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KODAI MATH. J.
**41** (2018), 240–263

# CYCLIC COVERINGS OF THE PROJECTIVE LINE BY MUMFORD CURVES IN POSITIVE CHARACTERISTIC

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**Abstract**

We study the rigid analytic geometry of cyclic coverings of the projective line. We determine the defining equation of cyclic coverings of degree $p$ of the projective line by Mumford curves over complete discrete valuation fields of positive characteristic $p$. Previously, Bradley studied that of any degree over non-archimedean local fields of characteristic zero.

## 1. Introduction

A geometrically connected smooth projective curve of genus $\geq 2$ over a complete discrete valuation field $(K, |\cdot|)$ is called a *Mumford curve* if it is analytically isomorphic to a rigid analytic space of the form $(\mathbf{P}^1 \backslash \mathcal{L})/\Gamma$, where $\Gamma \subset \mathrm{PGL}_2(K)$ is a Schottky group and $\mathcal{L} \subset \mathbf{P}^1$ is the set of limit points. Recall that a finitely generated torsion-free discontinuous subgroup of $\mathrm{PGL}_2(K)$ is called a *Schottky group* if it has infinitely many limit points in $\mathbf{P}^1$. Mumford curves are algebraically characterized by the property that they have split degenerate reduction [7, Theorem 3.3, Theorem 4.20]. Cyclic coverings of $\mathbf{P}^1$ by Mumford curves were studied by Bradley and van Steen; see [2], [9]. When $K$ is a non-archimedean local field of characteristic zero, Bradley studied the defining equation of cyclic coverings of any degree of $\mathbf{P}^1$ by Mumford curves [2, Theorem 4.3].

In this paper, we focus on cyclic coverings of degree $p$ of $\mathbf{P}^1$ by Mumford curves in characteristic $p > 0$. Let

$$\varphi : X \to \mathbf{P}^1$$

be a cyclic covering of degree $p$ over $K$. Assume that $K$ is of characteristic $p > 0$. In [9, Proposition 3.1], van Steen showed that if $X$ is a Mumford curve, by replacing $K$ by its finite extension, it is defined by an equation of the form

$$y^p - y = \sum_{i=1}^{r} \frac{\lambda_i}{x - a_i} \tag{1.1}$$

2010 *Mathematics Subject Classification*. Primary 14H50; Secondary 11D41; Tertiary 14G22.
*Key words and phrases*. Mumford curve, cyclic covering, rigid analytic geometry.
Received January 30, 2017; revised May 10, 2017.

for some $\lambda_i \in K^\times$ and $a_i \in K$ $(1 \leq i \leq r)$ satisfying $a_i \neq a_j$ for $i \neq j$. In the following, we assume that $X$ is defined by the equation (1.1). The cyclic covering $X$ has genus $(p-1)(r-1)$ [9, Proposition 1.3]. Thus, we also assume $(p-1)(r-1) \geq 2$, i.e., $r \geq 3$, or $r = 2$ and $p \geq 3$. The main theorem of this paper is the following:

THEOREM 1.1. *Let $(K, |\cdot|)$ be a complete discrete valuation field of positive characteristic $p > 0$. Let $\varphi : X \to \mathbf{P}^1$ be a cyclic covering of degree $p$ over $K$ defined by the equation (1.1) for $r \geq 3$, or $r = 2$ and $p \geq 3$. Then the following conditions are equivalent*:

- *$X$ is a Mumford curve over a finite extension of $K$.*
- *$|\lambda_i \lambda_j| < |a_i - a_j|^2$ for any $i \neq j$.*

Previously, van Steen studied the defining equation of hyperelliptic curves which are Mumford curves [10]. When $p = r = 2$, the cyclic covering $X$ has genus 1 and van Steen obtained results similar to Theorem 1.1; see [10, Section 4]. Tsushima told the author that for any $p$, Theorem 1.1 for $r = 2$ can also be proved by computing reductions explicitly.

Note that for any cyclic covering $\varphi : X \to \mathbf{P}^1$ by a Mumford curve $X$ over $K$, there exists a surjective homomorphism from a discrete subgroup of $\mathrm{PGL}_2(K)$ generated by finitely many elements of finite order to the Galois group of $\varphi$; see [5, Chapter 8]. Since the order of any element of finite order of $\mathrm{PGL}_2(K)$ is not divisible by $p^2$, the degree $\deg \varphi$ is not divisible by $p^2$. When $p$ does not divide $\deg \varphi$, we can use Bradley's method in [2] to study the defining equation of $X$.

The organization of this paper is as follows. In Section 2, we review some basic properties of Mumford curves. In Section 3, we summarize some facts about cyclic coverings of $\mathbf{P}^1$ by Mumford curves proved by van Steen [9]. The proof of Theorem 1.1 is given in Section 4 and Section 5.

## 2. Basic properties of Mumford curves

In this paper, we use the language of rigid analytic geometry. We refer to [6] for basic notations on rigid analytic geometry and [5] for those on Mumford curves used in this paper.

Let $(K, |\cdot|)$ be a complete discrete valuation field of characteristic $p > 0$ and $K^\circ$ (resp. $k$) its valuation ring (resp. residue field). We fix a uniformizer $\pi \in K^\circ$. We fix an algebraic closure $\overline{K}$ of $K$. We also denote the extension of the valuation $|\cdot|$ on $K$ to $\overline{K}$ by the same symbol. We denote by $\mathrm{val}_K(\cdot)$ the normalized additive valuation on $K$, i.e., we have $\mathrm{val}_K(\pi) = 1$.

Let $A$ be an affinoid algebra over $K$. For an element $f \in A$, let

$$|f|_{\mathrm{sp}} := \sup\{|f(x)| \mid x \in \mathrm{Sp}\, A\}$$

be the *spectral seminorm* of $f$. (It is called the *supremum norm* in [6, Section 1.4].) We put

$$A^\circ := \{f \in A \mid |f|_{\mathrm{sp}} \le 1\},$$

$$A^{\circ\circ} := \{f \in A \mid |f|_{\mathrm{sp}} < 1\}.$$

We denote the residue ring of an affinoid algebra $A$ by $\bar{A} := A^\circ/A^{\circ\circ}$. The affine scheme $\operatorname{Spec} \bar{A}$ over $k$ is called the *canonical reduction* of the affinoid space $\operatorname{Sp} A$. We put $\overline{\operatorname{Sp} A} := \operatorname{Spec} \bar{A}$. (For details, see [6, Section 1.4].)

For an affinoid algebra $A$ over $K$, an algebra $B$ of topologically finite type over $K^\circ$ is called a *$K^\circ$-model* of $A$ if $B$ is flat over $K^\circ$ and $B \otimes_{K^\circ} K \cong A$; see [6, Definition 3.3.1].

A subgroup $N$ of $\mathrm{PGL}_2(K)$ is called *discontinuous* if the set of limit points of the canonical action of $N$ on $\mathbf{P}^1(K)$ does not equal to $\mathbf{P}^1(K)$ and the closure of $Na$ is compact for any $a \in \mathbf{P}^1(K)$. Obviously, a discontinuous subgroup is discrete. A finitely generated torsion-free discontinuous subgroup of $\mathrm{PGL}_2(K)$ is called a *Schottky group* if it has infinitely many limit points in $\mathbf{P}^1$. A Schottky group $\Gamma$ is a free group; see [5, Chapter 1]. We put

$$\Omega := \mathbf{P}^1 \backslash \{\text{the limit points of } \Gamma\},$$

which is a one-dimensional rigid analytic space over $K$. The quotient $\Omega/\Gamma$ is isomorphic to the analytification of a geometrically connected smooth projective curve $X_\Gamma$ of genus $\ge 2$ over $K$. A smooth projective curve of genus $\ge 2$ over $K$ which is isomorphic to $X_\Gamma$ for some Schottky group $\Gamma \subset \mathrm{PGL}_2(K)$ is called a *Mumford curve*. We identify projective curves over $K$ and their analytifications by the "GAGA"-correspondence. Concerning the automorphism group, we have a natural isomorphism

$$\mathrm{Aut}(X_\Gamma) \cong N_{\mathrm{PGL}_2(K)}(\Gamma)/\Gamma,$$

where $N_{\mathrm{PGL}_2(K)}(\Gamma)$ is the normalizer of $\Gamma$ in $\mathrm{PGL}_2(K)$; see [5, Chapter 7]. Mumford proved the following theorem:

Theorem 2.1 (Mumford [7, Theorem 3.3, Theorem 4.20]). *A geometrically connected smooth projective curve $X$ of genus $\ge 2$ over $K$ is a Mumford curve if and only if it has split degenerate reduction, i.e., there exists a proper flat scheme $Y$ over* $\operatorname{Spec} K^\circ$ *such that*

- $Y \times_{\operatorname{Spec} K^\circ} \operatorname{Spec} K \cong X$,
- *the normalizations of all the irreducible components of* $Y \times_{\operatorname{Spec} K^\circ} \operatorname{Spec} \bar{k}$ *are rational curves* (*where $\bar{k}$ is an algebraic closure of $k$*), *and*
- *all the singular points of the closed fiber* $Y \times_{\operatorname{Spec} K^\circ} \operatorname{Spec} k$ *are $k$-rational ordinary double points with two $k$-rational branches.*

We collect some properties of the Bruhat-Tits tree $\mathscr{T}$ of $\mathrm{PGL}_2(K)$ used in Section 5 of this paper; see [3, Section 2], [8, Chapter II] for details. The Bruhat-Tits tree $\mathscr{T}$ is a combinatorial graph defined as follows:

- The set of vertices $\mathrm{vert}(\mathscr{T})$ is the set of equivalence classes of $K^\circ$-lattices in $K \oplus K$. Here, two $K^\circ$-lattices $M_1$, $M_2$ are equivalent if $M_1 = aM_2$ for some $a \in K^\times$.

- Two vertices $w_1, w_2 \in \mathrm{vert}(\mathscr{T})$ are adjacent if and only if $\pi M_1 \subsetneq M_2 \subsetneq M_1$ for some $K^\circ$-lattices $M_1$ and $M_2$ in the equivalence classes $w_1$ and $w_2$, respectively.

