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Reflected BSDE with a constraint and its applications in an incomplete market

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In this paper, we study a type of reflected BSDE with a constraint and prove the existence of the smallest g-supersolution for this equation. We then demonstrate its applications in the pricing of American options in an incomplete market.

Keywords: American options in an incomplete market; backward stochastic differential equation with a constraint; reflected backward stochastic differential equation

1. Introduction

A backward stochastic differential equation (BSDE) driven by a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is of the form

$$\mathrm{d}y_t + g(t, y_t, z_t)\,\mathrm{d}t - z_t\,\mathrm{d}B_t = 0, \qquad t \in [0, T],$$

where g is a given function called the *generator* of the BSDE. Here, all processes are assumed to be square-integrable and progressively measurable with respect to the $(B_t)_{t\geq 0}$-filtration. For a given terminal condition $y_T = \xi$, a solution $(y_t, z_t)_{t\in[0,T]}$ is a pair of processes satisfying the above relation. We often call it a g-*solution* to specify the generator g. In the case where the generator g is a Lipschitz function of (y, z), the existence and uniqueness of such a BSDE was given by [15]. In this paper, we consider 1-dimensional BSDE, that is, g and y are assumed to be real-valued. We are interested in a new type of BSDE with the following type of singular generator:

$$g_\Gamma(t, \omega, y, z) = \begin{cases} g(t, y, z), & (y, z) \in \Gamma(t, \omega), \\ +\infty, & \text{otherwise}, \end{cases}$$

where, for each (t, ω), $\Gamma(t, \omega)$ is a given closed subset of $\mathbb{R} \times \mathbb{R}^d$. This type of g_Γ-solution $(y_t)_{t\in[0,T]}$ is formulated as the smallest g-supersolution constrained in Γ with a given terminal condition ξ. This type of BSDE and its application to the problem of option pricing with constrained portfolios was studied in Cvitanic and Karatzas [1,2] and Cvitanic, Karatzas and Soner

[4] for a convex constraint, and by Peng [16] and Peng and Yang [19] for more general situations. The framework of the present paper is based on [16].

In this paper, we mainly study g_Γ-reflected BSDEs, that is, BSDEs reflected by a lower obstacle $(L_t)_{t\in[0,T]}$ or an upper obstacle $(U_t)_{t\in[0,T]}$ with the above singular generator g_Γ. Our results non-trivially generalize the original paper of El Karoui *et al.* [6], as well as Hamadene [8], Hamadene and Lepeltier [10] and Lepeltier and Xu [14], in which the generators are assumed to be Lipschitz functions. Since the obstacles L and U can be very general L^2-processes, our results also generalize Peng and Xu [17].

Recently, the study of reflected BSDEs has been very active since it can be applied to optimal stopping, optimal switching, American option pricing and the related dynamic risk measures, stochastic differential controls and games with mixed strategies (e.g., Dynkin games). We refer to [6], also Cvitanic and Karatzas [3,8,10], Karatzas and Kou [11], Lepeltier and San Martín [13,14], Peng and Xu [18], as well as [17] for various situations involving reflected BSDEs with non-singular generators and their applications. In this paper, we also discuss how to apply our results on g_Γ-reflected BSDEs to American call and put options in an incomplete market with portfolio constraints.

Observe that a g_Γ-solution of a BSDE reflected by a lower obstacle $(L_t)_{t\in[0,T]}$ can also be considered as a BSDE with constraint $\Gamma_t \times \{y \in \mathbb{R} : y \geq L_t\}$. However, it is theoretically and practically important to separate the reflecting process $\bar{A}$ from the total increasing process $A + \bar{A}$ since the related (generalized) Skorokhod reflecting condition plays an important role (see Proposition 3.1). This type of separation is an important feature of our results.

This paper is organized as follows. In the next section, we present the main notation and conditions used throughout the paper. In Section 3, we present results and proofs of the existence and uniqueness of a reflected BSDE with the singular generator g_Γ. We then discuss some applications of our main results to the problem of pricing of American options in a market with portfolio constraints in Section 4. A monotonic limit theorem, its generalization and other results which are needed in the proofs of this paper are given in the Appendix.

2. g_Γ-solution: the smallest g-supersolution of a BSDE with constraint Γ

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B = (B^1, B^2, \ldots, B^d)^{\mathrm{T}}$ a d-dimensional Brownian motion defined on $[0, \infty)$. The natural filtration generated by this Brownian motion is denoted by

$$\mathcal{F}_t = \sigma\big\{\{B_s; 0 \leq s \leq t\} \cup \mathcal{N}\big\},$$

where $\mathcal{N}$ is the collection of all P-null sets of $\mathcal{F}$. The Euclidean norm of an element $x \in \mathbb{R}^m$ is denoted by $|x|$. We also need the following notation, for $p \in [1, \infty)$:

- $\mathbf{L}^p(\mathcal{F}_t; \mathbb{R}^m) := \{\mathbb{R}^m$-valued $\mathcal{F}_t$-measurable random variables X such that $E[|X|^p] < \infty\}$;
- $\mathbf{L}^p_{\mathcal{F}}(0, t; \mathbb{R}^m) := \{\mathbb{R}^m$-valued and $\mathcal{F}_t$-progressively measurable processes φ defined on $[0, t]$ such that $E\int_0^t |\varphi_s|^p \,\mathrm{d}s < \infty\}$;
- $\mathbf{D}^p_{\mathcal{F}}(0, t; \mathbb{R}^m) := \{\mathbb{R}^m$-valued and RCLL $\mathcal{F}_t$-progressively measurable processes φ defined on $[0, t]$ such that $E[\sup_{0\leq s\leq t} |\varphi_s|^p] < \infty\}$;

- $\mathbf{A}^p_{\mathcal{F}}(0,t) :=$ {increasing processes A in $\mathbf{D}^p_{\mathcal{F}}(0,t;\mathbb{R})$ with $A(0)=0$}.

When $m=1$, we simply use $\mathbf{L}^p(\mathcal{F}_t)$, $\mathbf{L}^p_{\mathcal{F}}(0,t)$ and $\mathbf{D}^p_{\mathcal{F}}(0,t)$. In this section, we consider BSDE on the interval $[0,T]$ with a fixed $T>0$.

We consider a function

$$g(\omega,t,y,z):\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$$

which always plays the role of the generator of our BSDE. It satisfies the following classical assumptions: there exists a constant $\mu>0$, such that, for each y, y' in $\mathbb{R}$ and z, z' in $\mathbb{R}^d$, we have

$$\begin{aligned} &\text{(i)}\quad g(\cdot,y,z)\in\mathbf{L}^2_{\mathcal{F}}(0,T);\\ &\text{(ii)}\quad |g(t,\omega,y,z)-g(t,\omega,y',z')|\le\mu(|y-y'|+|z-z'|),\quad \mathrm{d}P\times\mathrm{d}t\text{-a.s.}\end{aligned}\tag{1}$$

The constraint Γ of our BSDE is a mapping $\Gamma(t,\omega):\Omega\times[0,T]\to\mathcal{C}(\mathbb{R}\times\mathbb{R}^d)$, where $\mathcal{C}(\mathbb{R}\times\mathbb{R}^d)$ is the collection of all closed subsets of $\mathbb{R}\times\mathbb{R}^d$. Γ is assumed to be $\mathcal{F}_t$-adapted, namely,

$$\begin{aligned} &\text{(i)}\quad (y,z)\in\Gamma(t,\omega)\text{ if and only if } d_{\Gamma(t,\omega)}(y,z)=0,\ t\in[0,T],\text{ a.s.};\\ &\text{(ii)}\quad d_{\Gamma_\cdot}(y,z)\text{ is an }\mathcal{F}_t\text{-adapted process for each }(y,z)\in\mathbb{R}\times\mathbb{R}^d,\end{aligned}\tag{2}$$

where $d_{\Gamma_t}(\cdot,\cdot)$ is a distance function from (y,z) to Γ: for $t\in[0,T]$,

$$d_{\Gamma_t}(y,z):=\inf_{(y',z')\in\Gamma_t}(|y-y'|^2+|z-z'|^2)^{1/2}\wedge 1.$$

$d_{\Gamma_t}(y,z)$ is a Lipschitz function: for each y, y' in $\mathbb{R}$ and z, z' in $\mathbb{R}^d$, we always have

$$|d_{\Gamma_t}(y,z)-d_{\Gamma_t}(y',z')|\le(|y-y'|^2+|z-z'|^2)^{1/2}.$$

Remark 2.1. The above type of constraint Γ was first considered in Peng [16]. In fact, Peng's constraint is formulated as

$$\Gamma_t(\omega)=\{(y,z)\in\mathbb{R}^{1+d}:\Phi(\omega,t,y,z)=0\},\tag{3}$$

where $\Phi(\omega,t,y,z):\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\to[0,\infty)$ is a given non-negative function satisfying similar conditions as (1). In this paper, we always consider the case

$$\Phi(t,y,z)=d_{\Gamma_t}(y,z).$$

In fact, these two definitions are equivalent. In [4], the constraint is assumed to be convex.

We are then within the framework of supersolution and subsolution of BSDEs of the following type.

Definition 2.1* (g*-supersolution*, g*-subsolution; cf. El Karoui, Peng and Quenez (1997) [7] and Peng (1999) [16]). *A process $y\in\mathbf{D}^2_{\mathcal{F}}(0,T)$ is called a* g-supersolution (*resp.* g-subsolution) *if there exist a process* $z \in \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ *and an increasing RCLL process* $A \in \mathbf{A}^2_{\mathcal{F}}(0,T)$ (*resp.* $K \in \mathbf{A}^2_{\mathcal{F}}(0,T)$) *such that for* $t \in [0,T]$,

$$\begin{gathered} y_t = y_T + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s + A_T - A_t - \int_t^T z_s\,\mathrm{d}B_s \\ \left(\text{resp. } y_t = y_T + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - (K_T - K_t) - \int_t^T z_s\,\mathrm{d}B_s \right). \end{gathered} \tag{4}$$

Here, z *and* A (*resp.* K) *are called the* martingale representing part *and* increasing part *of* y, *respectively.* y *is called a* g-solution *if* $A_t = K_t = 0$, $t \in [0,T]$. y *is called a* Γ-constrained g-supersolution *if* y *and its corresponding martingale representing part* z *satisfy*

$$(y_t, z_t) \in \Gamma_t \qquad \left(\text{or } d_{\Gamma_t}(y_t, z_t) = 0\right), \qquad \mathrm{d}P \times \mathrm{d}t\text{-a.s. in } \Omega \times [0,T]. \tag{5}$$

Remark 2.2. We observe that if $y \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ is a g-supersolution or g-subsolution, then the pair (z, A) in (4) is uniquely determined since the martingale representing part z is uniquely determined. Occasionally, we also call the triple (y, z, A) a g-supersolution or g-subsolution.

