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ON PROPER HOLOMORPHIC SELF-MAPPINGS OF GENERALIZED COMPLEX ELLIPSOIDS AND GENERALIZED HARTOGS TRIANGLES

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Abstract

In this paper, we study proper holomorphic self-mappings of generalized complex ellipsoids and generalized Hartogs triangles. By making use of our previous result on the holomorphic automorphism group of a generalized complex ellipsoid and Monti-Morbidelli's result on the extendability of a local CR-diffeomorphism between open subsets contained in the strictly pseudoconvex part of the boundary of a generalized complex ellipsoid, we obtain natural generalizations of some results due to Landucci, Chen-Xu and Zapalowski.

1. Introduction and results

Let D_1 and D_2 be two domains in $\mathbf{C}^n$. A continuous mapping $f : D_1 \to D_2$ is said to be *proper* if $f^{-1}(K)$ is compact in D_1 for every compact subset K of D_2. Proper holomorphic mappings between bounded domains have been studied from various points of view. (See, for instance, Bedford [5], Jarnicki-Pflug [13].) In connection with this, there is a fundamental question as follows:

Question. *Let D be a bounded domain in $\mathbf{C}^n$ with $n > 1$. Then, is it true that every proper holomorphic mapping $f : D \to D$ must be biholomorphic?*

The answer to this question is negative, in general, without any other assumptions on the domain D or on the mapping f. However, there already exist articles solving this question affirmatively.

In this paper, we would like to study this question in the case where D is a generalized complex ellipsoid or a generalized Hartogs triangle. In order to state our precise results, let us start with defining our generalized complex ellipsoids

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and generalized Hartogs triangles. For any positive integers ℓ_i, m_j and any positive real numbers p_i, q_j with $1 \le i \le I$, $1 \le j \le J$, we set

$$\ell = (\ell_1, \dots, \ell_I), \quad m = (m_1, \dots, m_J), \quad p = (p_1, \dots, p_I), \quad q = (q_1, \dots, q_J)$$

and define a *generalized complex ellipsoid* $\mathscr{E}_\ell^p$ and a *generalized Hartogs triangle* $\mathscr{H}_{\ell,m}^{p,q}$ by

$$\mathscr{E}_\ell^p = \left\{ z \in \mathbf{C}^{|\ell|}; \sum_{i=1}^{I} \|z_i\|^{2p_i} < 1 \right\} \quad \text{and}$$

$$\mathscr{H}_{\ell,m}^{p,q} = \left\{ (z,w) \in \mathbf{C}^N; \sum_{i=1}^{I} \|z_i\|^{2p_i} < \sum_{j=1}^{J} \|w_j\|^{2q_j} < 1 \right\},$$

respectively, where

$$\begin{aligned} &z = (z_1, \dots, z_I) \in \mathbf{C}^{\ell_1} \times \cdots \times \mathbf{C}^{\ell_I} = \mathbf{C}^{|\ell|}, \quad |\ell| = \ell_1 + \cdots + \ell_I, \\ &w = (w_1, \dots, w_J) \in \mathbf{C}^{m_1} \times \cdots \times \mathbf{C}^{m_J} = \mathbf{C}^{|m|}, \quad |m| = m_1 + \cdots + m_J, \\ &\text{and} \quad \mathbf{C}^N = \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}, \quad N = |\ell| + |m|. \end{aligned}$$

For convenience and no loss of generality, in this paper we always assume that

$$p_2, \dots, p_I \neq 1, \quad q_2, \dots, q_J \neq 1$$

if $I \ge 2$ or $J \ge 2$. Hence, if $I = 1$, then $\mathscr{E}_\ell^p = B^{\ell_1}$, the unit ball in $\mathbf{C}^{\ell_1}$, whether $p_1 = 1$ or not; and if $I \ge 2$, then $\mathscr{E}_\ell^p$ is different from the unit ball $B^{|\ell|}$ in $\mathbf{C}^{|\ell|}$. In general, both the domains $\mathscr{E}_\ell^p$ and $\mathscr{H}_{\ell,m}^{p,q}$ are not geometrically convex and their boundaries are not smooth. Notice that $\partial\mathscr{H}_{\ell,m}^{p,q}$ contains the origin 0 of $\mathbf{C}^N$.

Let us now return to our question above in the case where D is a generalized complex ellipsoid or a generalized Hartogs triangle. Then we have already known the following: If all the exponents p_i are positive integers, then $\mathscr{E}_\ell^p$ is a bounded pseudoconvex domain with real-analytic boundary. Hence, by a direct consequence of Bedford-Bell [6], every proper holomorphic self-mapping of $\mathscr{E}_\ell^p$ is a biholomorphic mapping. Independently, Landucci [18] studied the structure of proper holomorphic mappings between generalized complex ellipsoids $\mathscr{E}_\ell^p$ and $\mathscr{E}_{\ell'}^{p'}$ with $\ell_i, \ell_i' = 1$, $p_i, p_i' \in \mathbf{N}$ $(1 \le i \le I)$, and proved that every proper holomorphic self-mapping of such a generalized complex ellipsoid $\mathscr{E}_\ell^p$ must be a biholomorphic mapping. If some of p_i's are not integers, then the boundary of $\mathscr{E}_\ell^p$ is no longer real-analytic. However, as is shown by Dini-Primicerio [11], even in such a case the same conclusion holds for $\mathscr{E}_\ell^p$, provided that all the ℓ_i's are equal to 1. On the other hand, for the generalized Hartogs triangles, Landucci also studied in [19] the structure of proper holomorphic mappings between generalized Hartogs triangles $\mathscr{H}_{\ell,m}^{p,q}$ and $\mathscr{H}_{\ell',m'}^{p',q'}$ with $\ell_i, \ell_i' = 1$, $p_i, p_i' \in \mathbf{N}$

