hold.

Remark 3.3. For any $q \in [2, \infty[$, the condition (14) is satisfied if we assume that

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/q}} \|\mathbf{E}(\xi_n|\mathcal{F}_0)\|_q < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/q}} \|\xi_{-n} - \mathbf{E}(\xi_{-n}|\mathcal{F}_0)\|_q < \infty. \tag{16}$$

The fact that (16) implies (14) extends [19], Corollary 2, and also [4], Corollary 5, from the case $q = 2$ to more general situations.

For causal linear processes, Shao and Wu [23] also showed that the weak invariance principle holds under the condition (14), as long as the coefficients of the linear processes satisfy a certain regularity condition. To be more precise, their condition on the coefficients of the linear processes lead either to $\beta > 1$ or $\beta < 1$. For this last case, they specified the coefficients $(a_i)_{i\geq 0}$ as follows: for $1 < \alpha < 3/2$, $a_j = j^{-\alpha}\ell(j)$ for $j \geq 1$ (where $\ell(i)$ is a slowly varying function) and $\sum_{j=0}^{\infty} a_j = 0$ (see, e.g., their Lemma 4.1). For this choice, v_n^2 is regularly varying with coefficient $\beta = 3 - 2\alpha < 1$. Our Theorem 3.2 does not require conditions on the coefficients, but only the fact that the variance is regularly varying, which is a necessary condition.

3.2. Convergence to a mixture of Brownian motions

The case $\beta = 1$ deserves special attention. For this case, the limit is a mixture of Brownian motions.

As an immediate consequence of Theorem 3.2, we formulate the following corollary for causal linear processes, under a recent condition introduced by Wu and Woodroofe [29].

Corollary 3.1. *Let ξ_0 be a regular random variable such that $\|\xi_0\|_q < \infty$ for some $q > 2$ and let $\xi_i = \xi_0 \circ T^i$. Assume, in addition, that*

$$\sum_{j\in\mathbf{Z}} \|P_0(\xi_j)\|_q < \infty. \tag{17}$$

Let $(a_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2 such that $a_i = 0$ for $i < 0$. Let $b_j = a_0 + \cdots + a_j$. Define $(X_k)_{k\geq 1}$ as above and assume that

$$\sum_{k=0}^{n-1} b_k^2 \to \infty \qquad \text{as } n \to \infty, \tag{18}$$

and that

$$\sum_{j=0}^{\infty} (b_{n+j} - b_j)^2 = \mathrm{o}\left(\sum_{k=0}^{n-1} b_k^2\right). \tag{19}$$

Then $v_n^2 \sim nh(n)$, where $h(n)$ is a slowly varying function. Moreover, the process $\{v_n^{-1} S_{[nt]}, t \in [0,1]\}$ converges in $D([0,1])$ to $\sqrt{\eta}W$, where W is a standard Brownian motion, independent of η, and $\eta = \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0\xi_k|\mathcal{I})$. In addition, (13) *holds.*

To prove this result, it suffices to apply Theorem 3.2 and to use the fact that under (18) and (19), $v_n^2 \sim nh(n)$ (see [29]). Under the same conditions (18) and (19), Wu and Min [28], in their Theorem 1, also proved the weak invariance principle, but under the stronger condition $\sum_{j\geq 0} j\|P_0(\xi_j)\|_q < \infty$ (in their paper, the random variables ξ_j are adapted to the filtration $\mathcal{F}_j$).

Remark 3.4. The above result fails if, in (17), we take $q = 2$; see [29] and also [16], Example 1, page 657.

Let us make some comments on the case where the condition (1) is sufficient for weak convergence to the Brownian motion with the normalization $\sqrt{n}$. The first case is already known and the second case deserves a short proof.

Comment 3.1. *When $\sum_{i\in\mathbf{Z}} |a_i| < \infty$ (the short memory case) and condition (1) is satisfied, one can use the result from [18] in the adapted case, showing that the invariance principle for the linear process is inherited from the innovations at no extra cost. For this case, the process $\{n^{-1/2}S_{[nt]}, t \in [0, 1]\}$ converges in distribution in $D([0, 1])$ to $\sqrt{\eta}W$, where W is a standard Brownian motion, independent of η, and $\eta = A^2 \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0\xi_k|\mathcal{I})$ with $A = \sum_{i\in\mathbf{Z}} a_i$. Moreover, $\mathbf{E}(\max_{1\leq k\leq n} S_k^2) \leq Cn$. See [4], Corollaries 2 and 3, for the non-adapted case.*

Comment 3.2. *Let $(a_i)_{i\in Z}$ in ℓ^2 and assume that the series $\sum_{i\in\mathbf{Z}} a_i$ converges (meaning that the two series $\sum_{i\geq 0} a_i$ and $\sum_{i<0} a_i$ converge) and Heyde's [11] condition* (H) *holds:*

$$\text{(H)} \quad \sum_{n=1}^{\infty}\Big(\sum_{k\geq n} a_k\Big)^2 < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty}\Big(\sum_{k\leq -n} a_k\Big)^2 < \infty.$$

Assume, also, that condition (1) is satisfied. The same conclusion as in Comment 3.1 then holds.

Example 5. Heyde's condition allows the following possibility: $\sum_{i\in\mathbf{Z}} |a_i| = \infty$, but $\sum_{i\in\mathbf{Z}} a_i$ converges. For instance, if, for $n < 0$, $a_n = 0$ and, for $n \geq 1$, $a_n = (-1)^n u_n$, for some sequence $(u_n)_{n\geq 1}$ of positive coefficients decreasing to zero such that $\sum_{n\geq 1} u_n = \infty$, then condition (H) is satisfied as soon as $\sum_{n>0} u_n^2 < \infty$, which is a minimal condition. It is noteworthy to indicate that Heyde's condition implies (19).

Now, if $\sum_{j\in\mathbf{Z}} |a_j| = \infty$ and (H) does not hold, then condition (17) may still be weakened in some particular cases. The following result generalizes Corollary 4 in [4] to the case where the innovations of the linear process are not necessarily martingale difference sequences. We write

$$s_n^2 = n\Big(\sum_{i=-n}^{n} a_i\Big)^2. \tag{20}$$

Theorem 3.3. *Let $(a_i)_{i\in\mathbf{Z}}$ be a sequence of real numbers in ℓ^2, but not in ℓ^1, and let s_n^2 be defined by* (20). *Define $(X_k)_{k\geq 1}$ as above and assume that*

$$\limsup_{n\to\infty}\frac{\sum_{i=-n}^{n}|a_i|}{|\sum_{i=-n}^{n}a_i|}<\infty \quad \text{and} \quad \sum_{k=1}^{n}\sqrt{\sum_{|i|\geq k}a_i^2}=\mathrm{o}(s_n). \tag{21}$$

If one of the following two conditions holds,

(a) $\sum_{j\in\mathbf{Z}}\|P_0(\xi_j)\|_{\Psi_{2,\alpha}}<\infty,$ *where* $\Psi_{2,\alpha}(x)=x^2\log^{\alpha}(1+x^2)$ *and* $\alpha>2$,

or

(b) $\sum_{j\in\mathbf{Z}}\log(1+|j|)\|P_0(\xi_j)\|_2<\infty,$

then $\{s_n^{-1}S_{[nt]}, t\in[0,1]\}$ converges weakly in $D([0,1])$ to $\sqrt{\eta}W$, where W is a standard Brownian motion, independent of η, and $\eta=\sum_{k\in\mathbf{Z}}\mathbf{E}(\xi_0\xi_k|\mathcal{I})$. In addition, there exists a positive constant C (not depending on n) such that

$$\mathbf{E}\Big(\max_{1\leq k\leq n}S_k^2\Big)\leq Cs_n^2. \tag{22}$$

Remark 3.5. For two positive sequences of numbers, the notation $u_n\sim v_n$ means that $\lim_{n\to\infty}u_n/v_n=1$. According to [4], Remark 12, we have that

$$s_n^2\sim v_n^2\sim nh(n),$$

where $h(n)$ is a slowly varying function at infinity. In addition, if we assume the first part of condition (21) and $\sum_{j\in\mathbf{Z}}|a_j|=\infty$, then we get that $s_n/\sqrt{n}\to\infty$ as $n\to\infty$.