The graph $\mathscr{T}$ is actually a tree [8, Chapter II, Theorem 1]. The set of edges of $\mathscr{T}$ is denoted by $\mathrm{edge}(\mathscr{T})$. A sequence $w_1, w_2, w_3 \dots$ of distinct vertices of $\mathscr{T}$ gives a *half-line* on $\mathscr{T}$ if $w_i$, $w_{i+1}$ are adjacent for any $i \geq 1$. Two half-lines given by $w_1, w_2, w_3 \dots$ and $w_1', w_2', w_3' \dots$ are equivalent if there exist $i, j \geq 1$ such that $w_{i+r} = w_{j+r}'$ for any $r \geq 0$. An equivalence class of half-lines on $\mathscr{T}$ is called an *end* of $\mathscr{T}$. There is a natural bijection between $\mathbf{P}^1(K)$ and the set of ends of $\mathscr{T}$ as follows. For an element $a \in \mathbf{P}^1(K)$, let $V_a \subset K \oplus K$ be a 1-dimensional $K$-subspace corresponding to $a$. Let $w_i \in \mathrm{vert}(\mathscr{T})$ be the equivalence class of $K^\circ$-lattices containing

$$\pi^i(K^\circ \oplus K^\circ) + V_a \cap (K^\circ \oplus K^\circ).$$

Then the sequence $w_1, w_2, w_3 \dots$ gives the end of $\mathscr{T}$ corresponding to $a$. This bijection is equivariant with respect to the action of $\mathrm{PGL}_2(K)$. See [8, Chapter II, p. 72] for details.

For $v, w \in \mathrm{vert}(\mathscr{T})$ and $a, b \in \mathbf{P}^1(K)$, let $[v, w]$ (resp. $[v, a[$, $]a, b[$) be the path from $v$ to $w$ without backtracking (resp. the half-line from $v$ to $a$, the line from $a$ to $b$), where we regard $a$, $b$ as ends of $\mathscr{T}$. For $v, w \in \mathrm{vert}(\mathscr{T})$, the length of the path $[v, w]$ is called the *distance* from $v$ to $w$, and is denoted by $\mathrm{dist}(v, w)$; see [8, Section 1.2]. For subtrees $\mathscr{R}, \mathscr{S} \subset \mathscr{T}$, we put

$$\mathrm{dist}(\mathscr{R}, \mathscr{S}) := \min_{\substack{v \in \mathrm{vert}(\mathscr{R}) \\ w \in \mathrm{vert}(\mathscr{S})}} \mathrm{dist}(v, w).$$

We denote by $v_1 \in \mathrm{vert}(\mathscr{T})$ the vertex corresponding to the equivalence class of $K^\circ$-lattices containing $K^\circ e_1 \oplus K^\circ e_2$, where $\{e_1, e_2\}$ is the standard basis of $K \oplus K$. For $a \in K^\times$ (resp. $w \in \mathrm{vert}(\mathscr{T})$), the intersection of $]0, \infty[$, $]0, a[$, and $]a, \infty[$ (resp. $]0, \infty[$, $[w, 0[$, and $[w, \infty[$) consists of one vertex only, and we denote it by $v(0, \infty, a)$ (resp. $v(0, \infty; w)$).

- If $\mathrm{val}_K(a) \geq 0$, we have $v(0, \infty, a) \in [v_1, 0[$ and $\mathrm{dist}(v_1, v(0, \infty, a)) = \mathrm{val}_K(a)$.
- If $\mathrm{val}_K(a) \leq 0$, we have $v(0, \infty, a) \in [v_1, \infty[$ and $\mathrm{dist}(v_1, v(0, \infty, a)) = -\mathrm{val}_K(a)$.

Since $v(0, \infty, a) = v(0, \infty; w)$ for any $w \in ]0, a[ \cap ]a, \infty[$, we can compute $\mathrm{val}_K(a)$ by using $v(0, \infty; w)$.

For any discrete subgroup $N \subset \mathrm{PGL}_2(K)$ and any $v \in \mathrm{vert}(\mathscr{T})$, the stabilizer

$$N_v := \{\gamma \in N \mid \gamma(v) = v\}$$

is a finite group. For an element $\gamma \in \mathrm{PGL}_2(K)$ of finite order, let $M(\gamma) \subset \mathscr{T}$ be the smallest subtree generated by the vertices fixed by $\gamma$. The subtree $M(\gamma)$ is called the *mirror* of $\gamma$; see [3, Section 2]. An element $\gamma \in \mathrm{PGL}_2(K)$ of order $p$ is called a *parabolic element*. A parabolic element $\gamma \in \mathrm{PGL}_2(K)$ has a unique fixed point in $\mathbf{P}^1$. For a parabolic element $\gamma \in \mathrm{PGL}_2(K)$ and $v \in \mathrm{vert}(M(\gamma))$, the subset

$$\{e \in \mathrm{edge}(M(\gamma)) \mid v \text{ is an extremity of } e\}$$

consists of one element only or coincides with

$$\{e \in \mathrm{edge}(\mathscr{T}) \mid v \text{ is an extremity of } e\}.$$

For any parabolic element $\gamma \in \mathrm{PGL}_2(K)$ and any $v \in \mathrm{vert}(M(\gamma))$, the element $\gamma$ acts freely on the following set:

$$\{e \in \mathrm{edge}(\mathscr{T}) \backslash \mathrm{edge}(M(\gamma)) \mid v \text{ is an extremity of } e\}.$$

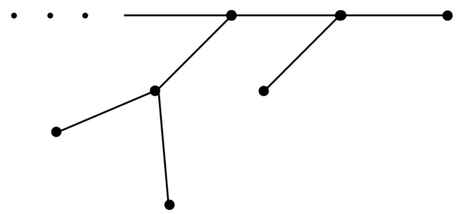

FIGURE. $M(\gamma)$ for a parabolic element $\gamma \in \mathrm{PGL}_2(K)$ when $k \cong \mathbf{F}_2$

## 3. Some facts about cyclic coverings of degree $p$ of the projective line by Mumford curves

In this section, we review some facts about cyclic coverings of degree $p$ of $\mathbf{P}^1$ proved by van Steen [9]. Let

$$\varphi : X \to \mathbf{P}^1$$

be a cyclic covering of degree $p$ over $K$. Let $a_1, a_2, \dots, a_r \in \mathbf{P}^1$ be the branch points of $\varphi$. We assume that $a_i \neq \infty$ for every $i$. By replacing $K$ by its finite extension, we may assume that $a_1, a_2, \dots, a_r$ are $K$-rational points on $\mathbf{P}^1$.

We denote the function field of $\mathbf{P}^1$ (resp. $X$) by $K(x)$ (resp. $F$). Since $F/K(x)$ is an Artin-Schreier extension, by replacing $K$ by its finite extension, there exists $y \in F$ such that $F = K(x, y)$ and

$$y^p - y = \sum_{i=1}^{r} \sum_{\substack{j=1 \\ j \not\equiv 0 \bmod p}}^{n_i} \frac{\lambda_{ij}}{(x - a_i)^j}$$

for some $\lambda_{ij} \in K^{\times}$. Using this equation, we embed $X$ into $\mathbf{P}^1 \times \mathbf{P}^1$. If $X$ is a Mumford curve, we have $n_i = 1$ for every $i$ [9, Proposition 3.1]. We assume that $n_i = 1$ for every $i$ and put $\lambda_i := \lambda_{i1}$. The cyclic covering $X$ has genus $(p-1)(r-1)$ [9, Proposition 1.3]. Thus, we also assume $(p-1)(r-1) \geq 2$, i.e.,

$r \geq 3$, or $r = 2$ and $p \geq 3$. Hence $X$ is defined by

$$y^p - y = \sum_{i=1}^{r} \frac{\lambda_i}{x - a_i}.$$

If $X$ is a Mumford curve, there exist $s_1, s_2, \ldots, s_r \in \mathrm{PGL}_2(K)$ satisfying the following conditions [9, Proposition 2.2, Section 3]:

- $s_i$ $(1 \leq i \leq r)$ is an element of order $p$,
- the subgroup $N \subset \mathrm{PGL}_2(K)$ generated by $s_i$ $(1 \leq i \leq r)$ is discontinuous and isomorphic to the free product of $\langle s_i \rangle$ $(1 \leq i \leq r)$, (this implies the subgroup $\Gamma \subset \mathrm{PGL}_2(K)$ generated by $s_i^n s_{i+1}^{-n}$ $(1 \leq i \leq r-1,\ 1 \leq n \leq p-1)$ is a Schottky group satisfying $N \subset N_{\mathrm{PGL}_2(K)}(\Gamma)$,)
- $X \cong \Omega/\Gamma$, $\mathbf{P}^1 \cong \Omega/N$, and the covering $\varphi : X \to \mathbf{P}^1$ coincides with the natural projection $\Omega/\Gamma \to \Omega/N$, where
$$\Omega := \mathbf{P}^1 \backslash \{\text{the limit points of } \Gamma\},$$
- the fixed point $P_i \in \mathbf{P}^1$ of $s_i$ is an element of $\Omega$,
- the image of $P_i$ under the natural projection $\Omega \to \Omega/N \cong \mathbf{P}^1$ is the branch point $a_i \in \mathbf{P}^1$,
- $s_i(y) = y + 1$ $(1 \leq i \leq r)$, where we consider $s_i$ as an element of $\mathrm{Aut}(X) \cong N_{\mathrm{PGL}_2(K)}(\Gamma)/\Gamma$.

In particular, we have

$$N/\Gamma \cong \mathrm{Gal}(F/K(x)) \cong \mathbf{Z}/p\mathbf{Z}.$$

We note that $M(s_i) \cap M(s_j) = \emptyset$ for any $i \neq j$. In fact, if there exists a vertex $v \in \mathrm{vert}(M(s_i) \cap M(s_j))$, it is fixed by infinitely many elements $s_{l_1}^{n_1} \cdots s_{l_m}^{n_m}$ for $m \geq 0$, $l_k \in \{i, j\}$, and $1 \leq n_k \leq p-1$ $(1 \leq k \leq m)$ with $l_k \neq l_{k+1}$ $(1 \leq k \leq m-1)$, but, since $N$ is discrete, the stabilizer $N_v$ is a finite group. The contradiction shows $M(s_i) \cap M(s_j) = \emptyset$.