$(1 \le i \le I)$ and $m, m' = 1$, $q, q' \in \mathbf{N}$. In particular, he found the existence of a generalized Hartogs triangle $\mathscr{H}^{p,q}_{\ell,m}$ admitting a proper non-biholomorphic self-mapping. Landucci's result was later extended by Chen-Xu [9], [10] and Zapalowski [22] to the class of generalized Hartogs triangles $\mathscr{H}^{p,q}_{\ell,m}$ with ℓ_i, $m_j = 1$, $0 < p_i$, $q_j \in \mathbf{R}$ for all i, j and $J > 1$.

In view of these results, it would be naturally expected that the same conclusion as in the case where ℓ_i, $m_j = 1$ for all i, j is also valid for our generalized complex ellipsoids $\mathscr{E}^p_\ell$ with $\ell_i \ge 1$ or generalized Hartogs triangles $\mathscr{H}^{p,q}_{\ell,m}$ with ℓ_i, $m_j \ge 1$. This cannot be achieved in full generality at this moment. However, under the assumption that all the exponents p_i and q_j are greater than or equal to 1, we can give an affirmative answer to this. Before stating our results, observe that the boundary of $\mathscr{E}^p_\ell$ is C^2-smooth if and only if $p_i \ge 1$ for all $i = 1, \dots, I$. Therefore, in connection with our question, it would be the class of generalized complex ellipsoids $\mathscr{E}^p_\ell$ with $p_i \ge 1$ for all $i = 1, \dots, I$ that we should study first.

The main purpose of this paper is to establish the following theorems. (For the explicit descriptions of holomorphic automorphisms of $\mathscr{E}^p_\ell$, see Section 2.)

Theorem 1. *Let $\mathscr{E}^p_\ell$ be a generalized complex ellipsoid in $\mathbf{C}^{|\ell|}$ with $|\ell| \ge 2$. Assume that $1 \le p_i \in \mathbf{R}$ for all $i = 1, \dots, I$. Then every proper holomorphic mapping $f : \mathscr{E}^p_\ell \to \mathscr{E}^p_\ell$ is necessarily a holomorphic automorphism of $\mathscr{E}^p_\ell$.*

It should be emphasized that if $1 \le p_i \in \mathbf{R}$ for all i, then $\mathscr{E}^p_\ell$ is a geometrically convex bounded domain with C^2-smooth (but not C^3-smooth) boundary $\partial\mathscr{E}^p_\ell$, in general, and our $\mathscr{E}^p_\ell$ in Theorem 1 admits the case where some of ℓ_i's are greater than 1. Therefore our theorem is not an immediate consequence of any other papers.

The structure of proper holomorphic self-mappings of $\mathscr{H}^{p,q}_{\ell,m}$ with $|\ell|\,|m| = 1$, that is, $\mathscr{H}^{p,q}_{\ell,m} \subset \mathbf{C}^2$, is already discussed in [19], [22], in detail. So, in this paper, we would like to study our question in the case where D is a generalized Hartogs triangle $\mathscr{H}^{p,q}_{\ell,m}$ with $|\ell|\,|m| > 1$. Then, our Theorem 1 can be applied to prove the following theorems:

Theorem 2. *Let $\mathscr{H}^{p,q}_{\ell,m}$ be a generalized Hartogs triangle in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}$ with $|\ell| \ge 2$, $|m| \ge 2$. Assume that $1 \le p_i$, $q_j \in \mathbf{R}$ for all $i = 1, \dots, I$, $j = 1, \dots, J$. Then a holomorphic mapping $\Phi : \mathscr{H}^{p,q}_{\ell,m} \to \mathscr{H}^{p,q}_{\ell,m}$ is proper if and only if Φ can be written in the form*

$$\Phi : (z_1, \dots, z_I, w_1, \dots, w_J) \mapsto (\tilde{z}_1, \dots, \tilde{z}_I, \tilde{w}_1, \dots, \tilde{w}_J),$$
$$\tilde{z}_i = A_i z_{\sigma(i)} \ \ (1 \le i \le I), \quad \tilde{w}_j = B_j w_{\tau(j)} \ \ (1 \le j \le J)$$

(think of z_i, w_j as column vectors), where $A_i \in U(\ell_i)$, $B_j \in U(m_j)$ and σ, τ are permutations of $\{1, \dots, I\}$, $\{1, \dots, J\}$ respectively, satisfying the condition: $\sigma(i) = s$, $\tau(j) = t$ can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$, $(m_j, q_j) = (m_t, q_t)$.

In particular, Φ is a holomorphic automorphism of $\mathscr{H}^{p,q}_{\ell,m}$.