Example 6. Consider the following choice of $(a_k)_{k\in\mathbf{Z}}$: $a_0=1$ and $a_i=1/|i|$ for $i\neq 0$. Then Theorem 3.3 applies. Indeed, for this choice, condition (21) holds and $s_n\sim 2\sqrt{n}(\log n)$.

We now give a useful sufficient condition for the validity of condition (b) of Theorem 3.3.

Remark 3.6. Condition (b) of Theorem 3.3 is satisfied if we assume that

$$\sum_{n=1}^{\infty}\log n\frac{\|\mathbf{E}(\xi_n|\mathcal{F}_0)\|_2}{\sqrt{n}}<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\log n\frac{\|\xi_{-n}-\mathbf{E}(\xi_{-n}|\mathcal{F}_0)\|_2}{\sqrt{n}}<\infty. \tag{23}$$

4. Application to isotonic regression

Let ϕ be a non-decreasing function on the unit interval and let

$$y_k=\phi\Big(\frac{k}{n}\Big)+X_k, \qquad k=1,2,\ldots,n, \tag{24}$$

where (X_k) is a strictly stationary sequence of random variables such that $\mathbf{E}(X_k) = 0$ and $\mathbf{E}(X_k^2) < \infty$. The problem is then to estimate ϕ in a nonparametric way. We write $S_n = \sum_{k=1}^{n} X_k$.

Taking advantage of the monotonicity of the regression function, isotonic estimates have been suggested. Let $\mu_k = \phi(k/n)$. It is well known that the least-squares estimator

$$\hat{\mu} = \operatorname{argmin}\left\{\sum_{k=1}^{n}(y_k - \mu_k)^2, \mu_1 \leq \cdots \leq \mu_n\right\}$$

is such that

$$\hat{\mu}_k = \max_{i\leq k}\min_{j\geq k}\frac{y_i + \cdots + y_j}{j - i + 1}.$$

In addition, setting

$$Y_n(t) = \frac{1}{n}\left(\sum_{k=1}^{[nt]} y_k\right) \quad \text{and} \quad \widetilde{Y}_n = \mathrm{GCM}(Y_n),$$

where GCM designates the greatest convex minorant, we have

$$\hat{\mu}_k = \widetilde{Y}_n'\left(\frac{k}{n}\right),$$

where the derivative in taken on the left (see [21]). Now, let $\hat{\phi}_n(\cdot)$ be the left-continuous step function on [0, 1] such that $\hat{\phi}_n(k/n) = \hat{\mu}_k$ at the knots k/n for $k = 1, \ldots, n$.

When the error process (X_k) in the model (24) is short-range dependent and satisfies suitable weak dependence conditions, Zhao and Woodroofe [30] have obtained the asymptotic behavior of $\hat{\phi}_n(t)$. In their paper, an application to global warming is given. Some other situations are considered in [1]: in their Theorem 3(iii), they consider the case where (X_k) can exhibit long-range dependence and they assume that X_k is a function of a Gaussian process such that its Hermite polynomial expansion is of rank greater than one. When no shape assumption is imposed on the regression function, nonparametric regression analysis when data can exhibit long-range dependence has been also studied by other authors (see, e.g., [22] or, more recently, Gao and Wang [6] wherein random designs are introduced in the nonparametric trend model). The motivation for studying such models is that, in order to avoid misrepresenting the mean function or the conditional mean function of long-range dependent data, one should let the data "speak for themselves" in terms of specifying the true form of the mean function or the conditional mean function. Situations where the error process (X_k) in the model (24) is long-range dependent often occur when considering financial or climatological time series. For instance, the annual series of winter means of the NAO index (North Atlantic Oscillation index) exhibits long-range dependence (see [24]) and also an increasing trend for the last decade (which can possibly be explained by global warming). Concerning financial time series, we refer to the paper by Pesee [20], where daily exchange rate data are studied. For instance, the daily changes of the US dollar against the Deutsche Mark constitute a financial series that exhibits long-range dependence with a long period of monotonic trend. For other data examples of long-memory processes, we refer to the book by Beran [2]. In particular, concerning the monthly temperature for the northern hemisphere, Beran suggests (page 29 of his book) that the series could be long-range dependent (see his Figure 1.12a–c, page 31).

The aim of this section, then, is to derive the asymptotic behavior of $\hat{\phi}_n(t)$ when X_k is a linear process which can exhibit short or long memory. Recall that, by the well-known Wold decomposition, a stationary process in $\mathbf{L}^2$ that is purely non-deterministic and such that its one-step mean squared error is positive can be represented by a linear process generated by orthogonal random variables.

As is implicitly mentioned in [1] and elucidated in [30], the two main tools to obtain the asymptotic behavior of $\hat{\phi}_n(t)$ are a weak invariance principle for the partial sums process $\{S_{[nt]}, t \in [0,1]\}$, properly normalized, and a suitable moment inequality for $\max_{1\le k\le n} S_k^2$.

Theorem 4.1. *Let $(a_i)_{i\in\mathbf{Z}}$ and $(\xi_i)_{i\in\mathbf{Z}}$ be as in Comments 3.1 or 3.2. Let us consider the model (24) with X_k defined by (10). For any $t \in (0,1)$ such that $\phi'(t) > 0$,*

$$n^{1/3}\kappa^{-1}\bigl(\hat{\phi}_n(t) - \phi(t)\bigr) \quad \Longrightarrow \quad \bigl(\sqrt{\eta}\bigr)^{2/3} \operatorname{argmin}\{B(s) + s^2, s \in \mathbf{R}\},$$

where B denotes a standard two-sided Brownian motion independent of η, $\eta = \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0\xi_k|\mathcal{I})$ and $\kappa = 2(\frac{1}{2}A^2\phi'(t))^{1/3}$ with $A = \sum_{j\in\mathbf{Z}} a_j$.