A Γ-constrained g-supersolution can also be regarded as a supersolution of the BSDE with a singular generator g_Γ defined by

$$g_\Gamma(t, y, z) = g(t, y, z) 1_{\Gamma_t}(y, z) + (+\infty) \cdot 1_{\Gamma_t^C}(y, z),$$

so we define the smallest Γ-constrained g-supersolution as the g_Γ-solution.

Definition 2.2 ($\mathbf{g}_\Gamma$-solution). *A* g*-supersolution* $(y_t, z_t, A_t)_{0 \le t \le T}$ *is called a* g_Γ-supersolution *on* $[0,T]$ *with a given terminal condition* X *if* $d_{\Gamma_t}(y_t, z_t) = 0$, $\mathrm{d}P \times \mathrm{d}t$ *almost surely. The smallest* g_Γ*-supersolution* $(y_t, z_t, A_t)_{0 \le t \le T}$ *with a given terminal condition* $y_T = X$ *is called the* g_Γ-solution. *Here*, *"smallest" means that* $y_t \ge y'_t$, $t \in [0,T]$, *for any* g_Γ*-supersolution* $(y'_t, z'_t, A'_t)_{0 \le t \le T}$ *with* $y'_T = X$.

Remark 2.3. The above definition is meaningful since, by [16] (see Theorem A.2 in the Appendix), if there exists at least one g_Γ-supersolution, then the smallest Γ-constrained g-supersolution also exists.

Remark 2.4. By the above definition, if $(y_t, z_t, A_t)_{0 \le t \le T}$ is a g_Γ-solution on $[0,T]$ with terminal condition y_T, then for each $T_1 \le T$, $(y_t, z_t, A_t)_{0 \le t \le T_1}$ is also a g_Γ-solution on $[0, T_1]$ with terminal condition y_{T_1}. The above definition does not imply that the increasing process A is also the smallest one. In fact, the following example shows that there may exists a different g_Γ-supersolution $(\bar{y}, \bar{z}, \bar{A})$ on $[0,T]$ with the same terminal condition such that $A_t > \bar{A}_t$ for some t.

Example 2.1. Consider the case when $[0,T] = [0,2]$, $X = 0$, $g = 0$ and $\Gamma_t = \{(y,z) : y \ge 1_{[0,1]}(t)\}$. The g_Γ-solution of this equation is the solution of the reflected BSDE with the lower obstacle $1_{[0,1]}(t)$. This g_Γ-solution is expressed as $y_t = 1_{[0,1)}(t)$, $z_t = 0$, $A_t = 1_{[1,2]}(t)$ on $[0,T]$.

One can also check that $\bar{y}_t = 1_{[0,2)}(t)$ with $\bar{z}_t = 0$; $\bar{A}_t = 1_{\{t=2\}}(t)$ is also a g_Γ-supersolution with the same terminal condition $\bar{y}_T = 0$. However, we have $A_t > \bar{A}_t$ for $t \in [1, 2)$.

3. g_Γ-reflected BSDEs

3.1. Existence of g_Γ-reflected BSDEs: definitions and results

In this section, we consider the smallest g_Γ-supersolution with a lower (resp. upper) reflecting obstacle L (resp. U). We assume that the two reflected obstacles L and U satisfy

$$L, U \in \mathbf{L}^2_{\mathcal{F}}(0, T) \quad \text{and} \quad ess \sup_{0 \le t \le T} L_t^+, ess \sup_{0 \le t \le T} U_t^- \in \mathbf{L}^2(\mathcal{F}_T). \tag{6}$$

We only study the case of the constraint Γ not depending on y, only on z. In such a case, $\Gamma(t, \omega) = \mathbb{R} \times \Gamma_z(t, \omega)$, where $\Gamma_z(t, \omega)$ is a closed subset of $\mathbb{R}^d$. For more general situations, see Remarks 3.1 and 3.3.

First, let us introduce the definition of g_Γ-reflected solutions with a lower obstacle.

Definition 3.1. *A g_Γ-reflected solution with a lower obstacle L is a quadruple of processes $(y, z, A, \bar{A})$ satisfying*:

(i) $(y, z, A, \bar{A}) \in \mathbf{D}^2_{\mathcal{F}}(0, T) \times \mathbf{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^d) \times (\mathbf{A}^2_{\mathcal{F}}(0, T))^2$ *verifies*

$$\begin{aligned} y_t &= X + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T z_s\,\mathrm{d}B_s, \\ d_{\Gamma_t}(z_t) &= 0, \mathrm{d}P \times \mathrm{d}t \ a.s.; \end{aligned} \tag{7}$$

(ii) $y_t \ge L_t$, $\mathrm{d}P \times \mathrm{d}t$*-a.s., and the following generalized Skorokhod reflecting condition is satisfied*: *for each* $L^* \in \mathbf{D}^2_{\mathcal{F}}(0, T)$ *such that* $y_t \ge L_t^* \ge L_t$, $\mathrm{d}P \times \mathrm{d}t$*-a.s., we have*

$$\int_0^T (y_{s-} - L^*_{s-})\,\mathrm{d}\bar{A}_s = 0 \qquad a.s.; \tag{8}$$

(iii) *y is the smallest one*, *that is*, *for any quadruple* $(y^*, z^*, A^*, \bar{A}^*)$ *satisfying* (i) *and* (ii), *we have*

$$y_t \le y_t^* \qquad \forall t \in [0, T], a.s.$$

In the above formulation, we need to find two increasing processes A and $\bar{A}$ in order to keep the solution in the constraints $y_t \ge L_t$ and $z_t \in \Gamma$. In fact, these two increasing processes play different roles. A is used to keep the process z staying in the constraint Γ, while $\bar{A}$ is the reflecting force to keep y above the obstacle L. Actually, each of them has a different meaning in finance.

Our first main result in this paper is the following theorem.

Theorem 3.1. *Suppose that* (1), (2) *and* (6) *hold. For a given terminal condition* $X \in \mathbf{L}^2(\mathcal{F}_T)$, *we assume that there exists a triple* $(y^*, z^*, A^*) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T) \times \mathbf{A}^2_{\mathcal{F}}(0,T)$ *such that* $\mathrm{d}A^* \geq 0$ *and the following holds*:

$$y_t^* = X + \int_t^T g(s, y_s^*, z_s^*)\,\mathrm{d}s + (A_T^* - A_t^*) - \int_t^T z_s^*\,\mathrm{d}B_s,$$
$$(y_t^*, z_t^*) \in [L_t, \infty) \times \Gamma_t,\ \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \tag{9}$$

There then exists the g_Γ*-reflected solution* $(y, z, A, \bar{A})$ *with the barrier* L *of Definition* 3.1.

Remark 3.1. This theorem can be generalized to the case where Γ also depends on y. In fact, the basic idea of the proof of this theorem is based on a penalization method which still works for the y-dependence situation (cf. the proof of Theorem 3.1).

Under certain assumptions, condition (9) can be easily verified.

Example 3.1. Assume that there exists a constant C_0, large enough such that for $\forall y \geq C_0$,

$$g(t, y, 0) \leq C_0 + \mu|y|, \qquad 0 \in \Gamma_t, \tag{10}$$

and there exists a deterministic process $a(t)$ such that $L_t \leq a(t)$ on $[0, T]$. For X a random variable in $\mathbf{L}^2_{+,\infty}(\mathcal{F}_T)$, that is, $X \in \mathbf{L}^2(\mathcal{F}_t)$, $X^+ \in \mathbf{L}^\infty(\mathcal{F}_t)$, the triple of processes

$$(y_t^*, z_t^*, A_t^*) := \left(y_t^0, 0, \int_0^t [C_0 + \mu|y_s^0| - g(s, y_s^0, 0)]\,\mathrm{d}s + A_t^0 + A_t^1 \right)$$

is a solution of (9). Here, (y_t^0, A_t^0) is the solution of the ODE associated with coefficient $g(y) = C_0 + \mu|y|$, barrier $a(t)$ and terminal value $(\|X^+\|_\infty \vee C_0)\mathrm{e}^{\mu(T-t)} + C_0(T-t)$, and $A_t^1 = (X - \|X^+\|_\infty \vee C_0)1_{\{t=T\}}$.

The smallest g_Γ-reflected solution with an upper obstacle U is relatively more complicated than the case of the lower obstacle.

Definition 3.2. *A* g_Γ*-reflected solution with an upper obstacle* U *is a quadruple of processes*

$$(y, z, A, K) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times (\mathbf{A}^2_{\mathcal{F}}(0,T))^2$$

satisfying:

(i)

$$y_t = X + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s + A_T - A_t - (K_T - K_t) - \int_t^T z_s\,\mathrm{d}B_s,$$
$$d_{\Gamma_t}(z_t) = 0,\ \mathrm{d}P \times \mathrm{d}t\text{-a.s. } \mathcal{V}_{[0,T]}[A - K] = \mathcal{V}_{[0,T]}[A + K], \tag{11}$$

where $\mathcal{V}_{[0,T]}(\varphi)$ *denotes the total variation of a process* φ *on* $[0, T]$;

(ii) $y_t \le U_t$, $\mathrm{d}P \times \mathrm{d}t$-*a.s., and the generalized Skorokhod reflecting condition is satisfied*:

$$\int_0^T (U^*_{t-} - y_{t-})\,\mathrm{d}K_t = 0, \qquad a.s., \text{ for any } U^* \in \mathbf{D}^2_{\mathcal{F}}(0,T), \text{ s.t. } y_t \le U^*_t \le U_t, \mathrm{d}P \times \mathrm{d}t\text{-}a.s.;$$

(iii) *for any other quadruple* (y^*, z^*, A^*, K^*) *satisfying* (i) *and* (ii), *we have*

$$y_t \le y^*_t, \qquad 0 \le t \le T, a.s.$$

Remark 3.2. The relation $\mathcal{V}_{[0,T]}[A-K] = \mathcal{V}_{[0,T]}[A+K]$ in (11) implies that A and K never increase at same time. This relation can help us to characterize the solution. Indeed, it is easy to check that the quadruple $(y, z, A+K, 2K)$ satisfies all of the above relations except this one.

We also have the existence of a g_Γ-reflected solution with an upper obstacle U, given by the following result.

Theorem 3.2. *Assume that* (1) *and* (2) *hold for* g *and the constraint* Γ, *respectively, and that* U *is an* $\mathcal{F}_t$*-adapted RCLL process satisfying* (6). *Then, for each given terminal condition* $X \in \mathbf{L}^2(\mathcal{F}_T)$, *there exists a* g_Γ*-reflected solution* (y, z, A, K) *with upper obstacle* U *of Definition* 3.2.