THEOREM 3. *Let $\mathscr{H}_{\ell,m}^{p,q}$ be a generalized Hartogs triangle in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}$ with $|\ell| = 1$, $|m| \geq 2$. Assume that $1 \leq q_j \in \mathbf{R}$ for all $j = 1, \dots, J$. Then a holomorphic mapping $\Phi : \mathscr{H}_{\ell,m}^{p,q} \to \mathscr{H}_{\ell,m}^{p,q}$ is proper if and only if Φ can be written in the form*

$$\Phi : (z, w_1, \dots, w_J) \mapsto (\tilde{z}, \tilde{w}_1, \dots, \tilde{w}_J),$$

$$\tilde{z} = Az, \quad \tilde{w}_j = B_j w_{\tau(j)} \ (1 \leq j \leq J),$$

where $A \in \mathbf{C}$ with $|A| = 1$, $B_j \in U(m_j)$ and τ is a permutation of $\{1, \dots, J\}$ satisfying the condition: $\tau(j) = t$ can only happen when $(m_j, q_j) = (m_t, q_t)$.

In particular, Φ is a holomorphic automorphism of $\mathscr{H}_{\ell,m}^{p,q}$.

THEOREM 4. *Let $\mathscr{H}_{\ell,m}^{p,q}$ be a generalized Hartogs triangle in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}$ with $|\ell| \geq 2$, $|m| = 1$. Assume that $1 \leq p_i \in \mathbf{R}$ for all $i = 1, \dots, I$. Then a holomorphic mapping $\Phi : \mathscr{H}_{\ell,m}^{p,q} \to \mathscr{H}_{\ell,m}^{p,q}$ is proper if and only if Φ is a transformation*

$$\Phi : (z_1, \dots, z_I, w) \mapsto (\tilde{z}_1, \dots, \tilde{z}_I, \tilde{w})$$

of the following form:

CASE I. $I = 1$.

(I.1) $q/p \in \mathbf{N}$: *In this case, putting $r = q/p$, we have*

$$\tilde{z}_1 = w^{kr} H(z_1/w^r), \quad \tilde{w} = Bw^k,$$

where $k \in \mathbf{N}$, $H \in \mathrm{Aut}(B^{\ell_1})$ and $B \in \mathbf{C}$ with $|B| = 1$.

(I.2) $q/p \notin \mathbf{N}$: *In this case, putting $r = q/p$, we have*

$$\tilde{z}_1 = w^{(k-1)r} A z_1, \quad \tilde{w} = Bw^k,$$

where $k \in \mathbf{N}$, $A \in U(\ell_1)$, $(k-1)r \in \mathbf{Z}$ and $B \in \mathbf{C}$ with $|B| = 1$.

CASE II. $I \geq 2$.

(II.1) $p_1 = 1$, $q \in \mathbf{N}$: *In this case, we have*

$$\tilde{z}_1 = w^{kq} H(z_1/w^q), \quad \tilde{z}_i = w^{(k-1)q/p_i} \gamma_i(z_1/w^q) A_i z_{\sigma(i)} \ (2 \leq i \leq I), \quad \tilde{w} = Bw^k,$$

where

(1) $H \in \mathrm{Aut}(B^{\ell_1})$;

(2) *γ_i's are nowhere vanishing holomorphic functions on B^{ℓ_1} defined by*

$$\gamma_i(z_1) = \left(\frac{1 - \|a\|^2}{(1 - \langle z_1, a \rangle)^2} \right)^{1/2p_i}, \quad a = H^{-1}(o) \in B^{\ell_1},$$

where $o \in B^{\ell_1}$ is the origin of $\mathbf{C}^{\ell_1}$;

(3) *$k \in \mathbf{N}$, $A_i \in U(\ell_i)$, $(k-1)q/p_i \in \mathbf{Z}$ $(2 \leq i \leq I)$ and $B \in \mathbf{C}$ with $|B| = 1$;*

(4) *σ is a permutation of $\{2, \dots, I\}$ satisfying the following: $\sigma(i) = s$ can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$.*

(II.2) $p_1 = 1$, $q \notin \mathbf{N}$: *In this case, we have*

$$\tilde{z}_1 = w^{(k-1)q} A z_1, \quad \tilde{z}_i = w^{(k-1)q/p_i} A_i z_{\sigma(i)} \ (2 \leq i \leq I), \quad \tilde{w} = Bw^k,$$

where $k \in \mathbf{N}$, $A \in U(\ell_1)$, $(k-1)q \in \mathbf{Z}$, $A_i \in U(\ell_i)$, $(k-1)q/p_i \in \mathbf{Z}$ $(2 \le i \le I)$, $B \in \mathbf{C}$ *with* $|B| = 1$, *and* σ *is a permutation of* $\{2, \dots, I\}$ *satisfying the condition*: $\sigma(i) = s$ *can only happen when* $(\ell_i, p_i) = (\ell_s, p_s)$.

(II.3) $p_1 \neq 1$: *In this case, we have*

$$\tilde{z}_i = w^{(k-1)q/p_i} A_i z_{\sigma(i)} \ (1 \le i \le I), \quad \tilde{w} = B w^k,$$

where $k \in \mathbf{N}$, $A_i \in U(\ell_i)$, $(k-1)q/p_i \in \mathbf{Z}$ $(1 \le i \le I)$, $B \in \mathbf{C}$ *with* $|B| = 1$, *and* σ *is a permutation of* $\{1, \dots, I\}$ *satisfying the condition*: $\sigma(i) = s$ *can only happen when* $(\ell_i, p_i) = (\ell_s, p_s)$.