Let $\beta \in]0,2]$ and let h be a slowly varying function at infinity. Now, let

$$L(x) = \left(\frac{1}{h(x^{2/(4-\beta)})}\right)^{1/2} \tag{25}$$

and note that $L(x)$ is also a slowly varying function at infinity. Denote by L^* the asymptotic conjugate of L, which means that L^* satisfies

$$\lim_{x\to\infty} L^*(x)L(xL^*(x)) = 1. \tag{26}$$

Then define

$$d_n = \frac{1}{n^{(2-\beta)/(4-\beta)}}\ell(n), \qquad \text{where } \ell(n) = (L^*(n))^{2/(4-\beta)}. \tag{27}$$

Theorem 4.2. *Let $(a_i)_{i\in\mathbf{Z}}$ and $(\xi_i)_{i\in\mathbf{Z}}$ be as in Theorem 3.3. For $\beta = 1$ and $h(n) = |\sum_{i=-n}^{n} a_i|^2$, let d_n be defined by (27). Let us consider the model (24) with X_k defined by (10). For any $t \in (0,1)$ such that $\phi'(t) > 0$,*

$$d_n^{-1}\kappa^{-1}\bigl(\hat{\phi}_n(t) - \phi(t)\bigr) \quad \Longrightarrow \quad \bigl(\sqrt{\eta}\bigr)^{2/3} \operatorname{argmin}\{B(s) + s^2, s \in \mathbf{R}\},$$

where B denotes a standard two-sided Brownian motion independent of η, $\eta = \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0\xi_k|\mathcal{I})$ and $\kappa = 2(\frac{1}{2}\phi'(t))^{1/3}$.

Example 7. In the case of the linear process defined in Example 6, Theorem 4.2 applies with $d_n = n^{-1/3}(4\ln(n)/3)^{2/3}$.

Theorem 4.3. *Let $(a_i)_{i\in\mathbf{Z}}$ and $(\xi_i)_{i\in\mathbf{Z}}$ be as in Theorem 3.1 or 3.2 for some $\beta\in]0,2[$. By assumption, v_n^2 defined by (11) is regularly varying with exponent β. For this β and for $h(n)=v_n^2n^{-\beta}$, let d_n be defined by (27). Let us consider the model (24) with X_k defined by (10). Then, for any $t\in(0,1)$ such that $\phi'(t)>0$, we have*

$$d_n^{-1}\kappa_\beta^{-1}\big(\hat\phi_n(t)-\phi(t)\big)\quad\Longrightarrow\quad\big(\sqrt{\eta}\big)^{1/(2-H)}\operatorname{argmin}\{B_H(s)+s^2,s\in\mathbf{R}\},$$

where B_H denotes a standard two-sided fractional Brownian motion, independent of η, with Hurst index $H=\beta/2$, $\eta=\sum_{k\in\mathbf{Z}}\mathbf{E}(\xi_0\xi_k|\mathcal{I})$ and where the constant κ_β is given by $\kappa_\beta=2(\phi'(t)/2)^{(2-\beta)/(4-\beta)}$.

Example 8. In the case of the linear process defined in Example 1, Theorem 4.3 applies with $\beta=2d+1$ and $d_n=\tau_d n^{(1-2d)/(3-2d)}$, where τ_d is a positive constant depending only on d.

Proofs of Theorems 4.1–4.3. For any $t\in(0,1)$ and any $s\in[-td_n^{-1},d_n^{-1}(1-t)]$, let

$$Z_n(s)=d_n^{-2}\big(Y_n(t+d_ns)-Y_n(t)-\phi(t)d_ns\big).$$

Then $d_n^{-1}(\hat\phi_n(t)-\phi(t))=\widetilde{Z}_n'(0)$, the left-hand derivative of the GCM of Z_n at $s=0$. Hence, the key for establishing the result is the study of the GCM of the process Z_n. This can be done by following the arguments given in [1], Section 3, and also in [30]. More precisely, a careful analysis of the proofs given in both of these papers shows that the following lemma is valid.

Lemma 4.1. *Assume that there exists a positive sequence $m_n\to\infty$ satisfying, for any $t>0$,*

$$m_{[nt]}/m_n\to t^H,\qquad \text{where } H\in]0,1[,\tag{28}$$

and such that:

1. *the process $\{m_n^{-1}S_{[nt]},t\in[0,1]\}$ converges in $D([0,1])$ to $\sqrt{\eta}W_H$, where η is a positive random variable and W_H is a standard fractional Brownian motion (with Hurst index H) independent of η;*
2. $\mathbf{E}(\max_{1\le k\le n}S_k^2)\le Cm_n^2$.

Then, for any positive sequence $d_n\to0$ such that $nd_n\to\infty$ and $d_n^{-2}n^{-1}m_{[nd_n]}\to1$, and for any $t\in(0,1)$ such that $\phi'(t)>0$,

$$d_n^{-1}\kappa_H^{-1}\big(\hat\phi_n(t)-\phi(t)\big)\quad\Longrightarrow\quad\big(\sqrt{\eta}\big)^{1/(2-H)}\operatorname{argmin}\{B_H(s)+s^2,s\in\mathbf{R}\},$$

where $B_H(\cdot)$ denotes a standard two-sided fractional Brownian motion, independent of η, with Hurst index $H\in]0,1[$ and $\kappa_H=2(\phi'(t)/2)^{(1-H)/(2-H)}$.

Proof. We proceed as in the proof of Anevski and Hössjer [1], Theorem 3. The main point is then to verify their assumptions A1–A7 in order to apply their Corollary 1. Since $nd_n\to\infty$, assumption A2 follows from the arguments given in the proof of Anevski and Hössjer [1], Theorem 3(i). By the properties of our limiting process, $\sqrt{\eta}W_H$, the assumptions A5 and A7 are satisfied. Now, if assumption A1 holds, then, by Anevski and Hössjer [1], Proposition 2, and the properties of the fractional Brownian motion, assumption A6 also holds. Note that their Proposition 2 allows the continuous mapping theorem to be applied to the functional h from $D[-c,c]$ (the space of cadlag functions on $[-c,c]$) to $\mathbb{R}$, defined as the left-hand derivative of $\mathrm{GCM}(x)$ at 0. To verify their assumptions A3 and A4, it suffices to apply their Proposition 1. According to the proofs of their Lemmas B1 and B2, the condition (14) of their Proposition 1 is satisfied as soon as their condition (87) and our condition (28) are. Now, their condition (87) is clearly satisfied provided item 2 of Lemma 4.1 holds.