Remark 3.3. For the case where Γ depends on y, satisfying (2), Theorem 3.2 still holds under the following additional assumption: there exists a quadruple

$$(y^*, z^*, A^*, K^*) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times (\mathbf{A}^2_{\mathcal{F}}(0,T))^2$$

such that

$$\begin{aligned} y^*_t &= X + \int_t^T g(s, y^*_s, z^*_s)\,\mathrm{d}s + (A^*_T - A^*_t) - (K^*_T - K^*_t) - \int_t^T z^*_s\,\mathrm{d}B_s, \\ & d_{\Gamma_t}(y^*_t, z^*_t) = 0,\ y^*_t \le U_t, \text{ a.s. a.e.} \\ & \int_0^T (y^*_{t-} - U^*_{t-})\,\mathrm{d}K^*_t = 0 \text{ a.s., for any } U^* \in \mathbf{D}^2_{\mathcal{F}}(0,T),\ \text{ s.t. } y_t \le U^*_t \le U_t, \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \end{aligned} \tag{12}$$

In general, this assumption is not easy to verify. One typical example is $\Gamma_t = [L_t, +\infty) \times \mathbb{R}^d$. Indeed, the problem turns out to be a reflected BSDE with two barriers, L and U. By [17], we know that if there exists a semimartingale X such that $L \le X \le U$, $\mathrm{d}P \times \mathrm{d}t$-a.s., then condition (12) can be satisfied.

3.2. Existence of a g_Γ-reflected BSDE with a lower barrier: proof of Theorem 3.1

The main idea of the proof is a penalization method. We prove Theorem 3.1 by an approximation procedure. For given $m, n \in \mathbb{N}$, we consider the penalization equations

$$\begin{aligned} y_t^{m,n} = X &+ \int_t^T g(s, y_s^{m,n}, z_s^{m,n})\, \mathrm{d}s + m\int_t^T d_{\Gamma_s}(z_s^{m,n})\, \mathrm{d}s \\ &+ n\int_t^T (L_s - y_s^{m,n})^+\, \mathrm{d}s - \int_t^T z_s^{m,n}\, \mathrm{d}B_s. \end{aligned} \tag{13}$$

It is a classical BSDE with generator

$$g^{m,n}(t,y,z) := g(t,y,z) + m d_{\Gamma_t}(z) + n(L_t - y)^-,$$

which is a Lipschitz function. From [15], this equation admits a unique solution $(y^{m,n}, z^{m,n})$. We define $A_t^{m,n} := m\int_0^t d_{\Gamma_s}(y_s^{m,n}, z_s^{m,n})\, \mathrm{d}s$ and $\bar{A}_t^{m,n} := n\int_0^t (L_s - y_s^{m,n})^+\, \mathrm{d}s$. We have the following estimate.

Lemma 3.1. *Under the same assumptions as in Theorem 3.1, there exists a constant $C \in \mathbb{R}$ independent of m and n such that*

$$E\Big[\sup_{0\le t\le T} (y_t^{m,n})^2\Big] + E\int_0^T |z_s^{m,n}|^2\, \mathrm{d}s + E[(A_T^{m,n} + \bar{A}_T^{m,n})^2] \le C. \tag{14}$$

Proof. Setting $m = n = 0$, we then get a classical BSDE:

$$y_t^{0,0} = X + \int_t^T g(s, y_s^{0,0}, z_s^{0,0})\, \mathrm{d}s - \int_t^T z_s^{0,0}\, \mathrm{d}B_s.$$

For (y^*, z^*, A^*) given in (9), we have $d_{\Gamma_s}(z_s^*) \equiv 0$ and $(L_s - y_s^*)^+ \equiv 0$, thus

$$\begin{aligned} y_t^* = X &+ \int_t^T g(s, y_s^*, z_s^*)\, \mathrm{d}s + m\int_t^T d_{\Gamma_s}(z_s^*)\, \mathrm{d}s + n\int_t^T (L_s - y_s^*)^+\, \mathrm{d}s \\ &+ (A_T^* - A_t^*) - \int_t^T z_s^*\, \mathrm{d}B_s. \end{aligned}$$

By the comparison theorem, it follows that $y_t^* \ge y_t^{m,n} \ge y_t^{0,0}$ on $[0,T]$. Thus, $y^{m,n}$ satisfies the estimate

$$E\Big[\sup_{0\le t\le T} (y_t^{m,n})^2\Big] \le C_1 = \max\Big\{E\Big[\sup_{0\le t\le T} (y_t^*)^2\Big], E\Big[\sup_{0\le t\le T} (y_t^{0,0})^2\Big]\Big\}. \tag{15}$$

The rest of the proof can be obtained by applying the following lemma. □

Lemma 3.2. *Let* $(y^\alpha, z^\alpha, A^\alpha)_{\alpha\in\mathcal{A}}$ *be a family* g*-supersolution of the form*

$$y_t^\alpha = y_T^\alpha + \int_t^T g(s, y_s^\alpha, z_s^\alpha)\,\mathrm{d}s + (A_T^\alpha - A_t^\alpha) - \int_t^T z_s^\alpha\,\mathrm{d}B_s \tag{16}$$

such that, for each α, y_t^α *is continuous and such that* $E[\sup_{0\le t\le T}(y_t^\alpha)^2] \le C_1$, *where the constant* C_1 *is independent of* α. *There then exists a constant* C, *independent of* α, *such that*

$$E\int_0^T |z_s^\alpha|^2\,\mathrm{d}s + E[(A_T^\alpha)^2] \le C.$$

Proof. The method is borrowed from [16]. By applying Itô's formula to $|y_t^\alpha|^2$ on $[0, T]$ and taking expectations, with the Lipschitz property of g, we get

$$\begin{aligned}
&E[|y_t^\alpha|^2] + E\left[\int_t^T |z_s^\alpha|^2\,\mathrm{d}s\right] \\
&\quad\le E[(y_T^\alpha)^2] + E\int_t^T g^2(s,0,0)\,\mathrm{d}s + (2\mu+\mu^2)\int_t^T |y_s^\alpha|^2\,\mathrm{d}s + \frac{1}{2}E\left[\int_t^T |z_s^\alpha|^2\,\mathrm{d}s\right] \\
&\qquad + \frac{1}{\beta}E\left[\sup_{0\le t\le T}(y_t^\alpha)^2\right] + \beta E[(A_T^\alpha - A_t^\alpha)^2],
\end{aligned}$$

in view of $2ab \le \frac{1}{\beta}a^2 + \beta b^2$, where β is a real number to be fixed later. From integrability assumptions on $g(\cdot, 0, 0)$ and X, we get

$$E\int_0^T |z_s^\alpha|^2\,\mathrm{d}s \le C_2 + 2\beta E[(A_T^\alpha)^2]. \tag{17}$$

We then reformulate (16) as

$$A_T^\alpha = y_0^\alpha - y_T^\alpha - \int_0^T g(s, y_s^\alpha, z_s^\alpha)\,\mathrm{d}s + \int_0^T z_s^\alpha\,\mathrm{d}B_s$$

and take squares and expectations on both sides. With the Lipschitz condition on g, we get

$$\begin{aligned}
E[(A_T^\alpha)^2] &\le 4E[(y_0^\alpha)^2] + 4E[(y_T^\alpha)^2] + 16TE\int_0^T g^2(s,0,0)\,\mathrm{d}s \\
&\quad + 16\mu^2 TE\int_0^T |y_s^\alpha|^2\,\mathrm{d}s + (16\mu^2T + 4)E\int_0^T |z_s^\alpha|^2\,\mathrm{d}s.
\end{aligned}$$

It follows from (15), $X \in \mathbf{L}^2(\mathcal{F}_T)$ and $g(\cdot, 0, 0) \in \mathbf{L}^2_{\mathcal{F}}(0, T)$ that

$$E[(A_T^\alpha)^2] \le C + (16\mu^2T + 4)E\int_0^T |z_s^\alpha|^2\,\mathrm{d}s. \tag{18}$$

Setting $\beta = \frac{1}{32\mu^2 T+8}$ in (17) and substituting (18) into it, we deduce that $E\int_0^T |z_s^\alpha|^2\,\mathrm{d}s \le C$. Then, in view of (18), $E[(A_T^\alpha)^2] \le C$, and the proof is complete. □

We now give the proof of Theorem 3.1.

Proof of Theorem 3.1. In (13), we fix $m \in \mathbb{N}$ and set

$$g^m(t,y,z) := g(t,y,z) + m d_{\Gamma_t}(z).$$

Since g^m is a Lipschitz function and the condition (14) is satisfied, it follows from Theorem 4.1 in [17] that as $n \to \infty$, the triple $(y^{m,n}, z^{m,n}, \bar{A}^{m,n})$ converges to $(y^m, z^m, \bar{A}^m) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T) \times \mathbf{A}^2_{\mathcal{F}}(0,T)$, which is the solution of the following reflected BSDE whose coefficient is $g^m = g + m d_{\Gamma_\cdot}$:

$$\begin{aligned}
&y_t^m = X + \int_t^T g^m(s, y_s^m, z_s^m)\,\mathrm{d}s + \bar{A}_T^m - \bar{A}_t^m - \int_t^T z_s^m\,\mathrm{d}B_s,\\
&y_t^m \ge L_t,\ \text{a.s. a.e.}, \int_0^T (y_{t-}^m - L_{t-}^*)\,\mathrm{d}\bar{A}_t^m = 0, \qquad (19)\\
&\text{for each } L^* \in \mathbf{D}^2_{\mathcal{F}}(0,T), \text{ such that } y^m \ge L^* \ge L, \mathrm{d}P \times \mathrm{d}t\text{-a.s.}
\end{aligned}$$

We write $A_t^m = m\int_0^t d_{\Gamma_s}(z_s^m)\,\mathrm{d}s$. By (14), we have the following estimate:

$$E\Big[\sup_{0\le t\le T} (y_t^m)^2\Big] + E\int_0^T |z_s^m|^2\,\mathrm{d}s + E[(A_T^m + \bar{A}_T^m)^2] \le C.$$

Then, by (comparison) Theorem A.5 for reflected BSDEs, we have $y_t^m \le y_t^{m+1}$, $\bar{A}_t^m \ge \bar{A}_t^{m+1}$ and $\mathrm{d}\bar{A}_t^m \ge \mathrm{d}\bar{A}_t^{m+1}$ on $[0,T]$. Thus, when $m \to \infty$, $y_t^m \nearrow y_t$, $\bar{A}_t^m \searrow \bar{A}_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, for each $t \in [0,T]$, and $y_t \le y_t^*$. Thanks to Fatou's lemma, we get $E[\sup_{0\le t\le T}|y_t|^2] < \infty$ and thus $y^m \to y$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$. Since $\bar{A}^m$ is an RCLL process, we cannot directly apply the monotonic limit theorem, that is, Theorem A.2 or Theorem 2.1 in [16]. However, using similar techniques as used in the proof of Theorem 2.1 in [16], we know that the limit y can be written in the form

$$y_t = y_0 - \int_0^t g_s^0\,\mathrm{d}s - A_t - \bar{A}_t + \int_0^t z_s\,\mathrm{d}B_s,$$

where $z_\cdot$ and $g_\cdot^0$ (resp. A_t) are the weak limits of $z_\cdot^m$ and $g_s^m = g(s, y_s^m, z_s^m)$ (resp. A_t^m) in $\mathbf{L}^2_{\mathcal{F}}(0,T)$ (resp. $\mathbf{L}^2(\mathcal{F}_t)$). Since $A^m + \bar{A}^m$ is an increasing process, by Lemma 2.2 in [16], we know that y is RCLL. Applying Itô's formulae to $|y_t^m - y_t|^2$ on $[\sigma, \tau]$, with stopping times