In particular, Φ *is a holomorphic automorphism of* $\mathscr{H}^{p,q}_{\ell,m}$ *if and only if* $k = 1$ *in any cases.*

Considering the general case where ℓ_i, $m_j \ge 1$ in this paper, we obtain natural generalizations of some results due to Landucci [18], [19], Chen-Xu [9], [10] and Zapalowski [22]. Here it should be remarked that some of their techniques used in [9], [10], [18], [19] and [22] are not applicable to our case where $\ell_i \ge 1$ or $m_j \ge 1$. In fact, for instance, there is no several-variable analogue of the function $\lambda \mapsto \lambda^a$ $(\lambda \in \mathbf{C}^*,\ 0 < a \in \mathbf{R})$ that plays crucial roles in their papers.

Finally, we would like to point out the following: Let $\mathscr{H}^{p,q}_{\ell,m}$ be a generalized Hartogs triangle in $\mathbf{C} \times \mathbf{C}^{|m|}$ with $m_1 = \cdots = m_J = 1$ and $J \ge 2$. Then, according to [22; Theorem 3, (b)], one obtains the following result which contradicts our Theorem 3: *A holomorphic mapping* $\Phi : \mathscr{H}^{p,q}_{\ell,m} \to \mathscr{H}^{p,q}_{\ell,m}$ *is proper if and only if* Φ *has the form*

$$(\dagger) \qquad \Phi(z, w) = (\zeta z^k, h(w)), \quad (z, w) \in \mathscr{H}^{p,q}_{\ell,m},$$

where $\zeta \in \mathbf{C}$ *with* $|\zeta| = 1$, $k \in \mathbf{N}$ *and* $h : \mathscr{E}^q_m \to \mathscr{E}^q_m$ *is a proper holomorphic mapping such that* $h(0) = 0$. *In particular, there are non-trivial proper holomorphic self-mappings in such a* $\mathscr{H}^{p,q}_{\ell,m}$. But, this is obviously incorrect. In fact, consider, for instance, the generalized Hartogs triangle $\mathscr{H} := \mathscr{H}^{p,q}_{\ell,m}$ and the holomorphic mapping $\Phi : \mathscr{H} \to \mathscr{H}$ defined by

$$\mathscr{H} = \{(z, w) \in \mathbf{C} \times \mathbf{C}^2; |z| < \|w\|^2 < 1\}, \quad \Phi(z, w) = (z^2, w), \quad (z, w) \in \mathscr{H},$$

that is, $p = 1/2$, $q = (1, 1)$, $\zeta = 1$, $k = 2$ and $h = \mathrm{id}$, the identity mapping, in $(\dagger)$. Then Φ is holomorphic on $\mathbf{C}^3$ $(\supset \overline{\mathscr{H}})$ and, for the boundary point $(z_o, w_o) \in \partial\mathscr{H}$ given by $z_o = 1/2$, $w_o = (1/\sqrt{2}, 0)$, we have $\Phi(z_o, w_o) = (1/4, 1/\sqrt{2}, 0) \in \mathscr{H}$. Consequently, Φ is not proper, though it satisfies all the requirements of $(\dagger)$. From this, the assertion in [22; Corollary 8] may also be corrected.

Our proof of Theorem 1 above is based on our previous result on the structure of holomorphic automorphism groups of generalized complex ellipsoids [15] and an extension theorem of local CR-diffeomorphisms defined near a C^ω-smooth strictly pseudoconvex boundary point of a generalized complex ellipsoid due to Monti-Morbidelli [20]. Once Theorem 1 is proved, we can apply the same method used in our previous paper [16] to prove Theorems 2, 3 and 4. After some preparations in Sections 2 and 3, we prove our theorems in Section 4.

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Notation. Throughout this paper we use the following notation: For given points $z = (z_1, \dots, z_I) \in \mathbf{C}^{|\ell|}$, $w = (w_1, \dots, w_J) \in \mathbf{C}^{|m|}$ and $p = (p_1, \dots, p_I)$, $q = (q_1, \dots, q_J)$ as above, we set

$$z_i = (z_i^1, \dots, z_i^{\ell_i}) \ (1 \le i \le I), \quad w_j = (w_j^1, \dots, w_j^{m_j}) \ (1 \le j \le J),$$
$$\zeta = (\zeta_1, \dots, \zeta_N) = (z, w) \in \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N,$$
$$\zeta' = (\zeta_1, \dots, \zeta_{|\ell|}) = z, \quad \zeta'' = (\zeta_{|\ell|+1}, \dots, \zeta_N) = w \quad \text{and}$$
$$\rho^p(z) = \sum_{i=1}^{I} \|z_i\|^{2p_i}, \quad \rho^q(w) = \sum_{j=1}^{J} \|w_j\|^{2q_j}.$$

As usual, we write

$$\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_N^{\alpha_N} \quad \text{for } \zeta = (\zeta_1, \dots, \zeta_N) \in \mathbf{C}^N, \ \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{Z}^N.$$

For a given $n \in \mathbf{N}$, we denote by $U(n)$ the unitary group of degree n, and for a set $S \subset \mathbf{C}^n$, ∂S (resp. $\bar{S}$) stands for the boundary (resp. closure) of S. We denote by $\langle \cdot, \cdot \rangle$ the standard Hermitian inner product on $\mathbf{C}^n$, that is,

$$\langle \zeta, \eta \rangle = \sum_{j=1}^{n} \zeta_j \bar{\eta}_j \quad \text{for } \zeta = (\zeta_1, \dots, \zeta_n), \ \eta = (\eta_1, \dots, \eta_n) \in \mathbf{C}^n.$$

Let W be a domain in $\mathbf{C}^n$. Then we denote by $\mathrm{Aut}(W)$ the group of all holomorphic automorphisms of W equipped with the compact-open topology. For a given holomorphic mapping $F : W \to \mathbf{C}^n$, we denote by $J_F(\zeta)$ the Jacobian determinant of F at $\zeta \in W$ and put $V_F = \{\zeta \in W; J_F(\zeta) = 0\}$.