It remains to prove [1], assumption A1, namely, that the process

$$\left\{n^{-1}d_n^{-2}S_{[nd_nt]}, t \in [0,1]\right\}$$

converges in $D[0,1]$ to $\sqrt{\eta}W_H$, where η is a positive random variable and W_H is a standard fractional Brownian motion (with Hurst index H), independent of η. This holds by item 1 of Lemma 4.1 and the fact that $d_n^{-2}n^{-1}m_{[nd_n]} \to 1$. This completes the proof of Lemma 4.1. □

We go back to the proofs of Theorems 4.1–4.3. Note that the conditions of items 1 and 2 are clearly satisfied by using either Comment 3.1 or 3.2 (with $m_n = \sqrt{n}$), either Theorem 3.3 (with $m_n = \sqrt{n}|\sum_{i=-n}^{n} a_i|$) or Theorem 3.1 or 3.2 (with $m_n = v_n$). In addition, in all these situations, we have that $m_n = (n^\beta h(n))^{1/2}$ and the selection of d_n leads to

$$\begin{aligned}
d_n^{-2}n^{-1}m_{[nd_n]} &\sim d_n^{(\beta-4)/2}n^{(\beta-2)/2}\sqrt{h(nd_n)} \\
&\sim (L^*(n))^{-1}\sqrt{h\big((nL^*(n))^{2/(4-\beta)}\big)} \\
&\sim (L^*(n))^{-1}(L(nL^*(n)))^{-1},
\end{aligned}$$

which converges to 1 by (26). □

5. Proofs

5.1. Proof of Proposition 2.1

Without loss of generality, we shall assume that $D_\Psi = 1$ and $\sum_{j\in\mathbf{Z}} c_{m,j}^2 = 1$ since, otherwise, we can divide each coefficient $c_{m,j}$ by $(\sum_{j\in\mathbf{Z}} c_{m,j}^2)^{1/2}$ and each variable by D_Ψ. Start with the decomposition

$$Y_k = \sum_{j=-\infty}^{\infty} P_{k-j}(Y_k) = \sum_{j=-\infty}^{\infty} p_j P_{k-j}(Y_k)/p_j.$$

Then

$$S_m = \sum_{j=-\infty}^{\infty} p_j \sum_{k\in\mathbf{Z}} c_{m,k} P_{k-j}(Y_k)/p_j.$$

By using the facts that Ψ is convex and non-decreasing, and $p_j \geq 0$ with $\sum_{j\in\mathbf{Z}} p_j = D_\Psi = 1$, we obtain that

$$\mathbf{E}\Psi(|S_m|) \leq \sum_{j=-\infty}^{\infty} p_j \mathbf{E}\Psi\bigg(\bigg|\sum_{k\in\mathbf{Z}} c_{m,k} P_{k-j}(Y_k)/p_j\bigg|\bigg).$$

Consider the martingale difference $U_k = c_{m,k} P_{k-j}(Y_k)/p_j$, $k \in \mathbf{Z}$. By Burkholder's inequality (see [5], Theorem 6.6.2), we obtain that

$$\mathbf{E}\Psi\bigg(\bigg|\sum_{k\in\mathbf{Z}} c_{m,k} P_{k-j}(Y_k)/p_j\bigg|\bigg) \leq K_\alpha \mathbf{E}\Psi\bigg(\bigg(\sum_{k\in\mathbf{Z}} c_{m,k}^2 P_{k-j}^2(Y_k)/p_j^2\bigg)^{1/2}\bigg),$$

where K_α is a constant depending only on α. Let $\Phi(x) = \Psi(\sqrt{x})$. Since Φ is convex and $\sum_{k\in\mathbf{Z}} c_{m,k}^2 = 1$, it follows that

$$\begin{aligned}\mathbf{E}\Psi\bigg(\bigg|\sum_{k\in\mathbf{Z}} c_{m,k} P_{k-j}(Y_k)/p_j\bigg|\bigg) &\leq K_\alpha \mathbf{E}\Phi\bigg(\sum_{k\in\mathbf{Z}} c_{m,k}^2 P_{k-j}^2(Y_k)/p_j^2\bigg)\\ &\leq K_\alpha \sum_{k\in\mathbf{Z}} c_{m,k}^2 \mathbf{E}\Phi\big(P_{k-j}^2(Y_k)/p_j^2\big)\\ &\leq K_\alpha \sum_{k\in\mathbf{Z}} c_{m,k}^2 \mathbf{E}\big(\Psi\big(|P_{k-j}(Y_k)|/p_j\big)\big).\end{aligned}$$

Therefore,

$$\mathbf{E}\Psi(|S_m|) \leq K_\alpha \sum_{k\in\mathbf{Z}} c_{m,k}^2 \sum_{j=-\infty}^{\infty} p_j \mathbf{E}\big(\Psi\big(|P_{k-j}(Y_k)|/p_j\big)\big).$$

Now, note that $\|P_{k-j}(Y_k)\|_\Psi \leq p_j$, so using the fact that $\sum_{k\in\mathbf{Z}} c_{m,k}^2 = 1$ and $D_\Psi = \sum_{j=-\infty}^{\infty} p_j = 1$, we get

$$\mathbf{E}\Psi(|S_m|) \leq K_\alpha$$

and hence the desired result.

5.2. Proof of Proposition 2.2

Fix a positive integer m and define

$$\theta_{0,m} = \sum_{k=0}^{2m-2} \sum_{i=k-m+1}^{m-1} P_i(\xi_k) \quad \text{and} \quad \theta_{j,m} = \theta_{0,m} \circ T^j.$$

Observe that, by stationarity,

$$\|\theta_{0,m}\|_\Psi = \left\| \sum_{k=0}^{2m-2} \sum_{i=k-m+1}^{m-1} P_i(\xi_k) \right\|_\Psi \le 2m \sum_{i\in\mathbf{Z}} \|P_0(\xi_i)\|_\Psi < \infty.$$

Simple computations lead to the decomposition

$$\sum_{i=-m+1}^{m-1} P_i(\xi_0) - \sum_{\ell=1}^{2m-1} P_m(\xi_\ell) = \theta_{0,m} - \theta_{1,m},$$

implying that

$$\xi_0 - \left(\sum_k P_0(\xi_k)\right) \circ T^m = \theta_{0,m} - \theta_{1,m} + \sum_{|i|\ge m} P_i(\xi_0) - \left(\sum_{|k|\ge m} P_0(\xi_k)\right) \circ T^m.$$

With our notation ($d_0 = \sum_k P_0(\xi_k)$), we obtain

$$\xi_0 - d_0 = d_0 \circ T^m - d_0 + \theta_{0,m} - \theta_{1,m} + \sum_{|i|\ge m} P_i(\xi_0) - \left(\sum_{|k|\ge m} P_0(\xi_k)\right) \circ T^m. \tag{29}$$

By stationarity, we obtain similar decompositions for each $\xi_j - d_j$. We shall treat the terms from the error of approximation $\sum_{i\in\mathbf{Z}} c_{n,i}(\xi_i - d_i)$ separately. First, note that

$$\begin{aligned} R_1 := \sum_{j=-\infty}^{\infty} c_{n,j}(d_j \circ T^m - d_j) &= \sum_{j=-\infty}^{\infty} (c_{n,j-m} - c_{n,j}) d_j \\ &= \sum_{k=0}^{m-1} \sum_{j=-\infty}^{\infty} (c_{n,j-k-1} - c_{n,j-k}) d_j. \end{aligned}$$

According to Proposition 2.1,

$$\|R_1\|_\Psi \le C_\alpha m \|d_0\|_\Psi \left(\sum_{j=-\infty}^{\infty} (c_{n,j} - c_{n,j-1})^2 \right)^{1/2}.$$

To treat the second difference in the error, note that

$$R_2 := \sum_{i=-\infty}^{\infty} c_{n,i}(\theta_{i,m} - \theta_{i+1,m}) = \sum_{i=-\infty}^{\infty} (c_{n,i} - c_{n,i-1})\theta_{i,m}.$$

By the definition of $\theta_{0,m}$, we have that

$$\sum_{j\in\mathbf{Z}} \|P_j(\theta_{0,m})\|_\Psi \le \sum_{k=0}^{2m-2} \sum_{i=k-m+1}^{m-1} \sum_{j\in\mathbf{Z}} \|P_j(P_i(\xi_k))\|_\Psi.$$