$0 \le \sigma \le \tau \le T$, it then follows that

$$
\begin{aligned}
&E|y^m_\sigma - y_\sigma|^2 + E\int_\sigma^\tau |z^m_s - z_s|^2\,\mathrm{d}s \\
&\quad = E|y^m_\tau - y_\tau|^2 + E\sum_{t\in(\sigma,\tau]}[(\Delta A_t)^2 - (\bar{A}^m_t - \bar{A}_t)^2] - 2E\int_\sigma^\tau (y^m_s - y_s)(g^m_s - g^0_s)\,\mathrm{d}s \\
&\qquad + 2E\int_{(\sigma,\tau]}(y^m_s - y_s)\,\mathrm{d}A^m_s - 2E\int_{(\sigma,\tau]}(y^m_s - y_s)\,\mathrm{d}A_s + 2E\int_{(\sigma,\tau]}(y^m_{s-} - y_{s-})\,\mathrm{d}(\bar{A}^m_s - \bar{A}_s).
\end{aligned}
$$

Since $E\int_{(\sigma,\tau]}(y^m_s - y_s)\,\mathrm{d}A^m_s \le 0$ and $E\int_{(\sigma,\tau]}(y^m_{s-} - y_{s-})\,\mathrm{d}(\bar{A}^m_s - \bar{A}_s) \le 0$, we get

$$
\begin{aligned}
E\int_\sigma^\tau |z^m_s - z_s|^2\,\mathrm{d}s &\le E|y^m_\tau - y_\tau|^2 + E\sum_{t\in(\sigma,\tau]}(\Delta A_t)^2 + 2E\int_\sigma^\tau |y^m_s - y_s||g^m_s - g^0_s|\,\mathrm{d}s \\
&\quad + 2E\int_{(\sigma,\tau]}|y^m_s - y_s|\,\mathrm{d}A_s.
\end{aligned}
$$

We are now in the same situation as in the proof of the monotonic limit theorem (cf. [16], proof of Theorem 2.1). We can then follow the same approach to get the strong convergence of $z^m \to z$ in $\mathbf{L}^p_{\mathcal{F}}(0,T)$ for $p < 2$.

We pass to the limit on both sides of (19), using the above convergence results of $(y^m, z^m, A^m, \bar{A}^m)$. The limit $(y, z, A, \bar{A})$ satisfies

$$
y_t = X + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T z_s\,\mathrm{d}B_s.
$$

The estimate $E[(A^m_T)^2] \le C$ implies that $E[(\int_0^T d_{\Gamma_s}(z^m_s)\,\mathrm{d}s)^2] \le \frac{C}{m^2}$. When $m \to +\infty$, we get

$$
E\left[\int_0^T d_{\Gamma_s}(z_s)\,\mathrm{d}s\right] = 0, \qquad \text{thus } d_{\Gamma_t}(z_t) \equiv 0,\ \mathrm{d}P \times \mathrm{d}t\text{-a.s.}
$$

It remains to prove that (y, A) satisfies condition (ii) in Definition 3.1, that is, $y \ge L$ and

$$
\begin{aligned}
&\int_0^T (y_{t-} - L^*_{t-})\,\mathrm{d}\bar{A}_t = 0, \\
&\text{a.s., for any } L^* \in \mathbf{D}^2_{\mathcal{F}}(0,T) \text{ such that } y_t \ge L^*_t \ge L_t,\ \mathrm{d}P \times \mathrm{d}t\text{-a.s.}
\end{aligned}
\tag{20}
$$

From $y^m \ge L$, $m \in \mathbb{N}$, we have $y \ge L$. Thus, for each $L^* \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ such that $y \ge L^* \ge L$, we have

$$
\begin{aligned}
\int_0^T (y_{t-} - y^m_{t-} \wedge L^*_{t-})\,\mathrm{d}\bar{A}_t &= \int_0^T (y_{t-} - y^m_{t-})\,\mathrm{d}\bar{A}_t + \int_0^T (y^m_{t-} - y^m_{t-} \wedge L^*_{t-})\,\mathrm{d}\bar{A}^m_t \\
&\quad + \int_0^T (y^m_{t-} - y^m_{t-} \wedge L^*_{t-})\,\mathrm{d}(A_t - \bar{A}^m_t).
\end{aligned}
$$

As $m \to \infty$, the first term on the right-hand side tends to zero due to the Lebesgue dominated convergence theorem. The second term is null because of (19) and the fact that $y^m \geq y^m \wedge L^* \geq L$. For the third term, we have

$$
\begin{aligned}
E\left|\int_0^T (y_{t-}^m - y_{t-}^m \wedge L_{t-}^*)\,\mathrm{d}(A_t - \bar{A}_t^m)\right| &\leq E\Big[\sup_{t\in[0,T]} |y_t^m - y_{t-}^m \wedge L_{t-}^*|(A_T^m - A_T)\Big] \\
&\leq E\Big[\sup_{t\in[0,T]} |y_t^m - y_{t-}^m \wedge L_{t-}^*|^2\Big]^{1/2} E[(A_T^m - A_T)^2]^{1/2},
\end{aligned}
$$

which also converges also to zero in view of $E[(A_T^m - A_T)^2]^{1/2} \searrow 0$ and the boundedness of $E[\sup_{t\in[0,T]} |y_t^m|^2]$. This, with $y^m \wedge L^* \nearrow L^*$, yields (20).

For part (iii) of Definition 3.1, we consider a quadruple $(y^*, z^*, A^*, \bar{A}^*)$ which satisfies parts (i) and (ii) of Definition 3.1. Since $d_{\Gamma_s}(y_s^*, z_s^*) \equiv 0$, we have, for any $m \in \mathbb{N}$,

$$
y_t^* = X + \int_t^T g(s, y_s^*, z_s^*)\,\mathrm{d}s + m\int_t^T d_{\Gamma_s}(y_s^*, z_s^*)\,\mathrm{d}s + A_T^* - A_t^* + \bar{A}_T^* - \bar{A}_t^* - \int_t^T z_s\,\mathrm{d}B_s.
$$

Since $\mathrm{d}A^* \geq 0$, by (comparison) Theorem A.5, it follows that $y^* \geq y^m$ for all m. Thus, (iii) holds. □

Remark 3.4. If L is continuous or has only positive jumps ($L_{t-} \leq L_t$), then $\bar{A}$ is a continuous process. In this case, $\bar{A}^n$ in (19) are continuous and $\bar{A}_t^n \geq \bar{A}_t^{n+1}$, $\mathrm{d}\bar{A}_t^n \geq \mathrm{d}\bar{A}_t^{n+1}$, $0 \leq t \leq T$, with $E[(\bar{A}_T^n)^2] \leq C$. Thus, $\bar{A}_t^n \searrow \bar{A}_t$, $0 \leq t \leq T$. Moreover,

$$
0 \leq \bar{A}_t^n - \bar{A}_t \leq \bar{A}_T^n - \bar{A}_T.
$$

From

$$
E\Big[\sup_{0\leq t\leq T} (\bar{A}_t^n - \bar{A}_t)^2\Big] \leq E[(\bar{A}_T^n - \bar{A}_T)^2] \to 0 \qquad \text{as } n \to \infty,
$$

it follows that $\bar{A}_t^n \searrow \bar{A}_t$ uniformly. We can then pass to a limit on both sides of (19) to obtain the g_Γ-reflected BSDE with the lower obstacle L.

3.3. Comparison of different limits of $y^{m,n}$ to the g_Γ-reflected solution

The g_Γ-reflected BSDE with a lower barrier is a special type of constrained BSDE, in which y and z are constrained in $[L_t, +\infty)$ and Γ, respectively. Let us put the two constraints together and set $\widehat{\Gamma}_t = [L_t, +\infty) \times \Gamma_t \subset \mathbb{R} \times \mathbb{R}^d$. In this case, the penalization equation becomes

$$
\begin{aligned}
y_t^{n,n} &= X + \int_t^T g(s, y_s^{n,n}, z_s^{n,n})\,\mathrm{d}s + n\int_t^T d_{\widehat{\Gamma}_s}(y_s^{n,n}, z_s^{n,n})\,\mathrm{d}s - \int_t^T z_s^{n,n}\,\mathrm{d}B_s \\
&= X + \int_t^T g(s, y_s^{n,n}, z_s^{n,n})\,\mathrm{d}s + n\int_t^T d_{\Gamma_s}(z_s^{n,n})\,\mathrm{d}s + n\int_t^T (L_s - y_s^{n,n})^+\,\mathrm{d}s \qquad (21) \\
&\quad - \int_t^T z_s^{n,n}\,\mathrm{d}B_s.
\end{aligned}
$$

Let $\hat{A}_t^{n,n} = n\int_0^t d_{\hat{\Gamma}_s}(z_s^{n,n})\,\mathrm{d}s$. Again from the monotonic limit theorem, Theorem A.2, we know that $(y^{n,n}, z^{n,n}, \hat{A}^{n,n})$ converges to $(\hat{y}, \hat{z}, \hat{A}) \in \mathbf{L}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times \mathbf{A}^2_{\mathcal{F}}(0,T)$ as $n \to \infty$ and that the limit is the $g_{\hat{\Gamma}}$-solution, that is, the smallest g-supersolution constrained in $\hat{\Gamma}$:

$$\hat{y}_t = X + \int_t^T g(s, \hat{y}_s, \hat{z}_s)\,\mathrm{d}s + \hat{A}_T - \hat{A}_t - \int_t^T \hat{z}_s\,\mathrm{d}B_s,$$

with $d_{\hat{\Gamma}_t}(\hat{y}_t, \hat{z}_t) = 0$, a.e. a.s. on $[0,T]$.

Comparing this result to that of Theorem 3.1 for g_Γ-reflected BSDEs, we have the following.