2. Some known facts

In this section, for later purpose, we collect some known facts on the holomorphic automorphisms of generalized complex ellipsoids $\mathscr{E}_\ell^p$ in $\mathbf{C}^{|\ell|} = \mathbf{C}^{\ell_1} \times \cdots \times \mathbf{C}^{\ell_I}$.

If $I = 1$, then $\mathscr{E}_\ell^p$ is the unit ball B^{ℓ_1} in $\mathbf{C}^{\ell_1}$ and the structure of the holomorphic automorphism group $\mathrm{Aut}(B^{\ell_1})$ of B^{ℓ_1} is well-known. And, if $I \ge 2$ (hence, $p_i \ne 1$ for all $i = 2, \dots, I$ by our assumption), we have the following:

Theorem A (Kodama [15]). *The holomorphic automorphism group* $\mathrm{Aut}(\mathscr{E}_\ell^p)$ *consists of all transformations*

$$\Phi : (z_1, \dots, z_I) \mapsto (\tilde{z}_1, \dots, \tilde{z}_I)$$

of the following form:

CASE I. $p_1 = 1$. *In this case, we have*

$$\tilde{z}_1 = H(z_1), \quad \tilde{z}_i = \gamma_i(z_1) A_i z_{\sigma(i)} \ (2 \le i \le I),$$

(*think of z_i as column vectors*), *where*

(1) $H \in \mathrm{Aut}(B^{\ell_1})$;

(2) *γ_i's are nowhere vanishing holomorphic functions on B^{ℓ_1} defined by*

$$\gamma_i(z_1) = \left(\frac{1 - \|a\|^2}{(1 - \langle z_1, a \rangle)^2} \right)^{1/2p_i}, \quad a = H^{-1}(o) \in B^{\ell_1},$$

where $o \in B^{\ell_1}$ is the origin of $\mathbf{C}^{\ell_1}$;

(3) *$A_i \in U(\ell_i)$, the unitary group of degree ℓ_i*;

(4) *σ is a permutation of $\{2, \dots, I\}$ satisfying the following*: *$\sigma(i) = s$ can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$.*

CASE II. $p_1 \ne 1$. *In this case, we have*

$$\tilde{z}_i = A_i z_{\sigma(i)} \quad (1 \le i \le I),$$

where $A_i \in U(\ell_i)$ and σ is a permutation of $\{1, \dots, I\}$ satisfying the condition: *$\sigma(i) = s$ can only happen when $(\ell_i, p_i) = (\ell_s, p_s)$.*

Let $\mathscr{E}_\ell^p$ be a generalized complex ellipsoid in $\mathbf{C}^{|\ell|} = \mathbf{C}^{\ell_1} \times \cdots \times \mathbf{C}^{\ell_I}$ with $I \ge 2$ and assume that the exponents p_i and the integers ℓ_i satisfy the condition

$$(\ddagger) \qquad p_1 = 1, \quad \ell_1 \ge 1 \quad \text{and} \quad \mathbf{R} \ni p_i > 1, \quad \ell_i \ge 2 \ (2 \le i \le I).$$

Define here a subset $\mathscr{S}$ of $\partial \mathscr{E}_\ell^p$ by

$$\mathscr{S} = \{(z_1, z_2, \dots, z_I) \in \partial \mathscr{E}_\ell^p ; \|z_2\| \cdots \|z_I\| \ne 0\}.$$

By routine computations, it then follows that $\mathscr{S}$ is just the set consisting of all C^ω-smooth strictly pseudoconvex boundary points of $\mathscr{E}_\ell^p$. Note that $\mathscr{S}$ is a simply connected, connected real hypersurface in $\mathbf{C}^{|\ell|}$, since $\ell_i \ge 2$ for all $i = 2, \dots, I$. For this C^ω-smooth strictly pseudoconvex real hypersurface $\mathscr{S}$, we have the following:

THEOREM B (Monti-Morbidelli [20]). *Let $\mathscr{E}_\ell^p$ be a generalized complex ellipsoid in $\mathbf{C}^{|\ell|}$ satisfying the condition* ($\ddagger$). *Let O, O' be connected open subsets of $\mathscr{S}$ and let $f : O \to O'$ be a CR-diffeomorhism between O and O'. Then f extends to a global biholomorphic mapping $\hat{f} : \mathscr{E}_\ell^p \to \mathscr{E}_\ell^p$.*

In [20] they proved more: the extension $\hat{f}$ can be written as a composite mapping of four standard holomorphic automorphisms of $\mathscr{E}_\ell^p$, provided that all the exponents p_i are positive integers. Here, observe that they do not use essentially the fact that all the p_i's are positive integers except for the proofs of Propositions 3.4 and 5.1 in [20]. Moreover, if the condition ($\ddagger$) is satisfied, one can see that their proofs remain valid for these propositions even in the case where some of p_i's are not integers. Therefore, Theorem B has already been proved implicitly in [20]. Using power series expansion technique, Hayashimoto [12] gave an alternative proof of Monti-Morbidelli's theorem with some weaker conditions on the dimensions ℓ_i and the exponents $p_i \in \mathbf{N}$. However it seems difficult to apply the same technique to our general case where some of p_i's are not integers.