## 4. Proof of Theorem 1.1 (part 1)

In this section, we shall show that if the inequality

$$|\lambda_i \lambda_j| < |a_i - a_j|^2 \tag{4.1}$$

is satisfied for any $i \neq j$, then $X$ is a Mumford curve over a finite extension of $K$.

By replacing $K$ by its finite extension, for each $i$, there exists $\varepsilon_i \in |K^\times|$ satisfying

$$\varepsilon_i < |a_i - a_j| \quad \text{and} \quad |\lambda_i| < \frac{|a_i - a_j|^2}{|\lambda_j|} - \varepsilon_i$$

for any $j \neq i$. By replacing $K$ by its finite extension, for $i$, $j$, and $k$ satisfying $|a_i - a_j| < |a_i - a_k|$, there exists $\zeta_{i,j,k} \in |K^\times|$ satisfying

$$|a_i - a_j| < \zeta_{i,j,k} < |a_i - a_k|.$$

For each $1 \le i \le r$, let $\alpha_{i,1} \le \alpha_{i,2} \le \cdots \le \alpha_{i,M_i-1}$ be the following elements

$$\varepsilon_i,\ |\lambda_i|,\ |a_i - a_j|,\ \frac{|a_i - a_j|^2}{|\lambda_j|},\ \frac{|a_i - a_j|^2}{|\lambda_j|} - \varepsilon_i\ \ (j \ne i),$$
$$\zeta_{i,j,k}\ \ (j, k \text{ satisfying } |a_i - a_j| < |a_i - a_k|)$$

of $|K^\times|$ arranged in ascending order. We put $\alpha_{i,0} = 0$ and $\alpha_{i,M_i} = \infty$ for each $1 \le i \le r$.

We define sets $I$ and $J$ by

$$I := \{n = (n_1, \dots, n_r) \in \mathbf{Z}^r \mid 0 \le n_i \le M_i - 1\ \ (1 \le i \le r)\},$$
$$J := \{n = (n_1, \dots, n_r) \in I \mid |a_i - a_j| \ne \alpha_{i,n_i+1} \text{ or } |a_i - a_j| \ne \alpha_{j,n_j+1} \text{ for any } i \ne j\}.$$

For each $n = (n_1, \dots, n_r) \in I$, we define an affinoid open subvariety $U_n \subset \mathbf{P}^1$ by

$$U_n := \{x \in \mathbf{P}^1 \mid \alpha_{i,n_i} \le |x - a_i| \le \alpha_{i,n_{i+1}} \text{ for any } 1 \le i \le r\}.$$

Lemma 4.1. *$\{U_n\}_{n \in J}$ is an affinoid covering of $\mathbf{P}^1$.*

*Proof.* Since $\{U_n\}_{n \in I}$ is an affinoid covering of $\mathbf{P}^1$, it suffices to show that, for any $n \in I \backslash J$ and any $c \in U_n$, there exists $n' \in J$ satisfying $c \in U_{n'}$.

For $n \in I$, we put

$$M_n := \#\{1 \le i \le r \mid |a_i - a_j| = \alpha_{i,n_i+1} \text{ for some } j \ne i\}.$$

We prove Lemma 4.1 by induction on $M_n$.

We fix $n \in I \backslash J$ and $c \in U_n$. Since $n \in I \backslash J$, there exist distinct elements $i$, $j$ satisfying $|a_i - a_j| = \alpha_{i,n_i+1}$ and $|a_i - a_j| = \alpha_{j,n_j+1}$. In particular, we have $M_n \ge 2$. We have

$$U_n \subset \{x \in \mathbf{P}^1 \mid |x - a_i| \le |a_i - a_j| \text{ and } |x - a_j| \le |a_i - a_j|\}.$$

Hence we have $|c - a_i| = |a_i - a_j|$ or $|c - a_j| = |a_i - a_j|$.

We may assume $|c - a_i| = |a_i - a_j|$. We put $n'_k := n_k$ for $k \ne i$ and $n'_i := n_i + l'$, where we put

$$l' := \min\{l \ge 1 \mid \alpha_{i,n_i+1} < \alpha_{i,n_i+l+1}\}.$$

We put $n' := (n'_1, \dots, n'_r) \in I$. Then we have $c \in U_{n'}$. We have

$$\alpha_{i,n_i+l'+1} \le \zeta_{i,j,k} < |a_i - a_k|$$

for $k$ satisfying $|a_i - a_j| < |a_i - a_k|$. Hence we have $\alpha_{i,n_i+l'+1} \ne |a_i - a_k|$ for $k \ne i$. We have $M_{n'} < M_n$. By induction on $M_n$, there exists $n' \in J$ satisfying $c \in U_{n'}$. $\square$

We put $J' := \{n \in J \mid U_n \ne \emptyset\}$. For each $n \in J'$, we put $\beta_n := \min_{1 \le i \le r} \alpha_{i,n_i+1}$.

For any $n \in J'$, we have

$$U_n = \{x \in \mathbf{P}^1 \mid |x - a_{l(n,0)}| \le \beta_n\} \setminus \bigcup_{v=0}^{N_n} D_{n,v},$$

where

$$D_{n,v} := \{x \in \mathbf{P}^1 \mid |x - a_{l(n,v)}| < \alpha_{l(n,v),n_{l(n,v)}}\}$$

for some $N_n \ge 0$ and $1 \le l(n,v) \le r$ $(0 \le v \le N_n)$ with $|a_{l(n,0)} - a_{l(n,v)}| \le \beta_n$ $(1 \le v \le N_n)$. We may assume $D_{n,v} \cap D_{n,v'} = \emptyset$ for $v \ne v'$. We take $l(n,0)$ so that $\alpha_{l(n,0),n_{l(n,0)}} = \min_{0 \le v \le N_n} \alpha_{l(n,v),n_{l(n,v)}}$.

LEMMA 4.2. *For each* $n \in J'$, *we have* $\alpha_{l(n,v),n_{l(n,v)}} = |a_{l(n,0)} - a_{l(n,v)}|$ *for* $1 \le v \le N_n$. *Moreover, for* $1 \le v \le N_n$, *we have* $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,0),n_{l(n,0)}}$ *or* $|a_{l(n,0)} - a_{l(n,v)}| = \beta_n$.

*Proof.* We fix $1 \le v \le N_n$. Since $a_{l(n,v)} \notin D_{n,0}$, we have $\alpha_{l(n,0),n_{l(n,0)}} \le |a_{l(n,0)} - a_{l(n,v)}|$. We have

$$|a_{l(n,0)} - a_{l(n,v)}| \le \beta_n \le \alpha_{l(n,0),n_{l(n,0)}+1}.$$

Hence we have $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,0),n_{l(n,0)}}$ or $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,0),n_{l(n,0)}+1}$. Similarly, we have $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,v),n_{l(n,v)}}$ or $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,v),n_{l(n,v)}+1}$.

We assume that $|a_{l(n,0)} - a_{l(n,v)}| \ne \alpha_{l(n,v),n_{l(n,v)}}$. Then we have $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,v),n_{l(n,v)}+1}$ and $\alpha_{l(n,v),n_{l(n,v)}} < \alpha_{l(n,v),n_{l(n,v)}+1}$. Since $\alpha_{l(n,0),n_{l(n,0)}} \le \alpha_{l(n,v),n_{l(n,v)}}$, we have $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,0),n_{l(n,0)}+1}$, which contradicts the definition of $J$. Hence we have $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,v),n_{l(n,v)}}$.

If $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,0),n_{l(n,0)}+1}$, since $|a_{l(n,0)} - a_{l(n,v)}| \le \beta_n$, we have $\alpha_{l(n,0),n_{l(n,0)}+1} \le \beta_n$. Hence we have $\alpha_{l(n,0),n_{l(n,0)}+1} = \beta_n$. Consequently, we have $|a_{l(n,0)} - a_{l(n,v)}| = \alpha_{l(n,0),n_{l(n,0)}}$ or $|a_{l(n,0)} - a_{l(n,v)}| = \beta_n$. □

One can easily show that the admissible affinoid covering $\{U_n\}_{n \in J'}$ of $\mathbf{P}^1$ is a *formal analytic covering* in the sense of [6, Definition 3.1.6]; see [4, Proposition 2.2.6]. We see that $\mathcal{O}(U_n)$ is reduced and $|\mathcal{O}(U_n)|_{\mathrm{sp}} = |K|$; see [4, Proposition 2.2.6]. Then $\mathcal{O}(U_n)^\circ$ is a $K^\circ$-model of $\mathcal{O}(U_n)$ by [1, Theorem 1 of Section 6.4.3]. Hence the formal analytic covering $\{U_n\}_{n \in J'}$ of $\mathbf{P}^1$ defines a proper admissible formal scheme covered by $\{\mathrm{Spf}(\mathcal{O}(U_n)^\circ)\}_{n \in J'}$ by [6, Theorem 3.3.12]. Hence it is algebraic by Grothendieck's existence theorem. Consequently, the canonical reductions $\{\overline{U_n}\}_{n \in J'}$ define an algebraic reduction of $\mathbf{P}^1$. The canonical reductions $\{\overline{\varphi^{-1}(U_n)}\}_{n \in J'}$ define an algebraic reduction of $X$ over a finite extension of $K$.

In order to show that $X$ is a Mumford curve over a finite extension of $K$, it is enough to prove that, for each $n \in J'$, the affinoid open subvariety $\varphi^{-1}(U_n) \subset X$ satisfies the following conditions over a finite extension of $K$:

Condition 4.3.

- *All the irreducible components of the canonical reduction* $\overline{\varphi^{-1}(U_n)}$ *are rational curves, and*
- *all the singular points of the canonical reduction* $\overline{\varphi^{-1}(U_n)}$ *are ordinary double points.*

We shall show that $\varphi^{-1}(U_n)$ satisfies Condition 4.3 by calculating the canonical reductions $\overline{\varphi^{-1}(U_n)}$ explicitly. We fix an element $n \in J'$. We put $N := N_n$ and $l := l(n,0)$. We also put $D_\nu := D_{n,\nu}$, $d_\nu := a_{l(n,\nu)}$ for each $0 \le \nu \le N$. We take $b_1 \in K$ and $b_2 \in K^\times \cup \{\infty\}$ satisfying $|b_1| = \alpha_{l,n_l}$ and $|b_2| = \beta_n$. Then we have

$$U_n = \{x \in \mathbf{P}^1 \mid |x - d_0| \le |b_2|\} \setminus \bigcup_{\nu=0}^{N_n} D_\nu,$$

$$D_0 = \{x \in \mathbf{P}^1 \mid |x - d_0| < |b_1|\},$$

$$D_\nu = \{x \in \mathbf{P}^1 \mid |x - d_\nu| < |d_0 - d_\nu|\} \quad (1 \le \nu \le N_n).$$

For each $1 \le \nu \le N_n$, we have $|d_0 - d_\nu| = |b_1|$ or $|d_0 - d_\nu| = |b_2|$.