Proposition 3.1. *The above $g_{\hat{\Gamma}}$-solution of BSDE $(\hat{y}_t, \hat{z}_t, \hat{A}_t)_{t\in[0,T]}$ coincides with the g_Γ-reflected solution obtained in Theorem 3.1: $(\hat{y}_t, \hat{z}_t, \hat{A}_t) \equiv (y_t, z_t, A_t + \bar{A}_t)$.*

Proof. For $m \le n$, by the comparison theorem for (13) and (21), we have

$$y_t^{m,m} \le y_t^{m,n} \le y_t^{n,n}.$$

Letting $n \to \infty$ yields

$$y_t^{m,m} \le y_t^m \le \hat{y}_t,$$

then $m \to \infty$ yields

$$\hat{y}_t \le y_t \le \hat{y}_t.$$

Thus, the two g-supersolutions coincide with each other. $\square$

Let us consider another limit of $y^{m,n}$ by first letting $m \to \infty$. We have

$$y_t^{m,m} \ge y_t^{m,n} \ge y_t^{n,n}.$$

Once again from the monotonic limit theorem, Theorem A.2, when $m \to \infty$, the triple $(y^{m,n}, z^{m,n}, A^{m,n})$ converges to $(\widetilde{y}^n, \widetilde{z}^n, \widetilde{A}^n) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times \mathbf{A}^2_{\mathcal{F}}(0,T)$, which is the solution of the following g^n_Γ-supersolution of BSDE with $g^n = g + n(L_t - y)^+$, or

$$\begin{aligned} \tilde{y}_t^n &= X + \int_t^T g(s, \tilde{y}_s^n, \tilde{z}_s^n)\,\mathrm{d}s + \tilde{A}_T^n - \tilde{A}_t^n + n\int_t^T (L_s - \tilde{y}_s^n)^+\,\mathrm{d}s - \int_{ts}^T \tilde{z}_s\,\mathrm{d}B_s, \\ &(\tilde{z}_t^n) \in \Gamma_t, \mathrm{d}P \times \mathrm{d}t\text{-a.s.}, \mathrm{d}A^n \ge 0, \end{aligned} \tag{22}$$

and we have

$$\hat{y}_t \ge \tilde{y}_t^n \ge y_t^{n,n}.$$

By letting $n \to \infty$, we see that $\tilde{y}_t^n \uparrow \hat{y}_t = y_t$.

Remark 3.5. The proposition also holds for a general constraint $\Gamma(y,z)$ in $\mathbb{R} \times \mathbb{R}^d$.

3.4. Existence of a g_Γ-reflected solution with an upper barrier: proof of Theorem 3.2

The main idea is still based on the penalization method, but with more technicalities.

For each $n \in \mathbb{N}$, we consider the solution $(y^n, z^n, K^n) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times \mathbf{A}^2_{\mathcal{F}}(0,T)$ of the following reflected BSDE with the coefficient $g^n(t,y,z) = g(t,y,z) + nd_{\Gamma_t}(z)$ and the upper reflecting obstacle U:

$$
\begin{aligned}
& y_t^n = X + \int_t^T g^n(s, y_s^n, z_s^n)\,\mathrm{d}s - (K_T^n - K_t^n) - \int_t^T z_s^n\,\mathrm{d}B_s, \\
& y^n \le U, \mathrm{d}P \times \mathrm{d}t\text{-a.s.}, \mathrm{d}K \ge 0, \text{ and } \int_0^T (U_{t-}^* - y_{t-}^n)\,\mathrm{d}K_t^n = 0, \\
& \forall U^* \in \mathbf{D}^2_{\mathcal{F}}(0,T), \text{such that } y^n \le U^* \le U\,\mathrm{d}P \times \mathrm{d}t\text{-a.s.}
\end{aligned}
\tag{23}
$$

Since $g^n(t,y,z)$ is Lipschitz with respect to (y,z), from the existence theorem of [17] for reflected BSDEs with L^2-obstacle, this equation has a unique solution. We write $A_t^n = n\int_0^t d_{\Gamma_s}(z_s^n)\,\mathrm{d}s$.

In order to get an a priori estimate for (y^n, z^n, A^n, K^n), we need the following lemma.

Lemma 3.3. *For any $X \in \mathbf{L}^2(\mathcal{F}_T)$, there exists a quadruple of processes $(y^*, z^*, A^*, K^*) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times (\mathbf{A}^2_{\mathcal{F}}(0,T))^2$ satisfying*

$$
\begin{aligned}
& y_t^* = X + \int_t^T g(s, y_s^*, z_s^*)\,\mathrm{d}s + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T z_s^*\,\mathrm{d}B_s, \\
& d_{\Gamma_t}(z_t^*) = 0 \textit{ and } y_t^* \le U_t, \mathrm{d}P \times \mathrm{d}t\textit{-a.s., with } \int_0^T (y_{t-}^* - U_{t-}^*)\,\mathrm{d}K_t^* = 0, \textit{a.s.} \\
& \forall U^* \in \mathbf{D}^2_{\mathcal{F}}(0,T), \textit{ such that } y^* \le U^* \le U\,\mathrm{d}P \times \mathrm{d}t\textit{-a.s.}
\end{aligned}
\tag{24}
$$

Proof. Fix a process $\sigma_t \in \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ satisfying $\sigma_t \in \Gamma_t$, $t \in [0,T]$. We consider a forward SDE with the upper obstacle U_t. For $0 \le t \le T$,

$$
\begin{aligned}
& \mathrm{d}x_t = -g(t, x_t, \sigma_t)\,\mathrm{d}t - \mathrm{d}\bar{A}_t + \sigma_t\,\mathrm{d}B_t, \\
& x_0 = 1 \wedge U_0, \text{ with } x_t \le U_t, \text{ a.s. a.e.}
\end{aligned}
$$

Since $g(t,x,\sigma_t)$ is a Lipschitz function in x and $g(t,x,\sigma_t) \in \mathbf{L}^2_{\mathcal{F}}(0,T)$ with $\mathit{ess}\sup_{0\le t\le T} U_t^- \in \mathbf{L}^2(\mathcal{F}_T)$, this equation admits a solution $(x_t, \bar{A}_t)$ in $\mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{A}^2_{\mathcal{F}}(0,T)$. Set

$$
\begin{aligned}
& y_t^* = x_t, \qquad z_t^* = \sigma_t, \\
& A_t^* = \bar{A}_t + (x_T - X)^+ 1_{\{t=T\}}, \qquad K_t^* = (x_T - X)^+ 1_{\{t=T\}}.
\end{aligned}
$$

This quadruple is then exactly what we need. □

We have the following estimate.

Lemma 3.4. *There exists a constant $C > 0$, independent of n, such that*

$$E\Big[\sup_{0\le t\le T}(y_t^n)^2\Big] + E\int_0^T |z_s^n|^2\,\mathrm{d}s + E[(A_T^n)^2] + E[(K_T^n)^2] \le C. \tag{25}$$

Proof. Consider the following reflected BSDE:

$$\begin{aligned}
&y_t^0 = X + \int_t^T g(s, y_s^0, z_s^0)\,\mathrm{d}s - (K_T^0 - K_t^0) - \int_t^T z_s^0\,\mathrm{d}B_s, \qquad t\in[0,T],\\
&y_t^0 \le U_t, \mathrm{d}K_t^0 \ge 0, \int_0^T (y_{t-}^0 - U_{t-}^*)\,\mathrm{d}K_t^0 = 0\\
&\forall U^* \in \mathbf{D}_{\mathcal{F}}^2(0,T),\ \text{such that } y^0 \le U^* \le U\,\mathrm{d}P\times\mathrm{d}t\text{-a.s.}
\end{aligned}$$

This equation has a unique solution, $(y^0, z^0, K^0) \in \mathbf{D}_{\mathcal{F}}^2(0,T)\times\mathbf{L}_{\mathcal{F}}^2(0,T;\mathbb{R}^d)\times\mathbf{A}_{\mathcal{F}}^2(0,T)$. By the comparison theorem of reflected BSDEs, we have $y_t^n \ge y_t^0$ on $[0,T]$.

On the other hand, the quadruple (y^*, z^*, A^*, K^*) that we get from Lemma 3.3 satisfies

$$\begin{aligned}
&y_t^* = X + \int_t^T (g + nd_{\Gamma_s})(s, y_s^*, z_s^*)\,\mathrm{d}s + (A_T^* - A_t^*) - (K_T^* - K_t^*) - \int_t^T z_s^*\,\mathrm{d}B_s,\\
&y_t^* \le U_t, \text{ a.e. a.s. } \int_0^T (y_{t-}^* - U_{t-})\,\mathrm{d}K_t^* = 0, \text{ a.s.}
\end{aligned}$$

It follows from the comparison Theorem A.5 for reflected BSDEs that $y_t^n \le y_t^*$, $K_t^n \le K_t^*$ and $\mathrm{d}K_t^n \le \mathrm{d}K_t^*$ for each $n\in\mathbb{N}$, $t\in[0,T]$. Thus, there exists a constant $C>0$, independent of n, such that

$$E\Big[\sup_{0\le t\le T}(y_t^n)^2\Big] \le E\Big[\sup_{0\le t\le T}\{(y_t^0)^2 + (y_t^*)^2\}\Big] \le C \tag{26}$$

and

$$E[(K_T^n)^2] \le E[(K_T^*)^2] \le C. \tag{27}$$

To estimate (z^n, A^n), we just need to rewrite (23)

$$y_t^n - K_t^n = X - K_T^n + \int_t^T g(s, y_s^n, z_s^n)\,\mathrm{d}s + A_T^n - A_t^n - \int_t^T z_s^n\,\mathrm{d}B_s$$

and it is easy to check that Lemma 3.2 can be applied for the triple $(y_t^n - K_t^n, z_t^n, A_T^n)$ to get the estimates. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. In (23), since $g^n(t,y,z) \leq g^{n+1}(t,y,z)$, by (comparison) Theorem A.5 for reflected BSDEs, $y_t^0 \leq y_t^n \leq y_t^{n+1} \leq y_t^*$. Thus, $\{y_t^n\}_{n=1}^{\infty}$ increasingly converges to y_t as $n \to \infty$ and

$$E\Big[\sup_{0\leq t\leq T}(y_t)^2\Big] \leq E\Big[\sup_{0\leq t\leq T}\{(y_t^0)^2 + (y_t^*)^2\}\Big] \leq C.$$

It follows from the dominated convergence theorem that

$$\lim_{n\to\infty} E\left[\int_0^T |y_t^n - y_t|^2\,\mathrm{d}t\right] = 0.$$

We can also get from Theorem A.5 that $K_t^n \leq K_t^{n+1} \leq K_t^*$ and $\mathrm{d}K_t^n \leq \mathrm{d}K_t^{n+1} \leq \mathrm{d}K_t^*$, $0 \leq t \leq T$. It follows that $\{K^n\}_{n=1}^n$ increasingly converges to an increasing process $K \in \mathbf{A}^2_{\mathcal{F}}(0,T)$ with $E[(K_T)^2] \leq C$. Meanwhile, A^n are continuous increasing processes satisfying $E[(A_T^n)^2 + \int_0^T |z_s^n|^2\,\mathrm{d}s] \leq C$ and there exists a process $z \in \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ such that $z^n \to z$ weakly in $\mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

The conditions of the generalized monotonic limit theorem, Theorem A.1 or Theorem 3.1 in [17], are now satisfied. Therefore, we have $z^n \to z$ strongly in $\mathbf{L}^p_{\mathcal{F}}(0,T;\mathbb{R}^d)$ for $p < 2$. With the Lipschitz condition on g, the limit $y \in \mathbf{D}^2(0,T)$ can be written as

$$y_t = X + \int_t^T g(s,y_s,z_s)\,\mathrm{d}s + (A_T - A_t) - (K_T - K_t) - \int_t^T z_s\,\mathrm{d}B_s,$$

where, for each t, $A_t^n \to A_t$ weakly in $\mathbf{L}^2(\mathcal{F}_t)$, $K_t^n \to K_t$ strongly in $\mathbf{L}^2(\mathcal{F}_t)$ and $A, K \in \mathbf{A}^2_{\mathcal{F}}(0,T)$ are increasing processes.