3. Some lemmas

In this section, we shall prove several lemmas which will play crucial roles in our proofs of the theorems.

3.1. A Lemma for $\mathscr{E}_\ell^p$. In this Subsection, we write $\mathscr{E} = \mathscr{E}_\ell^p$ for the sake of simplicity, and $f : \mathscr{E} \to \mathscr{E}$ denotes an arbitrarily given proper holomorphic mapping.

First of all, since $\mathscr{E}$ is a bounded complete Reinhardt domain in $\mathbf{C}^{|\ell|}$, by a result of Bell [8] there exists a connected open neighborhood D of $\bar{\mathscr{E}}$ such that f extends to a holomorphic mapping $\hat{f} : D \to \mathbf{C}^{|\ell|}$. Therefore, replacing f by $\hat{f}$ if necessary, we may assume that f itself is a holomorphic mapping defined on D.

Under this assumption, we wish to prove the following:

Lemma 1. *Let $\mathscr{E}$ be a generalized complex ellipsoid in $\mathbf{C}^{|\ell|}$ with $I \geq 2$. Assume that $p_i > 1$ and $\ell_i \geq 2$ for all $i = 1, \dots, I$. Then the proper holomorphic mapping $f : \mathscr{E} \to \mathscr{E}$ is an automorphism of $\mathscr{E}$.*

Proof. Once it is shown that $V_f = \emptyset$, then $f : \mathscr{E} \to \mathscr{E}$ is an unbranched covering; and hence, it must be a biholomorphic mapping, since $\mathscr{E}$ is a simply connected domain. Assuming to the contrary that $V_f \neq \emptyset$, we wish to derive a contradiction. To this end, let us consider the functions $r(z)$ and $R(z)$ defined by

$$r(z) = \rho^p(z) - 1, \quad z \in \mathbf{C}^{|\ell|}, \quad \text{and} \quad R(z) = r(f(z)), \quad z \in D.$$

It then follows from the Hopf lemma that $R(z)$ is a C^2-smooth defining function for $\mathscr{E}$ as well as $r(z)$. Thus, if we set

$$D_\varepsilon = \{z \in D; R(z) < \varepsilon\} \quad \text{and} \quad D'_\varepsilon = \{z \in \mathbf{C}^{|\ell|}; r(z) < \varepsilon\}$$

for a sufficiently small $\varepsilon > 0$, then we have $\bar{\mathscr{E}} \subset D_\varepsilon \cap D'_\varepsilon$, $\overline{D_\varepsilon \cup D'_\varepsilon} \subset D$ and f gives rise to a proper holomorphic mapping, say again f, from D_ε onto D'_ε. Hence, for any irreducible component V of $V_f \cap D_\varepsilon$, it follows from Remmert's proper mapping theorem that $f(V)$ is a complex analytic subvariety of D'_ε and the restriction $\tilde{f} := f|V : V \to f(V)$ is also proper. In particular, V and $f(V)$ both have pure $\mathbf{C}$-dimension $|\ell| - 1$ and $\tilde{f}^{-1}(\mathrm{Sing}\, f(V))$ is nowhere dense in V. Therefore, by repeating exactly the same argument as in [4; p. 479], one can see that there exists a connected complex manifold M of $\mathbf{C}$-dimension $|\ell| - 1$ such that M is open dense in V and $\tilde{f}$ gives rise to a local biholomorphic mapping from M onto $\tilde{f}(M)$. Accordingly, both $M \cap \partial\mathscr{E}$ and $\tilde{f}(M) \cap \partial\mathscr{E}$ are C^2-differentiable submanifolds of $\partial\mathscr{E}$ with the same **R**-dimension $2|\ell| - 3$. Now let us set

$$\mathscr{S} = \{(z_1, \dots, z_I) \in \partial\mathscr{E}; \|z_1\| \cdots \|z_I\| \neq 0\} \quad \text{and}$$
$$\mathscr{W}_i = \{(z_1, \dots, z_I) \in \partial\mathscr{E}; z_i = 0\} \quad (1 \leq i \leq I).$$

Then $\mathscr{S}$ is the set of all C^2-smooth strictly pseudoconvex boundary points of $\mathscr{E}$ and $\partial\mathscr{E} \backslash \mathscr{S} = \bigcup_{i=1}^{I} \mathscr{W}_i$ is the set of all weakly pseudoconvex boundary points of $\mathscr{E}$. Note that each $\mathscr{W}_i$ is a C^2-differentiable submanifold of $\partial\mathscr{E}$ with $\dim_{\mathbf{R}} \mathscr{W}_i = 2|\ell| - 2\ell_i - 1 \leq 2|\ell| - 5$, because $\ell_i \geq 2$ by our assumption. Thus $\bigcup_{i=1}^{I} \mathscr{W}_i$ is too small to contain $M \cap \partial\mathscr{E}$; so that there exists a point $z^o \in \mathscr{S} \cap (M \cap \partial\mathscr{E}) \subset M \subset V_f$. On the other hand, by using the same method as in the proof of [8; Theorem 3], it can be checked that $J_f(z^o) \neq 0$ and f cannot be branched at the strictly pseudoconvex boundary point $z^o \in \mathscr{S}$; so that $z^o \notin V_f$. This is a contradiction; thereby, the proof is completed. $\square$

3.2. Lemmas for $\mathscr{H}^{p,q}_{\ell,m}$. Throughout this Subsection, we write $\mathscr{H} = \mathscr{H}^{p,q}_{\ell,m}$, where $\mathscr{H}^{p,q}_{\ell,m}$ is a generalized Hartogs triangle in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$ with $|\ell|\,|m| > 1$. And, $\Phi : \mathscr{H} \to \mathscr{H}$ denotes an arbitrarily given proper holomorphic mapping.