Now, $P_j(P_i(f)) = 0$ for $j \neq i$. It follows that

$$\sum_{j\in\mathbf{Z}} \|P_j(\theta_{0,m})\|_\Psi \le \sum_{k=0}^{2m-2} \sum_{\ell=k-m+1}^{m-1} \|P_0(\xi_\ell)\|_\Psi \le (2m-1) \sum_{\ell=-m+1}^{m-1} \|P_0(\xi_\ell)\|_\Psi$$

and, by Proposition 2.1, we conclude that

$$\|R_2\|_\Psi \le 2C_\alpha m \left(\sum_{j=-\infty}^{\infty} (c_{n,j} - c_{n,j-1})^2 \right)^{1/2} \sum_{\ell\in\mathbf{Z}} \|P_0(\xi_\ell)\|_\Psi .$$

For the term $R_3 := \sum_{i=-\infty}^{\infty} c_{n,i} (\sum_{|j|\ge m} P_j(\xi_0)) \circ T^i$, we apply Proposition 2.1 to get

$$\|R_3\|_\Psi \le C_\alpha \left(\sum_{i=-\infty}^{\infty} c_{n,i}^2 \right)^{1/2} \sum_{|j|\ge m} \|P_j(\xi_0)\|_\Psi .$$

To deal with the last term $R_4 := \sum_{i=-\infty}^{\infty} c_{n,i} (\sum_{|k|\ge m} P_0(\xi_k)) \circ T^{m+i}$, we again apply Proposition 2.1, which gives

$$\|R_4\|_\Psi \le C_\alpha \left(\sum_{i=-\infty}^{\infty} c_{n,i}^2 \right)^{1/2} \sum_{|k|\ge m} \|P_0(\xi_k)\|_\Psi .$$

Combining all the bounds, we obtain the desired approximation.

5.3. Proof of Lemma 2.1

For any $m \in [1, 2^N]$, write m in base 2 as follows:

$$m = \sum_{i=0}^{N} b_i(m) 2^i, \qquad \text{where } b_i(m) = 0 \text{ or } b_i(m) = 1.$$

Set $m_L = \sum_{i=L}^{N} b_i(m) 2^i$. So, for any $p \ge 1$, we have

$$|S_m|^p \le \left(\sum_{L=0}^{N} |S_{m_L} - S_{m_{L+1}}| \right)^p .$$

Hence, setting

$$\alpha_L = \|S_{2^L}\|_{\Psi_p} (\Psi^{-1}(2^{N-L}))^{1/p} \quad \text{and} \quad \lambda_L = \frac{\alpha_L}{\sum_{L=0}^{N} \alpha_L},$$

we get, by convexity,

$$|S_m|^p \le \sum_{L=0}^{N} \lambda_L^{1-p} |S_{m_L} - S_{m_{L+1}}|^p.$$

Now, $m_L \neq m_{L+1}$ only if $b_L(m) = 1$ and, in that case, $m_L = k_m 2^L$ with k_m odd. It follows that

$$\max_{1\le m\le 2^N} |S_m|^p \le \sum_{L=0}^{N} \lambda_L^{1-p} \max_{1\le k\le 2^{N-L}, k \text{ odd}} \left|S_{k2^L} - S_{(k-1)2^L}\right|^p.$$

Now, we apply [15], Lemma 11.3, to the variables

$$Z_k = \frac{|S_{k2^L} - S_{(k-1)2^L}|^p}{A^p}, \qquad \text{where } A = \|S_{2^L}\|_{\Psi_p},$$

and to the Young function Ψ. Since

$$\mathbf{E}(\Psi(Z_k)) = \mathbf{E}\Psi_p\left(\frac{|S_{2^L}|}{A}\right) \le 1$$

and since Ψ^{-1} is concave, we get that, for any measurable set B,

$$\mathbf{E}(Z_k \mathbf{1}_B) \le P(B)\Psi^{-1}\left(\frac{1}{P(B)}\right)$$

so that the assumptions of Ledoux and Talagrand [15], Lemma 11.3, are satisfied. It follows that

$$\mathbf{E}\Big(\max_{1\le k\le 2^{N-L}, k \text{ odd}} \left|S_{k2^L} - S_{(k-1)2^L}\right|^p\Big) \le A^p \Psi^{-1}(2^{N-L}).$$

Finally, we conclude that

$$\mathbf{E}\Big(\max_{1\le m\le 2^N} |S_m|^p\Big) \le \left(\sum_{L=0}^{N} \alpha_L\right)^p,$$

which is the desired result.

5.4. Proofs of Theorems 3.1 and 3.2

By the weak convergence theory of random functions, it suffices to establish the convergence of the finite-dimensional distributions and the tightness of $\{v_n^{-1} S_{[nt]}, t \in [0, 1]\}$. For the finite-dimensional distribution, we shall use the following proposition which was basically established in [17,19].

Proposition 5.1. *Let $\{\xi_k\}_{k\in\mathbf{Z}}$ be a strictly stationary sequence of centered and regular random variables in $\mathbf{L}^2$ such that $\sum_j \|P_0(\xi_j)\|_2 < \infty$. For any positive integer n, let $\{b_{n,i}, -\infty \le i \le \infty\}$ be a triangular array of numbers satisfying*

$$\sum_i b_{n,i}^2 \to 1 \quad \text{and} \quad \sum_j (b_{n,j} - b_{n,j-1})^2 \to 0 \qquad \text{as } n \to \infty \tag{30}$$

and

$$\sup_j |b_{n,j}| \to 0 \qquad \text{as } n \to \infty. \tag{31}$$

Then $\{S_n = \sum_j b_{n,j}\xi_j\}$ converges in distribution to $\sqrt{\eta}N$, where N is a standard Gaussian random variable, independent of η, and $\eta = \sum_{k\in\mathbf{Z}} \mathbf{E}(\xi_0\xi_k|\mathcal{I})$.

Proof. We give here the proof for completeness. By using Proposition 2.2, it suffices to prove this proposition with $d_j = d_0 \circ T^j$ in place of ξ_j, where $d_0 = \sum_j P_0(\xi_j)$. Hence, we just have to apply the central limit theorem for triangular arrays of martingales (see [8], Theorem 3.6). The Lindeberg condition has been established by Peligrad and Utev [17], provided that condition (31) and the first part of condition (30) are satisfied. Now, in the proof of Peligrad and Utev [19], Proposition 4, it is established that (30) implies that

$$\sum_j b_{n,j}^2 d_j^2 \to \eta \qquad \text{in probability as } n \to \infty,$$

which ends the proof of the proposition. □

We return to the proofs of Theorems 3.1 and 3.2. To prove the convergence of the finite-dimensional distributions, we shall apply the Cramér–Wold device. For all integer $1 \le \ell \le m$, let $n_\ell = [nt_\ell]$, where $0 < t_1 < t_2 < \cdots < t_m \le 1$. For $\lambda_1, \ldots, \lambda_m \in \mathbf{R}$, note that

$$\frac{\sum_{\ell=1}^m \lambda_\ell S_{n_\ell}}{v_n} = \sum_{j\in\mathbf{Z}} \left(\sum_{\ell=1}^m \frac{\lambda_\ell c_{n_\ell,j}}{v_n} \right) \xi_j, \tag{32}$$

where $c_{n,j} = a_{1-j} + \cdots + a_{n-j}$ for all $j \in \mathbf{Z}$ and $v_n^2 = \sum_{j\in\mathbf{Z}} c_{n,j}^2$. Let

$$b_{n,j} = \frac{1}{\Lambda_{m,\beta}} \sum_{\ell=1}^m \frac{\lambda_\ell c_{n_\ell,j}}{v_n}, \tag{33}$$

where

$$\Lambda_{m,\beta}^2 = \frac{1}{2} \sum_{\ell,k=1}^m \lambda_\ell \lambda_k (t_\ell^\beta + t_k^\beta - |t_k - t_\ell|^\beta).$$