From $E[(A_T^n)^2] = E[(n\int_0^T d_{\Gamma_s}(z_s^n)\,\mathrm{d}s)^2] \leq C$, it follows that

$$E\left[\left(\int_0^T d_{\Gamma_s}(z_s^n)\,\mathrm{d}s\right)^2\right] \leq \frac{C}{n^2}.$$

While $d_{\Gamma_s}(z_s^n) \geq 0$, we get that $\int_0^T d_{\Gamma_s}(z_s^n)\,\mathrm{d}s \to 0$ as $n \to \infty$. With the Lipschitz property of $d_{\Gamma_t}(z)$ and the convergence of z^n, we deduce that

$$d_{\Gamma_t}(z_t) = 0 \qquad \mathrm{d}P \times \mathrm{d}t\text{-a.s.}$$

We now prove that the quadruple (y,z,A,K) satisfies part (ii) of Definition 3.2. We have $y \leq U$ from $y^n \leq U$. Now, for each $U^* \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ such that $U \geq U^* \geq y \geq y^n$, since $\int_0^T (y_{t-}^n - U_{t-}^*)\,\mathrm{d}K_t^n = 0$, it follows that $\int_0^T (y_{t-} - U_{t-}^*)\,\mathrm{d}K_t^n = 0$. Moreover, we have $\mathrm{d}K_t^n \leq \mathrm{d}K_t$ and $K_T^n \nearrow K_T$ in $\mathbf{L}^2(\mathcal{F}_T)$. It then follows that

$$0 \leq \int_0^T (U_{t-}^* - y_{t-})\,\mathrm{d}(K_t - K_t^n) \leq \sup_{t\in[0,T]} (U_t^* - y_t)\cdot[K_T - K_T^n].$$

With this, (6) and the estimate of y, it follows that part (ii) of Definition 3.2 holds.

We now prove that part (iii) in Definition 3.2 holds true for the quadruple. In fact, for any other quadruple $(\bar{y}, \bar{z}, \bar{A}, \bar{K}) \in \mathbf{D}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d) \times (\mathbf{A}^2(0,T))^2$ satisfying

$$
\begin{aligned}
&\bar{y}_t = X + \int_t^T g(s,\bar{y}_s,\bar{z}_s)\,\mathrm{d}s + \bar{A}_T - \bar{A}_t - (\bar{K}_T - \bar{K}_t) - \int_t^T \bar{z}_s\,\mathrm{d}B_s,\\
&d_{\Gamma_t}(\bar{z}_t) = 0, \mathrm{d}P \times \mathrm{d}t\text{-a.s.}, \mathrm{d}\bar{A} \ge 0, \mathrm{d}\bar{K} \ge 0,\\
&\bar{y}_t \le U_t, \mathrm{d}P \times \mathrm{d}t\text{-a.s.}, \int_0^T (U^*_{t-} - \bar{y}_{t-})\,\mathrm{d}\bar{K}_t = 0, \ \text{a.s.},
\end{aligned}
\tag{28}
$$

for any $U^* \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ such that $\bar{y} \le U^* \le U$ $\mathrm{d}P \times \mathrm{d}t$-a.s. It then also satisfies, for all $n \in \mathbb{N}$,

$$
\bar{y}_t = X + \int_t^T g(s,\bar{y}_s,\bar{z}_s)\,\mathrm{d}s + n\int_t^T d_{\Gamma_s}(\bar{z}_s)\,\mathrm{d}s + \bar{A}_T - \bar{A}_t - (\bar{K}_T - \bar{K}_t) - \int_t^T \bar{z}_s\,\mathrm{d}B_s.
$$

Compare this to (23): since $\mathrm{d}\bar{A}_t \ge 0$, we have $\bar{y} \ge y^n$, and $\bar{K} \ge K^n$. Letting $n \to \infty$, it follows that

$$
\bar{y}_t \ge y_t, \qquad \bar{K}_t \ge K_t, \qquad \forall t \in [0,T], \text{a.s.} \tag{29}
$$

Therefore, y is the smallest process satisfying Definition 3.2(i) and (ii).

It remains to prove the relation $\mathcal{V}_{[0,T]}(A+K) = \mathcal{V}_{[0,T]}(A+K)$ in (11), namely, that A and K is the Jordan decomposition of $A-K$. For this, we set $\tilde{V}_t = \mathcal{V}_{[0,t]}(A-K)$ and define the Jordan decomposition of $A-K$ by

$$
\tilde{A}_t = \tfrac{1}{2}(\tilde{V}_t + A_t - K_t), \qquad \tilde{K}_t = \tfrac{1}{2}(\tilde{V}_t - A_t + K_t).
$$

We have $\mathrm{d}\tilde{K}_t = \frac{1}{2}\mathrm{d}(\tilde{V}_t - A_t + K_t) \le \mathrm{d}K_t$ and, thus, for each $U^* \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ with $U \ge U^* \ge y$, $\mathrm{d}P \times \mathrm{d}t$-a.s.,

$$
0 \le \int_0^T (y_{t-} - U^*_{t-})\,\mathrm{d}\tilde{K}_t \le \int_0^T (y_{t-} - U^*_{t-})\,\mathrm{d}K_t = 0.
$$

Therefore, the quadruple $(y, z, \tilde{A}, \tilde{K})$ also satisfies (28). It then follows from the second inequality of (29) that $\tilde{K} \ge K$. This, together with $\tilde{K} \le K$, yields $\tilde{K} \equiv K$ and thus A and K are indeed the Jordan decomposition of $A-K$. □

Remark 3.6. Since $y - K$ is the smallest process satisfying the BSDE associated with X, g^K and constraint Γ, it is the $(g^K)_\Gamma$-solution with terminal condition $X - K_T$, where

$$
g^K(t,y,z) = g(t, y + K_t, z).
$$

Remark 3.7. If U is continuous (or satisfies $U_{t-} \ge U_t$), then K is a continuous process. In fact, by [6], the solution y^n of (23) and the reflecting process K are continuous. This, together with $K^n \le K^{n+1}$ and $\mathrm{d}K^n \le \mathrm{d}K$, yields

$$
0 \le K_t - K^n_t \le K_T - K^n_T
$$

and thus

$$E\Big[\sup_{0\le t\le T}(K_t - K_t^n)^2\Big] \le E[(K_T - K_T^n)^2] \to 0.$$

The continuity of K then follows from the uniform convergence of K^n to K.

4. Applications of g_Γ-reflected BSDEs: American option pricing in an incomplete market

We follow the idea in El Karoui *et al.* [7]. In a financial market, we consider the wealth strategy and portfolio (Y_t, π_t) of an investor which is a pair of adapted processes in $\mathbf{L}^2_{\mathcal{F}}(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$. This pair solves the following BSDE:

$$-\mathrm{d}Y_t = g(t, Y_t, \pi_t^\tau \sigma_t)\,\mathrm{d}t - \pi_t^\tau \sigma_t\,\mathrm{d}B_t,$$

where g is a convex function of (y,π) satisfying the same Lipschitz condition given in (1). We suppose that the volatility matrix σ_t is invertible and that σ_t, $(\sigma_t)^{-1}$ are bounded. We are concerned with the problem of pricing an American contingent claim.

Let S be a continuous process satisfying $E[\sup_t (S_t^+)^2] < \infty$, which is a given continuous-time pay-off during $[t,T)$, and ξ be a given terminal pay-off at T. For a given $t \ge 0$, let $\mathcal{T}_t$ be the set of stopping times valued in $[t,T]$. The corresponding total pay-off at time $s \in [t,T]$ is

$$\tilde{S}_s = \xi 1_{\{s=T\}} + S_s 1_{\{s<T\}}.$$

According to [7], in a complete market, that is, a market without constraints on (Y,π), the price of the American contingent claim $(\tilde{S}_s)_{0\le s\le T}$ at time t is given by

$$Y_t = ess\sup_{\tau\in\mathcal{T}_t} Y_t(\tau, \tilde{S}_\tau).$$

Here, $Y_t(\tau,\tilde{S}_\tau)$ is the solution of the BSDE with terminal time τ and terminal condition $\tilde{S}_\tau$. In fact, the price $(Y_t)_{0\le t\le T}$ is the unique solution of the reflected BSDE associated with the terminal condition ξ and the obstacle S: there exists $(\pi_t) \in \mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ and an increasing continuous process (A_t) with $A_0 = 0$ such that

$$-\mathrm{d}Y_t = g(s, Y_t, \pi_t^\tau \sigma_t)\,\mathrm{d}s + \mathrm{d}A_t - \pi_t^\tau \sigma_t\,\mathrm{d}B_t, \qquad Y_T = \xi,$$
$$Y_t \ge S_t, 0 \le t \le T, \int_0^T (Y_t - S_t)\,\mathrm{d}A_t = 0.$$

Furthermore, the stopping time $D_t = \inf(t \le s \le T \mid \mathrm{d}A_s > 0) \wedge T$ is the biggest optimal time after t and

$$Y_t = Y_t(D_t, \tilde{S}_{D_t}).$$

Our problem is to price the American contingent claim $(\tilde{S}_s, 0 \le s \le T)$ for an incomplete market where the portfolios π_t are constrained in Γ_t, which is a closed subset of $\mathbb{R}^d$. This problem can be solved as follows. We set $\Gamma_t^1 = \{z \in \mathbb{R}^d : z^\tau \sigma_t^{-1} \in \Gamma_t\}$.

Theorem 4.1. *We assume that ξ is attainable, that is, there exists a g-supersolution (Y', z', A') on $[t, T]$ with $z'_t \in \Gamma_t^1$, t-a.e. and with the terminal condition ξ. Then the solution $(Y, z, A, \bar{A})$ of the g_{Γ^1}-reflected BSDE with lower obstacle S exists and Y is the price process of the American option in the incomplete market. The quadruple $(Y, z, A, \bar{A})$ solves*

$$\begin{aligned} Y_t &= \xi + \int_t^T g(s, Y_s, \pi_s^\tau \sigma_s)\,\mathrm{d}s + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T \pi_s^\tau \sigma_s\,\mathrm{d}B_s, \\ Y_t &\ge S_t, 0 \le t \le T, z^\tau \sigma_t^{-1} \in \Gamma_t^1, \int_0^T (Y_t - S_t)\,\mathrm{d}\bar{A}_t = 0. \end{aligned} \tag{30}$$

Furthermore, $\bar{A}$ is continuous and $D_0 = \inf(0 \le s \le T \mid \mathrm{d}\bar{A}_s > 0) \wedge T$ is the corresponding optimal stopping time.