Our proofs of the following lemmas will be carried out along the same lines as in [19], [9], [16], [22]; and some of them will be presented only in outline.

Let $S_{\mathscr{H}} = \{\alpha \in \mathbf{Z}^N; \zeta^\alpha \in \mathcal{O}(\mathscr{H}), \|\zeta^\alpha\|_{A^2(\mathscr{H})} < \infty\}$, where $\mathcal{O}(\mathscr{H})$ denotes the set of all holomorphic functions on $\mathscr{H}$ and $A^2(\mathscr{H})$ is the Bergman space of $\mathscr{H}$ with the norm $\|\cdot\|_{A^2(\mathscr{H})}$. Then it is known [3] that the Bergman kernel function $K = K_{\mathscr{H}}$ for $\mathscr{H}$ can be expressed as

$$K(\zeta, \eta) = \sum_{\alpha \in S_{\mathscr{H}}} c_\alpha \zeta^\alpha \bar{\eta}^\alpha, \quad \zeta, \eta \in \mathscr{H}, \tag{3.1}$$

with $c_\alpha > 0$ for each $\alpha \in S_{\mathscr{H}}$. By making use of this special form of $K(\zeta, \eta)$, we can show the following (cf. [16; Lemma 1]):

LEMMA 2. *The Bergman kernel function $K(\zeta, \eta)$ extends holomorphically in ζ and anti-holomorphically in η to an open neighborhood of $(\overline{\mathscr{H}} \backslash \{0\}) \times \mathscr{H}$ in $\mathbf{C}^{2N}$.*

Thanks to this lemma, we can prove the following:

LEMMA 3. *Let ζ_o be an arbitrary point of $\partial\mathscr{H} \backslash \{0\}$. Then there exists a connected open neighborhood U_{ζ_o} of ζ_o in $\mathbf{C}^N \backslash \{0\}$ such that Φ extends to a holomorphic mapping $\hat{\Phi} : \mathscr{H} \cup U_{\zeta_o} \to \mathbf{C}^N$.*

Proof. Let $P : L^2(\mathscr{H}) \to A^2(\mathscr{H})$ be the Bergman projection defined by

$$Pf(\zeta) = \int_{\mathscr{H}} K(\zeta, \eta) f(\eta)\, dV_\eta, \quad f \in L^2(\mathscr{H}).$$

It then follows from Lemma 2 that Pf can be extended to a holomorphic function, say $\hat{P}f$, defined on some domain $\mathscr{H} \cup O_{\zeta_o}$, where O_{ζ_o} is a connected open neighborhood of ζ_o contained in $\mathbf{C}^N \backslash \{0\}$.

Let $\phi \in C_0^\infty(\mathscr{H})$ be a non-negative function such that $\phi(\zeta_1, \dots, \zeta_N) = \phi(|\zeta_1|, \dots, |\zeta_N|)$ and $\int_{\mathscr{H}} \phi(\zeta)\, dV_\zeta = 1$. For any $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{Z}^N$ with $\alpha_j \geq 0$ $(1 \leq j \leq N)$, we set

$$\phi_\alpha(\zeta) = (c_\alpha \alpha!)^{-1}(-1)^{|\alpha|}\partial^{|\alpha|}\phi(\zeta)/\partial\bar{\zeta}_1^{\alpha_1}\cdots\partial\bar{\zeta}_N^{\alpha_N}, \quad \zeta \in \mathscr{H},$$

where c_α is the same constant appearing in (3.1) and $\alpha! = \alpha_1! \cdots \alpha_N!$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Then, thanks to the concrete description of the expansion of K as in (3.1), we can compute explicitly $P\phi_\alpha$ as $P\phi_\alpha(\zeta) = \zeta^\alpha$, $\zeta \in \mathscr{H}$. Consequently, by analytic continuation

$$\hat{P}\phi_\alpha(\zeta) = \zeta^\alpha, \quad \zeta \in \mathscr{H} \cup O_{\zeta_o}. \tag{3.2}$$

Now, express $\Phi = (\Phi_1, \dots, \Phi_N)$ with respect to the ζ-coordinate system in $\mathbf{C}^N$. Then, applying the transformation law by the Bergman projection under proper holomorphic mapping (cf. [7]) and using the fact (3.2), we have that

$$\begin{aligned}(J_\Phi \cdot (\Phi_1)^{\alpha_1} \cdots (\Phi_N)^{\alpha_N})(\zeta) &= (J_\Phi \cdot P\phi_\alpha \circ \Phi)(\zeta) \\ &= P(J_\Phi \cdot \phi_\alpha \circ \Phi)(\zeta) = \int_{\mathscr{H}} K(\zeta, \eta)(J_\Phi \cdot \phi_\alpha \circ \Phi)(\eta)\, dV_\eta\end{aligned}$$

for $\zeta \in \mathscr{H}$. Here, since the last term extends holomorphically to the function $\hat{P}(J_\Phi \cdot \phi_\alpha \circ \Phi)$ on $\mathscr{H} \cup O_{\zeta_o}$, we may assume that $J_\Phi \cdot (\Phi_1)^{\alpha_1} \cdots (\Phi_N)^{\alpha_N}$ is also a holomorphic function defined on $\mathscr{H} \cup O_{\zeta_o}$. In partiqular, considering the special case where $\alpha_j = 0$ for all j, we may assume that J_Φ is also a holomorphic function defined on $\mathscr{H} \cup O_{\zeta_o}$. Then, by the argument in the proof of [7; Theorem 1] using the fact that the ring $\mathscr{O}_{\zeta_o}$ of germs of holomorphic functions at ζ_o is a unique factorization domain, it can be shown that every component function Φ_j of Φ is actually holomorphic on some small open neighborhood U_{ζ_o} of ζ_o, as desired. □