We apply Proposition 5.1 to $b_{n,j}$ and the ξ_j's defined as $\Lambda_{m,\beta}\xi_j$. First, we have to calculate the limit over n of the quantity

$$\sum_{j\in\mathbf{Z}} b_{n,j}^2 = \frac{1}{\Lambda_{m,\beta}^2} \frac{\sum_{j\in\mathbf{Z}} \sum_{\ell=1}^m \sum_{k=1}^m \lambda_\ell \lambda_k c_{n_\ell,j} c_{n_k,j}}{v_n^2}.$$

For any $1 \le \ell \le k \le m$, by using the fact that for any two real numbers A and B, we have $A(A+B) = 1/2(A^2 + (A+B)^2 - B^2)$, we get that

$$\begin{aligned} \frac{1}{v_n^2} \sum_{j\in\mathbf{Z}} c_{n_\ell,j} c_{n_k,j} &= \frac{1}{2v_n^2} \sum_{j\in\mathbf{Z}} \big(c_{n_\ell,j}^2 + c_{n_k,j}^2 - (c_{n_\ell,j} - c_{n_k,j})^2\big) \\ &= \frac{1}{2v_n^2} \sum_{j\in\mathbf{Z}} (c_{n_\ell,j}^2 + c_{n_k,j}^2 - c_{n_k-n_\ell,j}^2). \end{aligned}$$

By now using condition (12), we derive that, for any $1 \le \ell \le k \le m$,

$$\frac{\sum_{j\in\mathbf{Z}} b_{n_\ell,j} b_{n_k,j}}{v_n^2} \to \frac{1}{2}\big(t_\ell^\beta + t_k^\beta - (t_k - t_\ell)^\beta\big). \tag{34}$$

It follows from (34) that

$$\lim_{n\to\infty} \sum_{j\in\mathbf{Z}} b_{n,j}^2 = 1. \tag{35}$$

As a consequence, the first part of condition (30) holds. On the other hand, by using Peligrad and Utev [19], Lemma A.1, the second part of condition (30) is satisfied. Now, by the proof of Corollary 2.1 in [17], we get that

$$\frac{\max_j |c_{n,j}|}{v_n} \to 0,$$

which, together with (12), implies (31). Now, applying Proposition 5.1, we derive that

$$\frac{\sum_{\ell=1}^m \lambda_\ell S_{n_\ell}}{v_n} \qquad \text{converges in distribution to } \Lambda_{m,\beta}\sqrt{\eta}N,$$

ending the proof of the convergence of the finite-dimensional distribution.

We now turn to the proof of the tightness of $\{v_n^{-1} S_{[nt]}, t \in [0,1]\}$. By using Proposition 2.1, we get, for $q \ge 2$, that

$$\|S_k\|_q \le C_q \bigg(\sum_{j\in\mathbf{Z}} b_{k,j}^2\bigg)^{1/2} \sum_{m\in\mathbf{Z}} \|P_0(\xi_m)\|_q = C_q v_k \sum_{m\in\mathbf{Z}} \|P_0(\xi_m)\|_q, \tag{36}$$

provided that $\sum_{m\in\mathbf{Z}} \|P_0(\xi_m)\|_q < \infty$. Therefore, the conditions of Taqqu [25], Lemma 2.1, page 290, are satisfied with $q > 2/\beta$ and the tightness follows.

Finally, to prove (13), we use (36), together with Lemma 2.1 applied with $\psi(x) = x$, by taking into account the fact that v_n^2 is regularly varying with exponent β.

5.5. Proof of Remarks 3.3 and 3.6

To prove Remark 3.3, we apply Lemma A.1 from the Appendix with $b_i = 1$ and $u_i = \|P_{-i}(\xi_0)\|_q$. Hence, we get

$$\sum_{n=1}^{\infty} \|P_{-n}(\xi_0)\|_q \le C_q \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} \|P_{-k}(\xi_0)\|_q^q \right)^{1/q}.$$

Applying the Rosenthal inequality given in [8], Theorem 2.12, we then derive that for any $q \in [2, \infty[$, there exists a constant c_q, depending only on q, such that

$$\sum_{k=n}^{\infty} \|P_{-k}(\xi_0)\|_q^q \le c_q \left\| \sum_{k=n}^{\infty} P_{-k}(\xi_0) \right\|_q^q = c_q \|\mathbf{E}(\xi_n|\mathcal{F}_0)\|_q^q.$$

The same argument works with $P_{-i}(\xi_0)$ replaced by $P_i(\xi_0)$, and the result follows by applying the Rosenthal inequality and noting that $\|\xi_{-n} - \mathbf{E}(\xi_{-n}|\mathcal{F}_0)\|_q = \| \sum_{k=n}^{\infty} P_{k+1}(\xi_0)\|_q$.

To prove Remark 3.6, we apply Lemma A.1 from the Appendix with $b_n = \log(n)$ and $u_n = \|P_0(\xi_n)\|_2$. We then get that

$$\sum_{n=1}^{\infty} \log n \|P_0(\xi_n)\|_2 \le C \sum_{n=1}^{\infty} \frac{\log n}{\sqrt{n}} \left(\sum_{k=n}^{\infty} \|P_0(\xi_k)\|_2^2 \right)^{1/2}.$$

Now, note that

$$\sum_{k=n}^{\infty} \|P_0(\xi_k)\|_2^2 = \|\mathbf{E}(\xi_n|\mathcal{F}_0)\|_2^2$$

and so

$$\sum_{n=1}^{\infty} \log n \|P_0(\xi_n)\|_2 \le C \sum_{n=1}^{\infty} \log n \frac{\|\mathbf{E}(\xi_n|\mathcal{F}_0)\|_2}{\sqrt{n}} < \infty.$$

The same argument works with $P_0(\xi_i)$ replaced by $P_0(\xi_{-i})$.