Proof. Let $\tau \in \mathcal{T}_t$ be any given stopping time and let $(\bar{Y}, \bar{z}, \bar{A})$ be a g_{Γ^1}-solution on $[0, \tau]$ with terminal condition $\tilde{S}_\tau$. By the comparison theorem, we know that $Y_t^n \le \bar{Y}_t$ on $[0, \tau]$, where Y^n is the solution of the reflected BSDE on $[0, T]$ associated with $(\xi, g + nd_{\Gamma_t^1}, S)$. Since Y^n upwardly converges to Y, we know $Y_t \le \bar{Y}_t$ on $[0, \tau]$. It follows that Y is the smallest g-supersolution constrained in Γ^1 among all g_{Γ^1}-solutions $\bar{Y}$ defined on $[t, \tau]$ with terminal condition $\bar{Y}_\tau = \tilde{S}_\tau$. Moreover Y is the g_{Γ^1}-solution defined on $[0, D_0]$. Thus, D_0 is the optimal stopping time. $\square$

4.1. Some examples of American call options

We study the American call option, setting $S_t = (X_t - k)^+$, $\xi = (X_T - k)^+$, where X is the price of underlying stock and k is the strike price. More precisely, X is the solution of

$$X_t = x_0 + \int_0^t \mu_s X_s\,\mathrm{d}s + \int_0^t \sigma_s X_s\,\mathrm{d}B_s. \tag{31}$$

Correspondingly, in (30), g is a linear function:

$$g(t, y, \pi) = -r_t y - (\mu_t - r_t)\pi^\tau \sigma_t.$$

Proposition 4.1. *If ξ is attainable, then the maturity time of an American call option in an incomplete market is still T.*

Proof. Consider the price process Y^0 of an American call option, without constraint, which is a solution of reflected the BSDE

$$Y_t^0 = \xi + \int_t^T g(s, Y_s^0, \pi_s^0)\,\mathrm{d}s + \bar{A}_T^0 - \bar{A}_t^0 - \int_t^T (\pi_s^0)^\tau \sigma_s\,\mathrm{d}B_s,$$

$$Y_t^0 \geq S_t, \int_0^T (Y_t^0 - S_t)\,\mathrm{d}\bar{A}_t^0 = 0.$$

Comparing it with (30), we have that $Y_t \geq Y_t^0$, $\bar{A}_t \leq \bar{A}_t^0$, $t \in [0, T]$.

In a complete market, an American call option always exercises at the terminal time T, which implies that $D_t^0 = T$, where $D_t^0 = \inf(t \leq s \leq T \mid \mathrm{d}A_s^0 > 0) \wedge T$. Therefore, we have $\bar{A}_t^0 = 0$ on $[0, T)$. It follows that $\bar{A}_t \leq \bar{A}_t^0 = 0$, $t \in [0, T]$. Then, by definition, $D_t = T$. □

From this proposition, we know that the seller's price process Y in an incomplete market is possibly greater than in a complete market. However, their exercise times are the same, that is, at T. So the seller's price is the same as the seller's price for the corresponding European contingent claim.

We now consider an interesting example.

Example 4.1 (No short-selling). In this case, $\Gamma_t = [0, \infty)$ for $t \in [0, T]$. We set $d = 1$. By Proposition 4.1 and Example 7.1 in [2], the price process of the American call option takes the same value as a European call option. This means that the constraint $K = [0, \infty)$ does not make any difference.

From this example, we know that the constraint $\Gamma_t = [0, \infty)$ does not influence the price processes of the American contingent claim. In fact, we have a more general result.

Proposition 4.2. *Consider the constraint $\Gamma_t = [0, \infty)$ for $t \in [0, T]$. If $\xi = \Phi(X_T)$, $S_t = l(X_t)$, where $\Phi, l : \mathbb{R} \to \mathbb{R}$ are both increasing in x, and σ satisfies the uniformly elliptic condition, then the price process Y takes the same value as in a complete market, that is, the constraint Γ does not influence the price.*

Proof. It is sufficient to prove that $\bar{\pi}_t \geq 0$ for the solution $(\bar{Y}, \bar{\pi}, \bar{A})$ of the following reflected BSDE:

$$\begin{aligned} \bar{Y}_t &= \Phi(X_T) + \int_t^T g(s, \bar{Y}_s, \bar{\pi}_s)\,\mathrm{d}s + \bar{A}_T - \bar{A}_t - \int_t^T \bar{\pi}_s^\tau \sigma_s\,\mathrm{d}B_s, \\ \bar{Y}_t &\geq l(X_t), \int_0^T \big(\bar{Y}_t - l(X_t)\big)\,\mathrm{d}\bar{A}_t = 0. \end{aligned} \tag{32}$$

We put $(X_s^{t,x}, \bar{Y}_s^{t,x}, \bar{\pi}_s^{t,x}, \bar{A}_s^{t,x})_{t \leq s \leq T}$ under the Markovian framework with (31). If we define

$$u(t, x) = \bar{Y}_t^{t,x},$$

then, by [6], we know that u is the viscosity solution of the PDE with an obstacle l,

$$\min\left\{u(t, x) - l(x), -\frac{\partial u}{\partial t} - \mathcal{L}u - g(t, x, u, \nabla u \sigma)\right\} = 0,$$
$$u(T, x) = \Phi(x),$$

where $\mathcal{L} = \frac{1}{2}(\sigma_s)^2 \frac{\partial^2}{\partial x \partial x} + \mu \frac{\partial}{\partial x}$. Since $(\bar{\pi}_r^{t,x})^\tau \sigma_r = \nabla u(r, X_r^{t,x})\sigma_r$ and σ_r is uniformly elliptic, we only need to prove that $\nabla u(t,x)$ is non-negative. Indeed, this is easy to obtain by the comparison theorem. For $x_1, x_2 \in \mathbb{R}$, with $x_1 \geq x_2$, we have $X_s^{t,x_1} \geq X_s^{t,x_2}$. It follows that $\Phi(X_T^{t,x_1}) \geq \Phi(X_T^{t,x_2})$ and $l(X_s^{t,x_1}) \geq l(X_s^{t,x_2})$, in view of our assumptions. By the comparison theorem for BSDEs, we get $\bar{Y}_t^{t,x_1} \geq \bar{Y}_t^{t,x_2}$, which implies that $u(t,x_1) \geq u(t,x_2)$. Therefore, u is increasing in x, that is, $\nabla u(t,x) \geq 0$, and it follows that $\bar{\pi}_t^{t,x} \geq 0$. □

4.2. Some examples of American put options

In this case, we set $S_t = (k - X_t)^+$, $\xi = (k - X_T)^+$, where X is the price of underlying stock given in (31) and k is the strike price. Parallel to Proposition 4.2, we have the following.

Proposition 4.3. *Consider the constraint $\Gamma_t = (-\infty, 0]$ for $t \in [0,T]$. If $\xi = \Phi(X_T)$, $S_t = l(X_t)$, where $\Phi, l : \mathbb{R} \to \mathbb{R}$ are both decreasing functions and σ satisfies the uniformly elliptic condition, then the price process Y takes the same value as in a complete market, that is, the constraint Γ has no influence on price process.*

Example 4.2 (No borrowing). $\Gamma_t = (-\infty, Y_t]$. Obviously, $Y_t \geq 0$, in view of $Y_t \geq S_t \geq 0$. Therefore, $\Gamma_t \supset (-\infty, 0]$, and by Proposition 4.3, we know that the price process Y takes the same value as in a complete market.

Under the "no short-selling" constraint, we will get a totally different result.

Example 4.3 (No short-selling). $\Gamma_t = [0, \infty)$ for $t \in [0,T]$. The pricing process Y with hedging π satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, \pi_s)\,\mathrm{d}s + A_T - A_t + \bar{A}_T - \bar{A}_t - \int_t^T \pi_s^* \sigma_s \,\mathrm{d}B_s,$$

$$Y_t \geq S_t, 0 \leq t \leq T, \int_0^T (Y_t - S_t)\,\mathrm{d}\bar{A}_t = 0, \pi_t \geq 0, t\text{-a.e.}$$

Note that $S_t = (k - X_t)^+ < k$. Hence, the g_Γ-reflected solution of the above equation is

$$Y_t = \begin{cases} k, & t \in [0,T), \\ (k - X_T)^+, & t = T, \end{cases}$$

$$\pi_t = 0,$$

$$A_t = \begin{cases} k \displaystyle\int_0^t r_s \,\mathrm{d}s, & t \in [0,T), \\ k \displaystyle\int_0^T r_s \,\mathrm{d}s + k - (k - X_T)^+, & t = T, \end{cases}$$

$$\bar{A}_t = 0.$$

In particular, the price of an American put option under the "no short-selling" constraint is $Y_0 = k$.

Appendix

In this appendix, we present the monotonic limit theorem introduced in [16] (a generalized version was introduced in [17]). We consider the following sequence of Itô processes:

$$y_t^i = y_0^i + \int_0^t g_s^i \, ds - A_t^i + K_t^i + \int_0^t z_s^i \, dB_s, \qquad t \in [0, T], i = 1, 2, \dots. \tag{33}$$

Here, $g^i \in \mathbf{L}^2_{\mathcal{F}}(0, T)$ and A^i, $K^i \in \mathbf{D}^2_{\mathcal{F}}(0, T)$ are given increasing processes. We assume that:

$$\begin{aligned} &\text{(i)} \quad (y_t^i) \text{ increasingly converges to } y \in \mathbf{L}^2_{\mathcal{F}}(0, T) \text{ with } E\Big[\sup_{0\le t\le T} |y_t|^2\Big] < \infty; \\ &\text{(ii)} \quad (g_t^i, z_t^i) \text{ weakly converges to } (g^0, z) \text{ in } \mathbf{L}^2_{\mathcal{F}}(0, T; \mathbb{R} \times \mathbb{R}^d); \\ &\text{(iii)} \quad A^i \text{ is continuous and increasing with } A_0^i = 0 \text{ and } E[(A_T^i)^2] < \infty. \end{aligned} \tag{34}$$

Furthermore, for K^i, we assume that:

$$\begin{aligned} &\text{(iv)} \quad K_t^j - K_s^j \ge K_t^i - K_s^i, \forall 0 \le s \le t \le T, \text{ a.s.}, \forall i \le j; \\ &\text{(v)} \quad \text{for each } t \in [0, T], K_t^j \nearrow K_t \text{ in } j, \text{ with } E[K_T^2] < \infty. \end{aligned} \tag{35}$$

An easy consequence is that:

$$\begin{aligned} &\text{(i)} \quad E\Big[\sup_{0\le t\le T} |y_t^i|^2\Big] \le C; \\ &\text{(ii)} \quad E\int_0^T |y_t^i - y_t|^2 \, ds \to 0. \end{aligned} \tag{36}$$

The generalized monotonic limit theorem given in [17] is as follows.