We put $z_i := \lambda_i (x - a_i)^{-1}$ for each $1 \le i \le r$. From the equation (1.1), we have

$$y^p - y = \sum_{i=1}^{r} z_i.$$

We put

$$\Lambda := \{i \mid |x - a_i| \le |\lambda_i| \text{ for some } x \in U_n\} = \{i \mid |z_i|_{\mathrm{sp}} \ge 1 \text{ on } U_n\}.$$

Lemma 4.4. *If $\Lambda = \emptyset$, the affinoid open subvariety $\varphi^{-1}(U_n)$ satisfies Condition 4.3.*

*Proof.* Since $\Lambda = \emptyset$, we have $|y|_{\mathrm{sp}} \le 1$ on $\varphi^{-1}(U_n)$. We embed $X$ into $\mathbf{P}^1 \times \mathbf{P}^1$ by $x$ and $y$. We have

$$\varphi^{-1}(U_n) \subset \{(x,y) \in \mathbf{P}^1 \times \mathbf{P}^1 \mid x \in U_n,\ |y| \le 1\}.$$

Hence we have

$$\mathcal{O}(\varphi^{-1}(U_n)) \cong \mathcal{O}(U_n)[y] \Big/ \left( y^p - y - \sum_{i=1}^{r} z_i \right).$$

Since $\mathcal{O}(U_n)^\circ$ is a $K^\circ$-model of $\mathcal{O}(U_n)$, the $K^\circ$-algebra

$$\mathcal{O}(U_n)^\circ[y] \Big/ \left( y^p - y - \sum_{i=1}^{r} z_i \right)$$

is a $K^{\circ}$-model of $\mathcal{O}(\varphi^{-1}(U_n))$. Since $|z_i| < 1$ on $U_n$ for each $1 \le i \le r$, the residue ring $\overline{\mathcal{O}(\varphi^{-1}(U_n))}$ is isomorphic to

$$\overline{\mathcal{O}(U_n)}[y]/(y^p - y),$$

and $\varphi^{-1}(U_n)$ satisfies Condition 4.3. $\square$

In the rest of this section, we assume $\Lambda \neq \emptyset$.

Lemma 4.5. *If $a_i \in U_n$ for some $1 \le i \le r$, we have*

$$U_n = \{x \in \mathbf{P}^1 \mid |x - a_i| \le \alpha_{i,1}\}$$

*and $a_j \notin U_n$ for $j \neq i$.*

*Proof.* Since $a_i \in U_n$, we have $n_i = 0$. For $j \neq i$, since $U_n$ is non-empty, we have

$$\{x \in \mathbf{P}^1 \mid |x - a_i| \le \alpha_{i,1}\} \cap \{x \in \mathbf{P}^1 \mid \alpha_{j,n_j} \le |x - a_j| \le \alpha_{j,n_j+1}\} \neq \emptyset.$$

Since $\varepsilon_i < |a_i - a_j|$, we have $\alpha_{i,1} < |a_i - a_j|$. Hence we have

$$\begin{aligned}\{x \in \mathbf{P}^1 \mid |x - a_i| \le \alpha_{i,1}\} &\cap \{x \in \mathbf{P}^1 \mid \alpha_{j,n_j} \le |x - a_j| \le \alpha_{j,n_j+1}\} \\ &= \{x \in \mathbf{P}^1 \mid |x - a_i| \le \alpha_{i,1}\}.\end{aligned}$$

We have

$$U_n = \bigcap_{j=1}^{r} \{x \in \mathbf{P}^1 \mid \alpha_{j,n_j} \le |x - a_j| \le \alpha_{j,n_j+1}\} = \{x \in \mathbf{P}^1 \mid |x - a_i| \le \alpha_{i,1}\}.$$

For $j \neq i$, since $a_j \notin \{x \in \mathbf{P}^1 \mid |x - a_i| \le \alpha_{i,1}\}$, we have $a_j \notin U_n$. $\square$

By Lemma 4.5, if $a_i \in U_n$ for some $i$, we have $b_1 = 0$ and $|b_2| = \alpha_{i,1}$. For each $a \in \mathbf{P}^1$, we put

$$\operatorname{dist}(a, U_n) := \inf_{u \in U_n} |a - u|.$$

Lemma 4.6. *For $1 \le i \le r$, we have* $\operatorname{dist}(a_i, U_n) = |b_1|$ *or* $\operatorname{dist}(a_i, U_n) \ge |b_2|$.

*Proof.* For $1 \le i \le r$, by Lemma 4.2, we have $\operatorname{dist}(a_i, U_n) = 0$, $\operatorname{dist}(a_i, U_n) = |b_1|$, or $\operatorname{dist}(a_i, U_n) \ge |b_2|$. Moreover, if $\operatorname{dist}(a_i, U_n) = 0$ for some $i$, we have $a_i \in U_n$, hence $b_1 = 0$. $\square$

Lemma 4.7. *If* $\operatorname{dist}(a_i, U_n) > |b_1|$ *for some $i$, we have $|x - a_i| = |d_0 - a_i|$ for every $x \in U_n$. In particular, $|x - a_i| = \operatorname{dist}(a_i, U_n)$ for every $x \in U_n$.*

*Proof.* By Lemma 4.6, we have $\operatorname{dist}(a_i, U_n) \ge |b_2|$. First, we assume $\operatorname{dist}(a_i, U_n) = |b_2|$. Then, by Lemma 4.2, we have $a_i \in D_{\nu(i)}$ for some $\nu(i)$ with

$|d_0 - d_{\nu(i)}| = |b_2|$. Since $|a_i - d_{\nu(i)}| < |b_2|$, we have $|d_0 - a_i| = |b_2|$. We have

$$\begin{aligned} U_n &\subset \{x \in \mathbf{P}^1 \mid |x - d_0| \leq |b_2| \text{ and } |x - d_{\nu(i)}| \geq |b_2|\} \\ &= \{x \in \mathbf{P}^1 \mid |x - a_i| = |b_2|\} \\ &= \{x \in \mathbf{P}^1 \mid |x - a_i| = |d_0 - a_i|\}. \end{aligned}$$

Next, we assume $\mathrm{dist}(a_i, U_n) > |b_2|$. Then, for any $x \in U_n$, we have

$$|x - d_0| \leq |b_2| < \mathrm{dist}(a_i, U_n) \leq |x - a_i|.$$

Hence we have $|x - a_i| = |d_0 - a_i|$ for any $x \in U_n$. □

LEMMA 4.8. *We have* $\mathrm{dist}(a_i, U_n) \neq \mathrm{dist}(a_j, U_n)$ *for any* $i, j \in \Lambda$ *with* $i \neq j$.

*Proof.* Assume that we have $\mathrm{dist}(a_i, U_n) = \mathrm{dist}(a_j, U_n)$ for some $i, j \in \Lambda$ with $i \neq j$. We put $d := \mathrm{dist}(a_i, U_n) = \mathrm{dist}(a_j, U_n)$. Then we have $d \leq |\lambda_i|$ and $d \leq |\lambda_j|$. By Lemma 4.6, we have $d = |b_1|$ or $d \geq |b_2|$. By Lemma 4.5, we have $d > 0$.

First, we assume $d = |b_1|$. By Lemma 4.2, we have $a_i \in D_{\nu(i)}$ for some $\nu(i)$ with $|d_0 - d_{\nu(i)}| = |b_1|$. Since $|a_i - d_{\nu(i)}| < |b_1|$, we have $|a_i - d_0| = |b_1|$. Similarly, we have $|a_j - d_0| = |b_1|$. Hence we have

$$|a_i - a_j| \leq \max\{|d_0 - a_i|, |d_0 - a_j|\} = |b_1| = d \leq \min\{|\lambda_i|, |\lambda_j|\},$$

which contradicts the inequality (4.1).

Next, we assume $d > |b_1|$. By Lemma 4.7, we have $|x - a_i| = d = |x - a_j|$ for any $x \in U_n$. Hence we have

$$|a_i - a_j| \leq \max\{|x - a_i|, |x - a_j|\} = d \leq \min\{|\lambda_i|, |\lambda_j|\}$$

for any $x \in U_n$, which contradicts the inequality (4.1). □

By Lemma 4.8, there exists a unique element $m \in \Lambda$ satisfying

$$\mathrm{dist}(a_m, U_n) = \min_{i \in \Lambda} \mathrm{dist}(a_i, U_n).$$

LEMMA 4.9. *For any* $i \in \Lambda \backslash \{m\}$, *we have*

$$\left| \frac{\lambda_i}{x - a_i} + \frac{\lambda_i}{a_i - a_m} \right| < 1$$

*for every* $x \in U_n$.

*Proof.* Since $\mathrm{dist}(a_m, U_n) < \mathrm{dist}(a_i, U_n)$, by Lemma 4.6, we have $\mathrm{dist}(a_i, U_n) > |b_1|$. By Lemma 4.7, we have $|x - a_i| = \mathrm{dist}(a_i, U_n)$ for every $x \in U_n$. For

$x \in U_n$ satisfying $|x - a_m| = \operatorname{dist}(a_m, U_n)$, we have $|x - a_m| < |x - a_i|$. Hence we have $|x - a_i| = |a_m - a_i|$ for every $x \in U_n$.

Since $m \in \Lambda$, we have $\alpha_{m,n_m} \le |\lambda_m|$. Since $|\lambda_m| < |a_m - a_i|^2 \cdot |\lambda_i|^{-1} - \varepsilon_i$, we have $\alpha_{m,n_m+1} \le |a_m - a_i|^2 \cdot |\lambda_i|^{-1} - \varepsilon_i$. Hence we have $|x - a_m| \le |a_m - a_i|^2 \cdot |\lambda_i|^{-1} - \varepsilon_i$ for every $x \in U_n$.