5.6. Proof of Theorem 3.3

For all $j \in \mathbf{Z}$, let $d_j = \sum_{\ell \in \mathbf{Z}} P_j(\xi_\ell)$. Note that if either condition (a) or condition (b) is satisfied, $(d_j)_{j \in \mathbf{Z}}$ is a sequence of martingale differences in $\mathbf{L}^2$. We set

$$Y_k = \sum_{i \in \mathbf{Z}} a_i d_{k-i} \quad \text{and} \quad T_n = \sum_{k=1}^{n} Y_k,$$

and apply [4], Corollary 4. By taking into account Remark 3.5, we derive that under (21),

$$\{s_n^{-1} T_{[nt]}, t \in [0,1]\} \quad \text{converges in distribution in } (D([0,1]), d) \text{ to } \quad \sqrt{\mathbf{E}(d_0^2|\mathcal{I})}W,$$

where W is a standard Brownian motion independent of $\mathcal{I}$. It follows that in order to prove that $\{s_n^{-1} S_{[nt]}, t \in [0,1]\}$ converges in distribution in $(D([0,1]), d)$ to $\sqrt{\mathbf{E}(d_0^2|\mathcal{I})}W$, it is sufficient to show that

$$\frac{\| \max_{1 \le k \le n} |S_k - T_k| \|_2}{s_n} \to 0 \qquad \text{as } n \to \infty. \tag{37}$$

Now, for any n, let N be such that $2^{N-1} < n \le 2^N$. By using Remark 3.5 and the properties of the slowly varying function, we get that $s_n \sim s_{2^N}$. So, the proof (37) is reduced to showing that

$$\frac{\| \max_{1 \le k \le 2^N} |S_k - T_k| \|_2}{s_{2^N}} \to 0 \qquad \text{as } N \to \infty. \tag{38}$$

We first prove that (38) holds under condition (a). By using Corollary 2.1, together with Lemma 2.1, we get that for any positive integer m,

$$\begin{aligned} \Big\| \max_{1 \le k \le 2^N} |S_k - T_k| \Big\|_2 &\le C_1 \sum_{|k| \ge m} \| P_0(\xi_k) \|_{\Psi_{2,\alpha}} \sum_{L=0}^{N} v_{2^L} (g^{-1}(2^{N-L}))^{1/2} \\ &\quad + C_2 m \sum_{L=0}^{N} (g^{-1}(2^{N-L}))^{1/2}, \end{aligned}$$

where $g(x) = x \log^\alpha(1+x)$. Noting that $g^{-1}(x) \sim \frac{x}{\log^\alpha(1+x)}$ as x goes to infinity, and taking into account Remark 3.5 and the first part of condition (21), we get that

$$\Big\| \max_{1 \le k \le 2^N} |S_k - T_k| \Big\|_2 \le C s_{2^N} \sum_{|k| \ge m} \| P_0(\xi_k) \|_{\Psi_{2,\alpha}} + C m \epsilon(N) s_{2^N}, \tag{39}$$

where $\epsilon(N) \to 0$ as $N \to \infty$. By now using (39) and first letting N tend to infinity and then m tend to infinity, we derive (38) under condition (a).

We now turn to the proof of (38) under condition (b). Taking $m = m_{2^L} = 2^{L/4}$ in Corollary 2.1 and using Lemma 2.1 with $p = 2$ and $\psi(x) = x$, we get that

$$\begin{aligned} &\frac{\| \max_{1 \le k \le 2^N} |S_k - T_k| \|_2}{s_{2^N}} \\ &\le C \frac{2^{N/2}}{s_{2^N}} \sum_{L=0}^{N} \frac{m_{2^L}}{2^{L/2}} + C \frac{2^{N/2}}{s_{2^N}} \sum_{L=0}^{N} \frac{v_{2^L}}{2^{L/2}} \sum_{|k| \ge m_{2^L}} \| P_0(\xi_k) \|_2 . \end{aligned} \tag{40}$$

By Remark 3.5, we have that $\lim_{N\to\infty} \frac{s_{2^N}}{2^{N/2}} = \infty$, which, together with the selection of m_{2^L}, implies that the first term on the right-hand side of the above inequality tends to zero as $n \to \infty$. Now, to treat the last term, we first fix a positive integer p and write

$$\begin{aligned} \frac{2^{N/2}}{s_{2^N}} \sum_{L=0}^{N} \frac{v_{2^L}}{2^{L/2}} \sum_{|k| \ge m_{2^L}} \| P_0(\xi_k) \|_2 &\le p \frac{2^{N/2}}{s_{2^N}} \max_{0 \le L < p} \frac{v_{2^L}}{2^{L/2}} \sum_{|k| \ge m_{2^L}} \| P_0(\xi_k) \|_2 \\ &\quad + \frac{2^{N/2}}{s_{2^N}} \sum_{L=p}^{N} \frac{v_{2^L}}{2^{L/2}} \sum_{|k| \ge m_{2^L}} \| P_0(\xi_k) \|_2 . \end{aligned}$$

Since $\lim_{N\to\infty} \frac{s_{2^N}}{2^{N/2}} = \infty$, the first term on the right-hand side of the above inequality tends to zero as $N \to \infty$. To treat the second one, we note that if N and p are large enough,

$$\frac{2^{N/2}}{s_{2^N}} \sum_{L=p}^{N} \frac{v_{2^L}}{2^{L/2}} \sum_{|k| \ge m_{2^L}} \| P_0(\xi_k) \|_2 \le C \sum_{L=p}^{N} \frac{h(2^L)}{h(2^N)} \sum_{|k| \ge m_{2^L}} \| P_0(\xi_k) \|_2 ,$$

where $h(n) = |\sum_{i=-n}^{n} a_i|$. By the first part of condition (21),

$$\limsup_{N\to\infty} \max_{p \le L \le N} \frac{h(2^L)}{h(2^N)} < \infty .$$

Hence, for N and p large enough and taking into account the selection of m_{2^L}, we get that

$$\frac{2^{N/2}}{s_{2^N}} \sum_{L=p}^{N} \frac{v_{2^L}}{2^{L/2}} \sum_{|k| \ge m_{2^L}} \| P_0(\xi_k) \|_2 \le C \sum_{|k| \ge 2^{p/4}} \log k \| P_0(\xi_k) \|_2 ,$$

which converges to zero as $p \to \infty$, by using condition (b). Hence, starting from (40) and taking into account the previous considerations, we get that (38) holds under condition (b). The proof of (22) is straightforward, following the arguments used to derive (37).

5.7. Proof of Comment 3.2

The justification of this result is due to the following coboundary decomposition. Define

$$Z_0 = \sum_{\ell=1}^{\infty}\sum_{k=\ell}^{\infty} a_k\xi_{-\ell} - \sum_{\ell=0}^{\infty}\sum_{k=-\infty}^{-\ell-1} a_k\xi_\ell. \tag{41}$$

Since condition (1) implies that the sequence $(\xi_i)_{i\in\mathbf{Z}}$ has a bounded spectral density, the random variable Z_0 is well defined in $\mathbf{L}^2$ under condition (H). Now,

$$Z_0 - Z_0\circ T = \sum_{\ell=1}^{\infty} a_\ell\xi_{-\ell} - \xi_0\sum_{k=1}^{\infty} a_k - \xi_0\sum_{k=1}^{\infty} a_{-k} + \sum_{\ell=1}^{\infty} a_{-\ell}\xi_\ell,$$

whence

$$A\xi_0 + Z_0 - Z_0\circ T = a_0\xi_0 + \sum_{j\in\mathbf{Z}\setminus\{0\}} a_j\xi_{-j} = X_0.$$

We derive that, for any $k \geq 1$,

$$S_k = A\sum_{i=1}^{k}\xi_i + Z_1 - Z_{k+1}, \tag{42}$$

where $Z_k = Z_0\circ T_k$. Since, under condition (1), the partial sums process $\{n^{-1/2}\sum_{k=1}^{[nt]}\xi_k, t \in [0,1]\}$ converges in distribution in $D([0,1])$ to $\sqrt{\lambda}W$ with $\lambda = \sum_{j\in\mathbf{Z}}\mathbf{E}(\xi_0\xi_j|\mathcal{I})$, we just have to show that

$$\limsup_{n\to\infty}\mathbf{P}\Big(\max_{1\leq k\leq n}|Z_{k+1}| \geq \varepsilon\sqrt{n}\Big) = 0,$$

which holds because $Z_0 \in \mathbf{L}^2$ (see [8], inequality (5.30)).