Theorem A.1. *We assume that* (34) *and* (35) *hold. The limit* y *of the sequence* $\{y^i\}_{i=1}^{\infty}$ *is then of the form*

$$y_t = y_0 + \int_0^t g_s^0 \, ds - A_t + K_t + \int_0^t z_s \, dB_s, \tag{37}$$

where A, $K \in \mathbf{A}^2_{\mathcal{F}}(0, T)$ *are increasing processes. Here, for each* $t \in [0, T]$, A_t (*resp.* K_t) *is the weak* (*resp. strong*) *limit of* $\{A_t^i\}_{i=1}^{\infty}$ (*resp.* $\{K_t^i\}_{i=1}^{\infty}$) *in* $\mathbf{L}^2(\mathcal{F}_t)$. *Furthermore, for any* $p \in [0, 2)$, $\{z^i\}_{i=1}^{\infty}$ *strongly converges to* z *in* $\mathbf{L}^p_{\mathcal{F}}(0, T; \mathbb{R}^d)$, *that is,*

$$\lim_{i\to\infty} E\int_0^T |z_s^i - z_s|^p \, ds = 0. \tag{38}$$

If, moreover, A is a continuous process, then we have

$$\lim_{i\to\infty} E\int_0^T |z_s^i - z_s|^2\,\mathrm{d}s = 0. \tag{39}$$

The monotonic limit theorem was originally obtained in [16]:

$$y_t^i = y_0^i + \int_0^t g_s^i\,\mathrm{d}s - A_t^i + \int_0^t z_s^i\,\mathrm{d}B_s, \qquad t\in[0,T], i=1,2,\dots. \tag{40}$$

Since this result is used in this paper, we state it as follows.

Theorem A.2. *We suppose that assumption (34) holds. The limit y of the sequence $\{y^i\}_{i=1}^\infty$ given in (40) then has the form*

$$y_t = y_0 + \int_0^t g_s^0\,\mathrm{d}s - A_t + \int_0^t z_s\,\mathrm{d}B_s, \qquad 0\le t\le T,$$

where $A\in\mathbf{A}^2_{\mathcal{F}}(0,T)$ is an increasing process. Here, for each $t\in[0,T]$, A_t is the weak limit of $\{A_t^i\}_{i=1}^\infty$ in $\mathbf{L}^2(\mathcal{F}_t)$. Furthermore, $\{z^i\}_{i=1}^\infty$ strongly converges to z in $\mathbf{L}^p_{\mathcal{F}}(0,T,\mathbb{R}^d)$, that is,

$$\lim_{i\to\infty} E\int_0^T |z_s^i - z_s|^p\,\mathrm{d}s = 0, \qquad p\in[0,2). \tag{41}$$

If, furthermore, $(A_t)_{t\in[0,T]}$ is continuous, then we have

$$\lim_{i\to\infty} E\int_0^T |z_s^i - z_s|^2\,\mathrm{d}s = 0. \tag{42}$$

The smallest g-supersolution with constraint Γ was first considered in [16], where Γ is defined as

$$\Gamma_t(\omega) = \{(y,z)\in\mathbb{R}^{1+d} : \Phi(\omega,t,y,z)=0\}.$$

Here, Φ is a non-negative, measurable Lipschitz function and $\Phi(\cdot,y,z)\in\mathbf{L}^2_{\mathcal{F}}(0,T)$ for $(y,z)\in\mathbb{R}\times\mathbb{R}^d$. Under the following assumption, the result of the existence of the smallest solution, obtained in [16], can be stated as follows.

Theorem A.3. *Suppose that the function g satisfies (1) and the constraint Γ satisfies (2). We assume that there is at least one Γ-constrained g-supersolution $y'\in\mathbf{D}^2_{\mathcal{F}}(0,T)$:*

$$\begin{gathered} y_t' = X' + \int_t^T g(s,y_s',z_s')\,\mathrm{d}s + A_T' - A_t' - \int_t^T z_s'\,\mathrm{d}B_s, \\ A'\in\mathbf{A}^2_{\mathcal{F}}(0,T),\ (y_t',z_t')\in\Gamma_t, t\in[0,T],\ \textit{a.s. a.e.} \end{gathered} \tag{43}$$

Then, for each $X \in \mathbf{L}^2(\mathcal{F}_T)$ with $X \leq X'$, a.s., there exists the g_Γ-solution $y \in \mathbf{D}^2_{\mathcal{F}}(0, T)$ with the terminal condition $y_T = X$ (defined in Definition 2.2). Moreover, this g_Γ-solution is the limit of a sequence of g^n-solutions with $g^n = g + nd_\Gamma$, that is,

$$y_t^n = X + \int_t^T (g + nd_\Gamma)(s, y_s^n, z_s^n)\,\mathrm{d}s - \int_t^T z_s^n\,\mathrm{d}B_s, \tag{44}$$

where the convergence is in the following sense:

$$\begin{gathered} y_t^n \nearrow y_t, \qquad \text{with } \lim_{n\to\infty} E[|y_t^n - y_t|^2] = 0, \\ \lim_{n\to\infty} E\int_0^T |z_t - z_t^n|^p\,\mathrm{d}t = 0,\ p \in (0, 2), \end{gathered} \tag{45}$$

$$A_t^n := n\int_0^t d_{\Gamma_s}(s, y_s^n, z_s^n)\,\mathrm{d}s \to A_t \qquad \text{weakly in } \mathbf{L}^2(\mathcal{F}_t), \tag{46}$$

where z and A are the corresponding martingale representing part and increasing part of y, respectively.

Proof. By the comparison theorem for BSDEs, $y_t^n \leq y_t^{n+1} \leq y_t'$. It follows that there exists a $y \leq y'$ such that, for each $t \in [0, T]$,

$$y_t^1 \leq y_t^n \nearrow y_t \leq y_t'.$$

Consequently, there exists a constant $C > 0$, independent of n, such that

$$E\Big[\sup_{0\leq t\leq T} (y_t^n)^2\Big] \leq C, \qquad \text{so } E\Big[\sup_{0\leq t\leq T} (y_t)^2\Big] \leq C.$$

Thanks to the (monotonic limit) Theorem A.2, we can take the limit on both sides of BSDE (44) and obtain

$$y_t = X + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s + A_T - A_t - \int_t^T z_s\,\mathrm{d}B_s.$$

On the other hand, by $E[(A_T^n)^2] = n^2 E[(\int_0^t d_{\Gamma_s}(y_s^n, z_s^n)\,\mathrm{d}s)^2] \leq C$, we have

$$d_{\Gamma_t}(y_t, z_t) \equiv 0, \qquad \mathrm{d}P \times \mathrm{d}t\text{-a.s.}$$

□

Remark A.1. If the constraint Γ is of the form $\Gamma_t = (-\infty, U_t] \times \mathbb{R}^d$, where $U_t \in \mathbf{L}^2(\mathcal{F}_t)$, then the g_Γ-solution with terminal condition $y_T = X$ exists, if and only if $d_{\Gamma_t}(Y_t, Z_t) \equiv 0$, a.s. a.e., where (Y, Z) is the solution of the BSDE

$$-\mathrm{d}Y_t = g(t, Y_t, Z_t)\,\mathrm{d}t - Z_t\,\mathrm{d}B_t, \qquad t \in [0, T], Y_T = X.$$

This follows easily by the comparison theorem.

We also have the following result.

Theorem A.4 (Comparison theorem for g_Γ-solutions). *We assume that g^1, g^2 satisfy (1) and Γ^1, Γ^2 satisfy (2). Further, we suppose that, for each $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, we have*

$$X^1\le X^2,\qquad g^1(t,y,z)\le g^2(t,y,z),\qquad \Gamma_t^1\supseteq\Gamma_t^2. \tag{47}$$

For $i=1,2$, let $Y^i\in\mathbf{D}^2_{\mathcal{F}}(0,T)$ be the $g^i_{\Gamma^i}$-solution with terminal condition $Y^i_T=X^i$. We then have

$$Y_t^1\le Y_t^2,\qquad \text{for } t\in[0,T], \text{a.s.}$$

Proof. Consider the penalization equations for the two constrained BSDEs: for $n\in\mathbf{N}$,

$$\begin{aligned} y_t^{1,n}&=X^1+\int_t^T g^{1,n}(s,y_s^{1,n},z_s^{1,n})\,\mathrm{d}s-\int_t^T z_s^{1,n}\,\mathrm{d}B_s,\\ y_t^{2,n}&=X^2+\int_t^T g^{2,n}(s,y_s^{2,n},z_s^{2,n})\,\mathrm{d}s-\int_t^T z_s^{2,n}\,\mathrm{d}B_s, \end{aligned} \tag{48}$$

where

$$\begin{aligned} g^{1,n}(t,y,z)&=g^1(t,y,z)+nd_{\Gamma_t^1}(y,z),\\ g^{2,n}(t,y,z)&=g^2(t,y,z)+nd_{\Gamma_t^2}(y,z). \end{aligned}$$

From (47), we have $g^{1,n}(t,y,z)\le g^{2,n}(t,y,z)$. It follows from the classical comparison theorem for BSDEs that $y_t^{1,n}\le y_t^{2,n}$. Since, as $n\to\infty$, $y_t^{1,n}\nearrow y_t^1$ and $y_t^{2,n}\nearrow y_t^2$, where y^1, y^2 are the respective g_Γ-solutions of the BSDEs, it follows that $y_t^1\le y_t^2$, $0\le t\le T$. □

The comparison theorem is a powerful tool and useful concept in BSDE theory (cf. [7]). Let us here recall the main theorem on reflected BSDEs and the related comparison theorem for the case of lower obstacle L. We do not repeat the case for the upper obstacle since it is essentially the same. This result, obtained in [17], is a generalized version of [6,8] and [14] for the existence part, and [9] for the comparison theorem part.

Theorem A.5 (Reflected BSDE and the related comparison theorem). *We assume that the coefficient g satisfies Lipschitz condition (1) and the lower obstacle L satisfies (6). Then, for each $X\in\mathbf{L}^2(\mathcal{F}_T)$ with $X\ge L_T$ there exists a unique triple $(y,z,A)\in\mathbf{D}^2_{\mathcal{F}}(0,T)\times\mathbf{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)\times\mathbf{A}^2_{\mathcal{F}}(0,T)$ such that*

$$y_t=X+\int_t^T g(s,y_s,z_s)\,\mathrm{d}s+A_T-A_t-\int_t^T z_s\,\mathrm{d}B_s$$

and the generalized Skorokhod reflecting condition is satisfied: for each $L^* \in \mathbf{D}^2_{\mathcal{F}}(0,T)$ *such that* $y_t \geq L^*_t \geq L_t$, $\mathrm{d}P \times \mathrm{d}t$*-a.s., we have*

$$\int_0^T (y_{s-} - L^*_{s-})\,\mathrm{d}A_s = 0 \qquad a.s.$$

Moreover, if a coefficient g', *an obstacle* L' *and terminal condition* X' *satisfy the same conditions as* g, L *and* X, *respectively,* $\forall(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, *then*

$$X' \leq X, \qquad g'(t,y,z) \leq g(t,y,z), \qquad L'_t \leq L_t, \qquad \mathrm{d}P \times \mathrm{d}t\text{-}a.s.$$

If the triple (y', z', A') *is the corresponding reflected solution, then we have*

$$Y'_t \leq Y_t, \qquad \forall t \in [0,T],\ a.s.$$

and if $L = L'$, *then for each* $0 \leq s \leq t \leq T$,

$$A'_t \geq A_t, \qquad A'_t - A'_s \geq A_t - A_s.$$

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