Consequently, for every $x \in U_n$, we have

$$\begin{aligned}\left|\frac{\lambda_i}{x - a_i} + \frac{\lambda_i}{a_i - a_m}\right| &= \frac{|\lambda_i| \cdot |x - a_m|}{|x - a_i| \cdot |a_i - a_m|} \\ &= \frac{|\lambda_i|}{|a_m - a_i|^2}\left(\frac{|a_m - a_i|^2}{|\lambda_i|} - \varepsilon_i\right) \\ &< 1. \end{aligned}$$ □

We put

$$\begin{aligned} f &:= \sum_{i \in \Lambda \setminus \{m\}} \left(z_i + \frac{\lambda_i}{a_i - a_m}\right) + \sum_{i \notin \Lambda} z_i, \\ C &:= -\sum_{i \in \Lambda \setminus \{m\}} \frac{\lambda_i}{a_i - a_m}. \end{aligned}$$

Then we have

$$y^p - y = \sum_{i=1}^{r} z_i = z_m + C + f.$$

By Lemma 4.9, we have $|f|_{\mathrm{sp}} < 1$ on $U_n$.

Lemma 4.10. *There exist $b_1' \in K^\times$, $b_2' \in K^\times \cup \{\infty\}$, and $C' \in K$ satisfying $|b_1'| \le |b_2'| \le 1$ or $1 \le |b_1'| \le |b_2'|$, and*

$$U_n = \{x \in \mathbf{P}^1 \mid |b_1'| \le |z_m + C'| \le |b_2'|\} \setminus \bigcup_{\nu=1}^{N} D_\nu',$$

*where*

$$D_\nu' = \{x \in \mathbf{P}^1 \mid |z_m + C' - d_\nu'| < |d_\nu'|\},$$

*for some $d_\nu' \in K^\times$ with $|d_\nu'| = |b_1'|$ or $|d_\nu'| = |b_2'|$.*

*Proof.* If $\operatorname{dist}(a_m, U_n) = |b_1|$, we put $b_1' := \lambda_m b_2^{-1}$, $b_2' := \lambda_m b_1^{-1}$, and $C' := 0$. (We put $b_2' := \infty$ if $b_1 = 0$.) If $\operatorname{dist}(a_m, U_n) \ne |b_1|$, we put $b_1' := \lambda_m b_1 (a_m - d_0)^{-2}$, $b_2' := \lambda_m b_2 (a_m - d_0)^{-2}$, and $C' := \lambda_m (a_m - d_0)^{-1}$. In both cases, we can check $b_1'$, $b_2'$, and $C'$ satisfy the conditions of Lemma 4.10. Since the computations are straightforward, we omit them. □

We put $z := z_m + C'$. We regard it as a coordinate function on $\mathbf{P}^1$. By replacing $K$ by its finite extension, there exists $C'' \in K$ satisfying $C''^p - C'' = C - C'$. We put $y' := y - C''$. Then we have $y'^p - y' = z + f$.

If $b_2' \neq \infty$, since

$$\mathcal{O}(\{z \in \mathbf{P}^1 \mid |b_1'| \leq |z| \leq |b_2'|\}) \cong K\langle b_2'^{-1}z, b_1'z^{-1}\rangle,$$

the residue ring $\overline{\mathcal{O}(U_n)}$ is isomorphic to a localization of $k[s,t]/(st - \overline{b_1'b_2'^{-1}})$.

If $b_2' = \infty$, since

$$\mathcal{O}(\{z \in \mathbf{P}^1 \mid |b_1'| \leq |z|\}) \cong K\langle b_1'z^{-1}\rangle,$$

the residue ring $\overline{\mathcal{O}(U_n)}$ is isomorphic to a localization of $k[t]$.

We consider the following two cases separately:

- $|b_1'| \leq |b_2'| \leq 1$.
- $1 \leq |b_1'| \leq |b_2'|$.

Lemma 4.11. *If $|b_1'| \leq |b_2'| \leq 1$, the affinoid open subvariety $\varphi^{-1}(U_n)$ satisfies Condition* 4.3 *over a finite extension of $K$.*

*Proof.* Since $|b_2'| \leq 1$, we have $|z|_{\mathrm{sp}} \leq 1$ on $U_n$. Similarly to the proof of Lemma 4.4, the residue ring $\overline{\mathcal{O}(\varphi^{-1}(U_n))}$ is isomorphic to a localization of

$$k[s,t,y']/(st - \overline{b_1'b_2'^{-1}}, y'^p - y' - \overline{b_2'}s)$$

If $|b_2'| < 1$, we have $\overline{b_2'} = 0$, and $\varphi^{-1}(U_n)$ satisfies Condition 4.3.

If $|b_2'| = 1$, we have

$$k[s,t,y']/(st - \overline{b_1'b_2'^{-1}}, y'^p - y' - \overline{b_2'}s) \cong k[t,y']/(t(y'^p - y') - \overline{b_1'}),$$

and $\varphi^{-1}(U_n)$ satisfies Condition 4.3. $\square$

Lemma 4.12. *If $1 \leq |b_1'| \leq |b_2'|$, the affinoid open subvariety $\varphi^{-1}(U_n)$ satisfies Condition* 4.3 *over a finite extension of $K$.*

*Proof.* Since $1 \leq |b_1'|$, we have $1 \leq |z|$ on $U_n$. By replacing $K$ by its finite extension, there exists $\xi, \xi' \in K$ such that $\xi^p = b_2'^{-1}$ and $\xi'^p = b_1'^{-1}$. (Here, we put $b_2'^{-1} := 0$ if $b_2' = \infty$.) We put $f' := b_2'^{-1}f$ and $f'' := z^{-1}f$. Since $1 \leq |b_2'|$, we have $|f'|_{\mathrm{sp}} < 1$ on $U_n$. Since $1 \leq |z|$ on $U_n$, we have $|f''|_{\mathrm{sp}} < 1$ on $U_n$. We also put $y'' := \xi y'$ and $w := \xi'^{-1}y'^{-1}$. Then we have

$$y''^p - \xi^{p-1}y'' = b_2'^{-1}z + f'$$

$$b_1'z^{-1}(1 - \xi'^{p-1}w^{p-1}) = w^p(1 + f'').$$

First, we assume $b_2' \neq \infty$. Similarly to the proof of Lemma 4.4, the residue ring $\overline{\mathcal{O}(\varphi^{-1}(U_n))}$ is isomorphic to a localization of

$$k[s,t,y'',w]/(st - \overline{b_1'b_2'^{-1}}, y''w - \overline{\xi\xi'^{-1}}, y''^p - \bar{\xi}^{p-1}y'' - s, t(1 - \overline{\xi'}^{p-1}w^{p-1}) - w^p),$$

which is a localization of $k[y'', w]/(y''w - \overline{\xi\xi'^{-1}})$, and $\varphi^{-1}(U_n)$ satisfies Condition 4.3.

Next, we assume $b_2' = \infty$. Similarly to the proof of Lemma 4.4, the residue ring $\overline{\mathcal{O}(\varphi^{-1}(U_n))}$ is isomorphic to a localization of

$$k[t, w]/(t(1 - \overline{\xi'}^{p-1} w^{p-1}) - w^p),$$

which is a localization of $k[w]$, and $\varphi^{-1}(U_n)$ satisfies Condition 4.3. $\square$

Consequently, $X$ is a Mumford curve over a finite extension of $K$.

## 5. Proof of Theorem 1.1 (part 2)

In this section, we shall show that if $X$ is a Mumford curve, the inequality $|\lambda_i \lambda_j| < |a_i - a_j|^2$ is satisfied for any $i \neq j$. Since the assertion is symmetric, we need only to prove the inequality

$$|\lambda_1 \lambda_2| < |a_1 - a_2|^2.$$

We use van Steen's method in [10, Section 3] and the Bruhat-Tits tree $\mathscr{T}$ of $\mathrm{PGL}_2(K)$.

Take $s_1, s_2, \dots, s_r \in \mathrm{PGL}_2(K)$ as in Section 3 of this paper. By replacing $K$ by its finite extension, we may assume that all the fixed points of $N$ on $\Omega$ are $K$-rational points.

Let $M \subset \mathscr{T}$ be the subtree generated by $M(s_i)$ $(1 \leq i \leq r)$. For each $i \neq j$, since $M(s_i) \cap M(s_j) = \emptyset$, there exist unique vertices $\xi_i(j) \in \mathrm{vert}(M(s_i))$ and $\xi_j(i) \in \mathrm{vert}(M(s_j))$ satisfying

$$\mathrm{dist}(M(s_i), M(s_j)) = \mathrm{dist}(\xi_i(j), \xi_j(i)).$$

For each $i \neq j$, let $e_i(j) \in \mathrm{edge}([\xi_i(j), \xi_j(i)])$ be the edge such that $\xi_i(j)$ is an extremity of $e_i(j)$.

Lemma 5.1. *There exist $s_i' \in N$ $(1 \leq i \leq r)$ satisfying the following conditions*:
- *For each $i$, the element $s_i'$ is $N$-conjugate to $s_i$. (This implies $s_i'$ is an element of order $p$ with $s_i'(y) = y + 1$.)*
- *$N$ is the free product of $\langle s_i' \rangle$ $(1 \leq i \leq r)$. (This implies $\Gamma$ is generated by $s_i'^n s_{i+1}'^{-n}$ $(1 \leq i \leq r-1,\ 1 \leq n \leq p-1)$.)*
- *We have $e \neq s_i'^n(e')$ for any $1 \leq i \leq r$, $0 \leq n \leq p-1$, and distinct edges $e, e' \in \mathrm{edge}(M')$, where $M' \subset \mathscr{T}$ is the subtree generated by $M(s_i')$ $(1 \leq i \leq r)$.*

*Proof.* We prove Lemma 5.1 by induction on

$$\sum_{1 \leq i, j \leq r} \mathrm{dist}(M(s_i), M(s_j)).$$

Since $M(s_i) \cap M(s_j) = \emptyset$ for $i \neq j$, we have

$$\sum_{1 \le i, j \le r} \operatorname{dist}(M(s_i), M(s_j)) \geq r(r-1).$$

We assume $e = s_m^n(e')$ for some distinct elements $e, e' \in \operatorname{edge}(M)$, $1 \le m \le r$, and $1 \le n \le p-1$. We fix $v_m \in \operatorname{vert}(M(s_m))$. There exists an extremity $v'$ of $e'$ with $v' \notin \operatorname{vert}(M(s_m))$. The vertex $s_m^n(v')$ is an extremity of $e$. There exist $k, l \in \{1, \dots, r\} \backslash \{m\}$ satisfying $e_m(k) \in \operatorname{edge}([v_m, s_m^n(v')])$ and $e_m(l) \in \operatorname{edge}([v_m, v'])$. We have $e_m(k) = s_m^n(e_m(l))$. In particular, we have $\xi_m(k) = \xi_m(l)$.