Appendix

A.1. A fact concerning series

Lemma A.1. *Let $q > 1$ and $\alpha = 2(q-1)/q$. Let $(b_j)_{j\in\mathbf{N}}$ be a sequence of non-negative numbers such that $n^\alpha b_n \leq K_\alpha\sum_{k=1}^{n} k^{\alpha-1}b_k$ for some positive constant K_α depending only on α. Then, for any sequence of non-negative numbers $(u_j)_{j\in\mathbf{N}}$, the following inequality holds*:

$$\sum_{n=1}^{\infty} b_n u_n \leq C_q\sum_{n=1}^{\infty} b_n\left(\frac{1}{n}\sum_{k=n}^{\infty} u_k^q\right)^{1/q},$$

where C_q is a constant depending only on q.

Proof. We write

$$\sum_{n=1}^{\infty} b_n u_n \le K_\alpha \sum_{n=1}^{\infty} n^{-\alpha} u_n \left(\sum_{k=1}^{n} b_k k^{\alpha-1}\right) \le K_\alpha \sum_{k=1}^{\infty} b_k k^{\alpha-1} \left(\sum_{n\ge k} n^{-\alpha} u_n\right).$$

Hölder's inequality then gives

$$\sum_{n=1}^{\infty} b_n u_n \le C_q' \sum_{k=1}^{\infty} b_k k^{\alpha-1} \left(\sum_{n\ge k} n^{-2}\right)^{\alpha/2} \left(\sum_{n\ge k} u_n^q\right)^{1/q}$$

and the result follows. □

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References

[1] Anevski, D. and Hössjer, O. (2006). A general asymptotic scheme for inference under order restrictions. *Ann. Statist.* **34** 1874–1930. MR2283721

[2] Beran, J. (1994). *Statistics for Long-Memory Processes. Monographs on Statistics and Applied Probability* **61**. New York: Chapman and Hall. MR1304490

[3] Davydov, Y.A. (1970). The invariance principle for stationary processes. *Theory Probab. Appl.* **15** 487–498. MR0283872

[4] Dedecker, J., Merlevède, F. and Volný, D. (2007). On the weak invariance principle for non-adapted sequences under projective criteria. *J. Theoret. Probab.* **20** 971–1004. MR2359065

[5] De La Peña, V. and Giné, E. (1999). *Decoupling: From Dependence to Independence: Randomly Stopped Processes, U-Statistics and Processes, Martingales and Beyond*. New York: Springer. MR1666908

[6] Gao, J. and Wang, Q. (2006). Long-range dependent time series specification. Preprint. Available at http://www.maths.usyd.edu.au/u/pubs/publist/preprints/2006/gao-26.pdf.

[7] Gordin, M.I. (1969). The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR* **188** 739–741. MR0251785

[8] Hall, P. and Heyde, C.C. (1980). *Martingale Limit Theory and Its Applications*. New York: Academic Press. MR0624435

[9] Hannan, E.J. (1973). Central limit theorems for time series regression. *Z. Wahrsch. Verw. Gebiete* **26** 157–170. MR0331683

[10] Heyde, C.C. (1974). On the central limit theorem for stationary processes. *Z. Wahrsch. Verw. Gebiete* **30** 315–320. MR0372955

[11] Heyde, C.C. (1975). On the central limit theorem and iterated logarithm law for stationary processes. *Bull. Austral. Math. Soc.* **12** 1–8. MR0372954

[12] Ibragimov, I.A. (1962). Some limit theorems for stationary processes. *Theory Probab. Appl.* **7** 349–382. MR0148125

[13] Konstantopoulos, T. and Sakhanenko, A. (2004). Convergence and convergence rate to fractional Brownian motion for weighted random sums. *Sib. Èlektron. Mat. Izv.* **1** 47–63. MR2132447

[14] Lamperti, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 62–78. MR0138128

[15] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Spaces. Isoperimetry and Processes.* Berlin: Springer. MR1102015

[16] Merlevède, F. and Peligrad, M. (2006). On the weak invariance principle for stationary sequences under projective criteria. *J. Theoret. Probab.* **19** 647–689. MR2280514

[17] Peligrad, M. and Utev, S. (1997). Central limit theorem for linear processes. *Ann. Probab*. **25** 443–456. MR1428516

[18] Peligrad, M. and Utev, S. (2006a). Invariance principle for stochastic processes with short memory. In *High Dimensional Probability* 18–32. *IMS Lecture Notes Monograph Series* **51**. Beachwood, OH: Inst. Math. Statist. MR2387758

[19] Peligrad, M. and Utev, S. (2006b). Central limit theorem for stationary linear processes. *Ann. Probab*. **34** 1608–1622. MR2257658

[20] Pesee, C. (2008). Long-range dependence of financial time series data. *WASET* **44** 163–167.

[21] Robertson, T., Wright, F.T. and Dykstra, R.L. (1988). *Order Restricted Statistical Inference.* Chichester: Wiley. MR0961262

[22] Robinson, P.M. (1997). Large-sample inference for nonparametric regression with dependent errors. *Ann. Statist.* **25** 2054–2083. MR1474083

[23] Shao, X. and Wu, W.B. (2006). Invariance principles for fractionally integrated nonlinear processes. In *Recent Developments in Nonparametric Inference and Probability* 20–30. *IMS Lecture Notes Monograph Series* **50**. Beachwood, OH: Inst. Math. Statist. MR2409061

[24] Stephenson, D.B., Pavan, V. and Bojariu, R. (2000). Is the North Atlantic Oscillation a random walk? *Int. J. Climatol.* **20** 1–18.

[25] Taqqu, M.S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* **31** 287–302. MR0400329

[26] Wang, Q., Lin, X-Y. and Gulati, C.M. (2001). Asymptotics for moving average processes with dependent innovations. *Statist. Probab. Lett.* **54** 347–356. MR1861379

[27] Wu, W.B. (2007). Strong invariance principles for dependent random variables. *Ann. Probab.* **35** 2294–2320. MR2353389

[28] Wu, W.B. and Min, W. (2005). On linear processes with dependent innovations. *Stochastic Process. Appl.* **115** 939–958. MR2138809

[29] Wu, W.B. and Woodroofe, M. (2004). Martingale approximations for sums of stationary processes. *Ann. Probab.* **32** 1674–1690. MR2060314

[30] Zhao, O. and Woodroofe, M. (2009). *Stationary Random Walks and Isotonic Regression: With Prediction for Time Series.* Saarbrücken: VDM. MR2627626

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