For $i, j \in \{1, \dots, r\} \backslash \{m\}$ with $e_m(i) \neq e_m(j)$, we have

$$\operatorname{edge}([\xi_i(m), \xi_m(i)] \cap [\xi_m(j), \xi_j(m)]) = \emptyset,$$

hence we have

$$[\xi_i(m), \xi_j(m)] = [\xi_i(m), \xi_m(i)] \cup [\xi_m(i), \xi_m(j)] \cup [\xi_m(j), \xi_j(m)].$$

In particular, we have $[\xi_i(m), \xi_j(m)] \cap M(s_m) \neq \emptyset$. Hence, for $i, j \in \{1, \dots, r\} \backslash \{m\}$ with $e_m(i) \neq e_m(j)$, we have $\xi_i(j) = \xi_i(m)$, $\xi_j(i) = \xi_j(m)$, and

$$\begin{aligned}\operatorname{dist}(M(s_i), M(s_j)) &= \operatorname{dist}(\xi_i(j), \xi_j(i)) \\ &= \operatorname{dist}(\xi_i(m), \xi_m(i)) + \operatorname{dist}(\xi_m(i), \xi_j(m)).\end{aligned}$$

For $i \neq m$, we put

$$I_i := \{1 \le j \le r \mid j \neq m \text{ and } e_m(j) = e_m(i)\}.$$

Then, for each $i \in I_k$ and $j \in I_l$, we have

$$e_m(i) = e_m(k) = s_m^n(e_m(l)) = s_m^n(e_m(j)) \in \operatorname{edge}([\xi_m(j), s_m^n(\xi_j(m))])$$

and $\xi_m(i) = \xi_m(j)$. Hence we have

$$e_m(i) \in \operatorname{edge}([\xi_i(m), \xi_m(i)]) \cap \operatorname{edge}([\xi_m(i), s_m^n(\xi_j(m))]).$$

For each $i \in I_l$, we put $s_i' := s_m^n s_i s_m^{-n}$. For each $i \notin I_l$, we put $s_i' := s_i$. Then the discrete subgroup $N \subset \mathrm{PGL}_2(K)$ is the free product of $\langle s_i' \rangle$ $(1 \le i \le r)$. We have $M(s_i') = s_m^n M(s_i)$ for $i \in I_l$.

We shall show

$$\sum_{1 \le i, j \le r} \operatorname{dist}(M(s_i'), M(s_j')) < \sum_{1 \le i, j \le r} \operatorname{dist}(M(s_i), M(s_j)).$$

To prove the above inequality, we estimate $\operatorname{dist}(M(s_i'), M(s_j'))$ for each $i$, $j$.

- For $i \in I_k$ and $j \in I_l$, we have $e_m(i) \neq e_m(j)$. We have

$$\begin{aligned}\operatorname{dist}(M(s_i'), M(s_j')) &= \operatorname{dist}(M(s_i), s_m^n M(s_j)) \\ &\le \operatorname{dist}(\xi_i(m), s_m^n \xi_j(m)) \\ &< \operatorname{dist}(\xi_i(m), \xi_m(i)) + \operatorname{dist}(\xi_m(i), s_m^n \xi_j(m))\end{aligned}$$

$$= \mathrm{dist}(\xi_i(m), \xi_m(i)) + \mathrm{dist}(\xi_m(i), \xi_j(m))$$
$$= \mathrm{dist}(M(s_i), M(s_j)).$$

- For $i, j \in I_l$, we have

$$\mathrm{dist}(M(s_i'), M(s_j')) = \mathrm{dist}(s_m^n M(s_i), s_m^n M(s_j)) = \mathrm{dist}(M(s_i), M(s_j)).$$

- For $i = m$ and $j \in I_l$, we have

$$\mathrm{dist}(M(s_m'), M(s_j')) = \mathrm{dist}(M(s_m), s_m^n M(s_j)) = \mathrm{dist}(M(s_m), M(s_j)).$$

- For $i \notin I_k \cup I_l \cup \{m\}$ and $j \in I_l$, since $e_m(i) \neq e_m(j)$ and $e_m(i) \neq e_m(k) = s_m^n e_m(j)$, we have

$$\begin{aligned} \mathrm{dist}(M(s_i'), M(s_j')) &= \mathrm{dist}(M(s_i), s_m^n M(s_j)) \\ &= \mathrm{dist}(\xi_i(m), \xi_m(i)) + \mathrm{dist}(\xi_m(i), s_m^n \xi_j(m)) \\ &= \mathrm{dist}(\xi_i(m), \xi_m(i)) + \mathrm{dist}(\xi_m(i), \xi_j(m)) \\ &= \mathrm{dist}(M(s_i), M(s_j)). \end{aligned}$$

- For $i, j \notin I_l$, since $s_i' = s_i$ and $s_j' = s_j$, we have

$$\mathrm{dist}(M(s_i'), M(s_j')) = \mathrm{dist}(M(s_i), M(s_j)).$$

Consequently, we have

$$\sum_{1 \le i, j \le r} \mathrm{dist}(M(s_i'), M(s_j')) < \sum_{1 \le i, j \le r} \mathrm{dist}(M(s_i), M(s_j)).$$

By induction, there exist $s_i' \in N$ $(1 \le i \le r)$ satisfying the conditions of Lemma 5.1. $\square$

We replace $s_i$ by $s_i'$ for every $1 \le i \le r$. Then we have $e \neq s_i^n(e')$ for any $1 \le i \le r$, $0 \le n \le p-1$, and any distinct elements $e, e' \in M$.

Recall that we put $v_1 := v(0, \infty, 1)$, and $P_i \in \Omega$ is the fixed point of $s_i$ for $1 \le i \le r$. By replacing $K$ by its finite extension and changing the coordinate of $\Omega \subset \mathbf{P}^1$, we may assume that the following conditions are satisfied:

- $P_1 = 0$ and $P_2 \neq \infty$.
- $|P_i| < |P_2|$ for any $i \neq 2$.
- The element $s_1 \in \mathrm{PGL}_2(K)$ is written as

$$s_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then we have $M(s_1) \cap ]0, \infty[ = [v_1, 0[$.

Since $s_2 \in \mathrm{PGL}_2(K)$ is an element of order $p$ fixing $P_2 \in \mathbf{P}^1(K) \backslash \{0, \infty\} = K^\times$, it is written as

$$s_2 = \begin{pmatrix} P_2(P_2 - \eta) & \eta P_2^2 \\ -\eta & P_2(P_2 + \eta) \end{pmatrix}$$

for some $\eta \in K^\times$.

LEMMA 5.2. *We have*

$$\mathrm{val}_K(\eta) = -\mathrm{dist}(M(s_1), M(s_2)) < 0.$$

*In particular, we have* $|\eta| > 1$.

*Proof.* Let

$$\gamma := \begin{pmatrix} P_2 & 0 \\ -1 & P_2 \end{pmatrix}.$$

Then we have

$$\gamma s_1 \gamma^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\gamma s_2 \gamma^{-1} = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{PGL}_2(K)$. We have $M(\gamma s_1 \gamma^{-1}) \cap ]0, \infty[ = [v_1, 0[$ and $M(\gamma s_2 \gamma^{-1}) \cap ]0, \infty[ = [v(0, \infty, \eta), \infty[$. Since $M(s_1) \cap M(s_2) = \emptyset$ and $M(\gamma s_i \gamma^{-1}) = \gamma M(s_i)$ $(i = 1, 2)$, we have $M(\gamma s_1 \gamma^{-1}) \cap M(\gamma s_2 \gamma^{-1}) = \emptyset$. Hence we have $|\eta| > 1$ and

$$\mathrm{val}_K(\eta) = -\mathrm{dist}(M(\gamma s_1 \gamma^{-1}), M(\gamma s_2 \gamma^{-1})) = -\mathrm{dist}(M(s_1), M(s_2)) < 0. \qquad \square$$

Since $\mathrm{val}_K(\eta) = -\mathrm{dist}(M(s_1), M(s_2))$ is invariant under $\mathrm{PGL}_2(K)$-conjugation, we may also assume $|\eta| < |P_2|$. Since $M(s_1) \cap ]0, \infty[ = [v_1, 0[$, we have $\xi_1(2) = v_1$, $\xi_2(1) = v(0, \infty, \eta)$, and

$$[v(0, \infty, \pi P_2), v(0, \infty, P_2)] \subset M(s_2).$$

LEMMA 5.3. *For any* $i \neq j$, *we have*

$$v(0, \infty; \xi_i(j)) \in \mathrm{vert}([v(0, \infty, \pi P_2), 0[).$$

*Proof.* For $i = 2$ and $j \neq 2$, since $|P_j| < |P_2|$ and

$$[v(0, \infty, \pi P_2), v(0, \infty, P_2)] \subset M(s_2),$$

we have

$$v(0, \infty; \xi_2(j)) \in \mathrm{vert}([v(0, \infty, \pi P_2), 0[).$$

For $i \neq 2$, since $|P_i| < |P_2|$ and $v(0, \infty, P_2) \in \mathrm{vert}(M(s_2))$, we have

$$v(0, \infty; w) \in \mathrm{vert}([v(0, \infty, \pi P_2), 0[)$$

for $w \in \mathrm{vert}(M(s_i))$. In particular, for $i \neq 2$ and $j \neq i$, we have

$$v(0, \infty; \xi_i(j)) \in \mathrm{vert}([v(0, \infty, \pi P_2), 0[). \qquad \square$$

By replacing $K$ by its finite extension, there exists a $K$-rational point $u \in \Omega$ such that $|u| = |u - P_2| = |P_2|$.

Lemma 5.4. *The following are satisfied*:
(1) *For* $1 \leq n \leq p-1$, *we have* $|s_2^n(P_1)| = |\eta|$.
(2) *For* $1 \leq n \leq p-1$, *we have* $|s_2^n(u)| = |P_2|$.
(3) *For* $1 \leq n \leq p-1$, *we have* $|s_1^n(u)| = 1$.
(4) *For any* $\gamma \in N$ *and* $i \neq 2$, *we have* $|\gamma(P_i)| < |P_2|$.
(5) *For any* $\gamma \in N$, *we have* $|\gamma(u) - P_2| = |P_2|$.
(6) *For any* $\gamma \in N$, *we have* $|\gamma(P_1)| \leq |\gamma(u)|$.

*Proof.* Since the path $[v(0, \infty, \pi P_2), v(0, \infty, P_2)]$ is contained in $M(s_2)$, every edge $e \in \mathrm{edge}(\mathscr{T})$ such that $v(0, \infty, P_2)$ is an extremity of $e$ is an edge of $M(s_2)$. For any $Q \in K^\times$ with $|Q| = |P_2|$ (i.e., $v(0, \infty, Q) = v(0, \infty, P_2)$), we have

$$\mathrm{edge}([v(0, \infty, Q), Q[ \cap M(s_2)) \neq \emptyset.$$

Hence, for $1 \leq n \leq p-1$, we have $v(0, \infty, s_2^n(Q)) = v(0, \infty, Q)$, i.e., $|s_2^n(Q)| = |P_2|$. In particular, we have

$$|s_2^n(u)| = |P_2| \quad \text{and} \quad |s_2^n(u) - P_2| = |P_2|.$$

The equality (2) is satisfied.

For any $Q \in K \backslash \{P_1, \ldots, P_r\}$, the intersection

$$M \cap \bigcap_{w \in M} [w, Q[$$

consists of one vertex only, and we denote it by $\xi(Q)$. Since the half-line $[v(0, \infty, P_2), 0[$ is contained in $M$, if

$$v(0, \infty; \xi(Q)) \in \mathrm{vert}([v(0, \infty, \pi P_2), 0[),$$

we have

$$v(0, \infty; \xi(Q)) = v(0, \infty, Q) \in \mathrm{vert}([v(0, \infty, \pi P_2), 0[),$$

hence $|Q| \leq |\pi P_2| < |P_2|$. In particular, by Lemma 5.3, for $Q \in K \backslash \{P_1, \ldots, P_r\}$ with $\xi(Q) = \xi_i(j)$ for some distinct elements $i$, $j$, we have $|Q| < |P_2|$.

For each $i$, we put

$$A_i := \{Q \in K \backslash \{P_1, \ldots, P_r\} \mid \xi(Q) \in \mathrm{vert}(M(s_i))\} \cup \{P_i\}.$$

We have $s_2^n(u) \in A_2$ since $|s_2^n(u)| = |P_2|$ $(0 \leq n \leq p-1)$, $|P_i| < |P_2|$ for $i \neq 2$, and $v(0, \infty, P_2) \in \mathrm{vert}(M(s_2))$.

For $i \neq j$, $Q \in A_j$, and $1 \leq n \leq p-1$, we have $e_i(j) \in \mathrm{edge}([\xi_i(j), Q[)$ and $s_i^n(e_i(j)) \notin \mathrm{edge}(M)$ by Lemma 5.1. Hence $M \cap [\xi_i(j), s_i^n(Q)[$ consists of $\xi_i(j)$ only. Hence we have $\xi(s_i^n(Q)) = \xi_i(j) \in \mathrm{vert}(M(i))$. In particular, we have $s_i^n(Q) \in A_i$.

Since $u \in A_2$ and $\xi_1(2) = v_1$, we have $|s_1^n(u)| = 1$ $(1 \leq n \leq p-1)$. The equality (3) is satisfied.

Since $P_1 \in A_1$ and $\xi_2(1) = v(0, \infty, \eta)$, we have $|s_2^n(P_1)| = |\eta|$ $(1 \leq n \leq p-1)$. The equality (1) is satisfied.

For an element

$$\gamma = s_{i_1}^{n_1} \cdots s_{i_m}^{n_m} \in N \quad (m \geq 2,\ 1 \leq n_l \leq p-1\ (1 \leq l \leq m),\ i_l \neq i_{l+1}\ (1 \leq l \leq m-1)),$$

by the above computations, we have $\xi(\gamma(P_i)) = \xi_{i_1}(i_2)$ and $\xi(\gamma(u)) = \xi_{i_1}(i_2)$. Hence we have $|\gamma(P_i)| < |P_2|$ for $i \neq 2$, $|\gamma(P_1)| = |\gamma(u)|$, and $|\gamma(u)| < |P_2|$. In particular, we have $|\gamma(u) - P_2| = |P_2|$. Hence (4), (5), and (6) are satisfied for this $\gamma$.

For $i \neq 2$, $j \neq i$, and $1 \leq n \leq p-1$, we have $\xi(s_j^n(P_i)) = \xi_j(i)$. Hence we have $|s_j^n(P_i)| < |P_2|$. For $i \neq 2$, we also have $|s_i^n(P_i)| = |P_i| < |P_2|$ for $0 \leq n \leq p-1$. Consequently, the inequality (4) is satisfied for any $\gamma \in N$.

For $j \neq 2$ and $1 \leq n \leq p-1$, we have $\xi(s_j^n(u)) = \xi_j(2)$. Hence we have $|s_j^n(u)| < |P_2|$. We also showed that $|s_2^n(u) - P_2| = |P_2|$ for $0 \leq n \leq p-1$. Consequently, the equality (5) is satisfied for any $\gamma \in N$.

For $i \neq 1, 2$, since $|P_1| < |P_2|$, we have

$$v(0, \infty; \xi_i(1)) \in \mathrm{vert}([v(0, \infty; \xi_i(2)), 0[).$$

Hence we have $|s_i^n(P_1)| \leq |s_i^n(u)|$. Since $s_1(P_1) = P_1 = 0$, we have $|s_1^n(P_1)| < |s_1^n(u)|$ $(0 \leq n \leq p-1)$. By (1) and (2), we have $|s_2^n(P_1)| = |\eta| < |P_2| = |s_2^n(u)|$ $(1 \leq n \leq p-1)$. Consequently, the inequality (6) is satisfied for any $\gamma \in N$. □

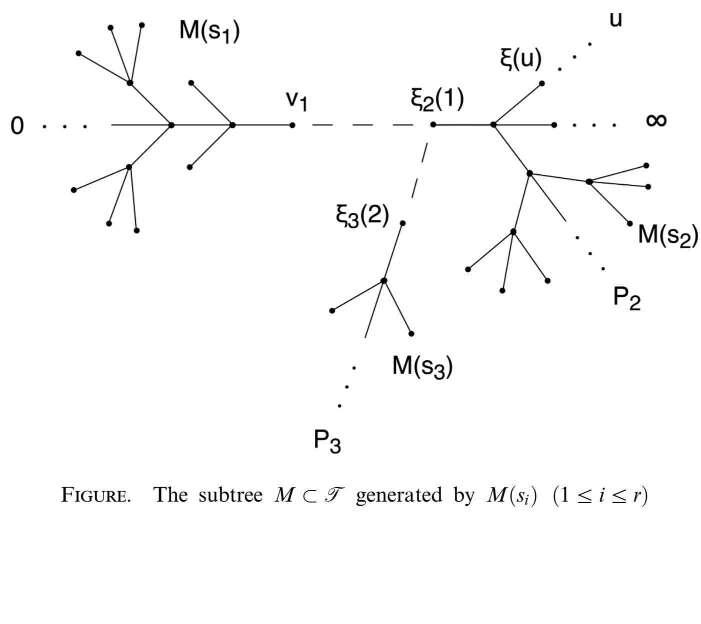


FIGURE. The subtree $M \subset \mathscr{T}$ generated by $M(s_i)$ $(1 \leq i \leq r)$

- Edges of $M(s_i)$ are denoted by solid line segments.
- Edges of $M \setminus \bigcup_i M(s_i)$ are denoted by dashed line segments.
- Half-lines are denoted by dots.

Recall that the function field of $\mathbf{P}^1$ (resp. $X$) is denoted by $K(x)$ (resp. $F = K(x, y)$). We treat $x$ as not only a function on $X$ and $\mathbf{P}^1$ but also an $N$-invariant function on $\Omega$ via the natural projection $\Omega \to \Omega/N \cong \mathbf{P}^1$. Similarly, we treat $y$ as not only a function on $X$ but also a $\Gamma$-invariant function on $\Omega$ via the natural projection $\Omega \to \Omega/\Gamma \cong \mathbf{P}^1$.

We also recall that for $1 \le i \le r$, the image of the fixed point $P_i \in \Omega$ of $s_i$ under the natural projection $\Omega \to \Omega/N \cong \mathbf{P}^1$ is the branch point $a_i \in \mathbf{P}^1$.

For any $\gamma \in N$ and $i \neq 2$, by Lemma 5.4 (4), we have $|\gamma(P_i)| < |P_2| = |u|$. Hence we have $\gamma(P_i) \neq u$. We have $x(u) \neq a_i$ for $i \neq 2$. By Lemma 5.4 (5), we have $\gamma(P_2) \neq u$ for any $\gamma \in N$. Hence we have $x(u) \neq a_2$.

There exists $\gamma \in \mathrm{PGL}_2(K)$ such that $\gamma(a_1) = 0$, $\gamma(a_2) = 1$, and $\gamma(x(u)) = \infty$. The inverse $\gamma^{-1}$ is written as

$$\gamma^{-1} = \begin{pmatrix} b & c \\ d & e \end{pmatrix} \in \mathrm{PGL}_2(K)$$

for some $b, c, d, e \in K$ satisfying $b - a_i d \neq 0$ $(1 \le i \le r)$. For each $i$, we have

$$\begin{aligned} \frac{\lambda_i}{x - a_i} &= \frac{\lambda_i}{\gamma^{-1}(\gamma(x)) - a_i} \\ &= \frac{\lambda_i d}{b - a_i d} + \frac{\lambda_i (be - cd)(b - a_i d)^{-2}}{\gamma(x) + (c - a_i e)(b - a_i d)^{-1}}. \end{aligned}$$

By replacing $K$ by its finite extension, there exists $C \in K$ satisfying

$$C^p - C = \sum_{i=1}^{r} \frac{\lambda_i d}{b - a_i d}.$$

We have

$$(y - C)^p - (y - C) = \sum_{i=1}^{r} \frac{\lambda_i (be - cd)(b - a_i d)^{-2}}{\gamma(x) + (c - a_i e)(b - a_i d)^{-1}}.$$

We also have

$$\frac{c - a_1 e}{b - a_1 d} - \frac{c - a_2 e}{b - a_2 d} = \frac{(a_2 - a_1)(be - cd)}{(b - a_1 d)(b - a_2 d)}.$$

Therefore, the inequality $|\lambda_1 \lambda_2| < |a_1 - a_2|^2$ is satisfied if and only if

$$\begin{aligned} \left| \frac{\lambda_1 (be - cd)}{(b - a_1 d)^2} \frac{\lambda_2 (be - cd)}{(b - a_2 d)^2} \right| &< \left| \frac{(a_2 - a_1)(be - cd)}{(b - a_1 d)(b - a_2 d)} \right|^2 \\ &= \left| \frac{c - a_1 e}{b - a_1 d} - \frac{c - a_2 e}{b - a_2 d} \right|^2 \end{aligned}$$

is satisfied. In the rest of this section, by replacing $x$ (resp. $y$) by $\gamma(x)$ (resp. $y - C$), we may assume $a_1 = 0$, $a_2 = 1$, and $x(u) = \infty$.

We put

$$\alpha := \prod_{\gamma \in N} \frac{P_2 - \gamma(u)}{P_2 - \gamma(P_1)},$$

which converges to an element of $K$; see [5, Section 8.1]. We have $|\alpha| = 1$ by Lemma 5.4 (4), (5). Let $z$ be a coordinate function on $\Omega \subset \mathbf{P}^1$. We have

$$x(z) = \alpha \prod_{\gamma \in N} \frac{z - \gamma(P_1)}{z - \gamma(u)}$$

since the both hand sides are $N$-invariant functions on $\Omega$ (i.e., functions on $\mathbf{P}^1 \cong \Omega/N$) having same zeros and poles and being 1 at $z = P_2$; see [5, Section 8.1].

We put

$$V_{i,\varepsilon} := \{z \in \Omega \mid |z - P_i| \leq \varepsilon\}$$

for $i = 1, 2$ and $\varepsilon \in |K^\times|$. Since $P_i \in \Omega$ is not a limit point of $N$, by replacing $K$ by its finite extension and taking $\varepsilon$ sufficiently small, we may assume $\varepsilon < |P_i - \gamma(u)|$ for any $\gamma \in N$.

We denote the power series expansion of $x$ on $V_{i,\varepsilon}$ by

$$x(z) = \alpha \sum_{n=0}^{\infty} c_{i,n}(z - P_i)^n.$$

Since $x(P_i) = a_i$, we have

$$x(z) - a_i = \alpha \sum_{n=1}^{\infty} c_{i,n}(z - P_i)^n.$$

Lemma 5.5. *We have* $\lambda_1 = \alpha c_{1,p}$ *and* $\lambda_2 = (-P_2^2\eta^{-1})^p \alpha c_{2,p}$.

*Proof.* We put

$$y_1(z) := \frac{1}{z},$$
$$y_2(z) := -\frac{P_2^2\eta^{-1}}{z - P_2}.$$

Then we have $y_i(s_i(z)) = y_i(z) + 1$ for $i = 1, 2$. We put $f_i := y_i - y$, which is an $s_i$-invariant function on $V_{i,\varepsilon}$. Since $y_i$ and $y$ have poles of order 1 at $P_i$ and we have $P_i = s_i(P_i)$, the function $f_i$ is holomorphic at $P_i$. We have

$$y_i^p - y_i = y^p - y + f_i^p - f_i = \frac{\lambda_i}{x - a_i} + h_i, \tag{5.1}$$

where we put

$$h_i := f_i^p - f_i + \sum_{j \neq i} \frac{\lambda_j}{x - a_j},$$

which is holomorphic at $P_i$.

For $i = 1$, by multiplying the both hand sides of (5.1) by $z^p(x - a_1)$, we have

$$(x - a_1) - z^{p-1}(x - a_1) = \lambda_1 z^p + z^p(x - a_1)h_1.$$

By comparing the degree 1 terms and the degree $p$ terms with respect to $z$, we have

$$\alpha c_{1,1} = 0,$$
$$\alpha c_{1,p} - \alpha c_{1,1} = \lambda_1.$$

Hence we have $\lambda_1 = \alpha c_{1,p}$.

For $i = 2$, by multiplying the both hand sides of (5.1) by $(z - P_2)^p(x - a_2)$, we have

$$\begin{aligned}(-P_2^2\eta^{-1})^p(x - a_2) &- (-P_2^2\eta^{-1})(z - P_2)^{p-1}(x - a_2) \\ &= \lambda_2(z - P_2)^p + (z - P_2)^p(x - a_2)h_2.\end{aligned}$$

By comparing the degree 1 terms and the degree $p$ terms with respect to $z - P_2$, we have

$$(-P_2^2\eta^{-1})^p\alpha c_{2,1} = 0,$$
$$(-P_2^2\eta^{-1})^p\alpha c_{2,p} - (-P_2^2\eta^{-1})\alpha c_{2,1} = \lambda_2.$$

Since $P_2 \neq 0$, $\eta \in K^\times$, and $\alpha \neq 0$, we have $c_{2,1} = 0$. Hence we have $\lambda_2 = (-P_2^2\eta^{-1})^p\alpha c_{2,p}$. □

LEMMA 5.6. *We have* $|\lambda_1| \leq |\eta|^{p-1} \cdot |P_2|^{-p}$ *and* $|\lambda_2| \leq |\eta|^{-p} \cdot |P_2|^p$.

*Proof.* For each $\gamma \in N$, $n \geq 1$, and $i = 1, 2$, we put

$$u_{i,0}^{(\gamma)} := \frac{P_i - \gamma(P_1)}{P_i - \gamma(u)},$$
$$u_{i,n}^{(\gamma)} := \frac{1 + u_{i,0}^{(\gamma)}}{(P_i - \gamma(u))^n}.$$

Since $\varepsilon < |P_i - \gamma(u)|$ for any $\gamma \in N$, we have

$$\frac{z - \gamma(P_1)}{z - \gamma(u)} = \sum_{n=0}^{\infty} u_{i,n}^{(\gamma)}(z - P_i)^n$$

on $V_{i,\varepsilon}$. (For this calculation, see [10, Section 3].) Hence we have

$$x(z) = \alpha \sum_{n=0}^{\infty} c_{i,n}(z - P_i)^n = \alpha \prod_{\gamma \in N} \frac{z - \gamma(P_1)}{z - \gamma(u)} = \alpha \prod_{\gamma \in N} \sum_{n=0}^{\infty} u_{i,n}^{(\gamma)}(z - P_i)^n.$$

We shall estimate $|c_{i,p}|$ by calculating $|u_{i,n}^{(\gamma)}|$.

For $i = 1$, since $s_1^j(P_1) = P_1$, we have $u_{1,0}^{(s_1^j)} = 0$ for $0 \le j \le p-1$. Hence we have

$$c_{1,p} = \left( \prod_{\gamma \in N \setminus \{s_1^j\}_{0 \le j \le p-1}} u_{1,0}^{(\gamma)} \right) \left( \prod_{j=0}^{p-1} u_{1,1}^{(s_1^j)} \right).$$

Recall that $P_1 = 0$. By Lemma 5.4 (6), we have $|u_{1,0}^{(\gamma)}| \le 1$ for any $\gamma \in N$. By Lemma 5.4 (1), (2), we have $|u_{1,0}^{(s_2^j)}| = |\eta| \cdot |P_2|^{-1}$ for $1 \le j \le p-1$. Since $u_{1,0}^{(s_1^j)} = 0$ for $1 \le j \le p-1$, by Lemma 5.4 (3), we have $|u_{1,1}^{(s_1^j)}| = |s_1^j(u)|^{-1} = 1$. We denote the identity element of $\mathrm{PGL}_2(K)$ by id. Since $u_{1,0}^{(\mathrm{id})} = 0$, we have $|u_{1,1}^{(\mathrm{id})}| = |P_2|^{-1}$. Consequently, we have

$$|c_{1,p}| \le |\eta|^{p-1} \cdot |P_2|^{-p}.$$

By Lemma 5.5, since $|\alpha| = 1$, we have

$$|\lambda_1| = |\alpha| \cdot |c_{1,p}| \le |\eta|^{p-1} \cdot |P_2|^{-p}.$$

For $i = 2$, by Lemma 5.4 (4), (5), we have $|u_{2,0}^{(\gamma)}| = 1$ for any $\gamma \in N$. By this equality and Lemma 5.4 (5), we have

$$|u_{2,n}^{(\gamma)}| = \frac{|1 + u_{2,0}^{(\gamma)}|}{|P_2 - \gamma(u)|^n} \le |P_2|^{-n}$$

for any $\gamma \in N$ and $n \ge 1$. Therefore, we have

$$|c_{2,p}| \le |P_2|^{-p}.$$

By Lemma 5.5, since $|\alpha| = 1$, we have

$$|\lambda_2| = |P_2|^{2p} \cdot |\eta|^{-p} \cdot |\alpha| \cdot |c_{2,p}| \le |\eta|^{-p} \cdot |P_2|^p. \qquad \square$$

By Lemma 5.2 and Lemma 5.6, we have

$$|\lambda_1 \lambda_2| \le |\eta|^{-1} < 1 = |a_1 - a_2|^2.$$

Recall that we have assumed $a_1 = 0$ and $a_2 = 1$.

Theorem 1.1 follows from this result and the result of Section 4.

*Acknowledgements*. The author would like to express his deepest gratitude to his adviser Tetsushi Ito for considerable and invaluable guidances. He is also grateful to Takahiro Tsushima for helpful comments about computation of reductions.

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