By Lemma 3 there exists a connected open neighborhood D of $\overline{\mathscr{H}} \backslash \{0\}$ in $\mathbf{C}^N$ such that Φ extends to a holomorphic mapping $\hat{\Phi} : D \to \mathbf{C}^N$. So, in the following part of this paper, we assume that Φ itself is holomorphic on D and V_Φ is a complex analytic subvariety of D (of $\dim_{\mathbf{C}} V_\Phi = N - 1$ if $V_\Phi \neq \emptyset$).

We now define the subsets $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$ of the boundary $\partial\mathscr{H}$ by setting

$$\begin{aligned}\mathscr{B}_1 &:= \{(z, w) \in \partial\mathscr{H}; \rho^p(z) < \rho^q(w) = 1\}, \\ \mathscr{B}_2 &:= \{(z, w) \in \partial\mathscr{H}; 0 < \rho^p(z) = \rho^q(w) < 1\}, \\ \mathscr{B}_3 &:= \{(z, w) \in \partial\mathscr{H}; \rho^p(z) = \rho^q(w) = 1\}.\end{aligned}$$

Then $\partial\mathscr{H} = \{0\} \cup \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ (disjoint union) and $\mathscr{B}_1$, $\mathscr{B}_2$ are open in $\partial\mathscr{H}$, while $\mathscr{B}_3$ is closed and nowhere dense in $\partial\mathscr{H}$.

Lemma 4. *In the notation above, we have*

$$\Phi(\mathscr{B}_1) \cap \mathscr{B}_2 = \emptyset, \quad \Phi(\mathscr{B}_2) \cap \mathscr{B}_1 = \emptyset \quad \text{and} \quad \Phi(\overline{\mathscr{B}}_1) \subset \overline{\mathscr{B}}_1, \quad \Phi(\mathscr{B}_2) \subset \overline{\mathscr{B}}_2.$$

Proof. To prove the first assertion, assuming the existence of a point $(a,b) \in \mathscr{B}_1$ such that $(\tilde{a},\tilde{b}) := \Phi(a,b) \in \mathscr{B}_2$, we wish to derive a contradiction. To this end, notice that $V_\Phi \cap \partial\mathscr{H}$ is nowhere dense in $\partial\mathscr{H}$. Thus, taking a nearby point of (a,b) if necessary, we may assume that $J_\Phi(a,b) \neq 0$ and every component of (a,b) is non-zero:

$$a_i^\alpha \neq 0 \ (1 \leq i \leq I, 1 \leq \alpha \leq \ell_i); \quad b_j^\mu \neq 0 \ (1 \leq j \leq J, 1 \leq \mu \leq m_j).$$

Accordingly, we can choose a small connected open neighborhood O of (a,b) in such a way that Φ gives rise to a biholomorphic mapping, say again, $\Phi : O \to \Phi(O) =: \tilde{O} \subset \mathbf{C}^N$ with $\Phi(O \cap \mathscr{H}) = \tilde{O} \cap \mathscr{H}$ and $\Phi(O \cap \mathscr{B}_1) = \tilde{O} \cap \mathscr{B}_2$. Without loss of generality, we may further assume that $O \cap \partial\mathscr{H} \subset \mathscr{B}_1$ and $O \cup \tilde{O} \subset (\mathbf{C}^*)^N$. Here define the functions $\gamma(z,w)$ and $r(z,w)$ by

$$\gamma(z,w) = \rho^q(w) - 1, \quad (z,w) \in O; \quad r(z,w) = \rho^p(z) - \rho^q(w), \quad (z,w) \in \tilde{O}.$$

It then follows that $\gamma(z,w)$ (resp. $r(z,w)$) is a C^ω-smooth defining function for $\mathscr{H}$ on the open neighborhood O (resp. $\tilde{O}$) of the point (a,b) (resp. $(\tilde{a},\tilde{b})$). And, by direct calculations we obtain that the complex tangent space $T^c_{(a,b)}(\mathscr{B}_1)$ to $\mathscr{B}_1$ at (a,b) and the Levi form $L_\gamma((a,b);(s,t))$ of γ for $(s,t) \in T^c_{(a,b)}(\mathscr{B}_1)$ are given, respectively, as follows:

$$T^c_{(a,b)}(\mathscr{B}_1) = \left\{ (s,t) \in \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}; \sum_{j=1}^{J} q_j \|b_j\|^{2(q_j-1)} \langle t_j, b_j \rangle = 0 \right\},$$

$$L_\gamma((a,b);(s,t)) = \sum_{j=1}^{J} q_j(q_j-1)\|b_j\|^{2(q_j-2)} |\langle t_j, b_j \rangle|^2 + \sum_{j=1}^{J} q_j \|b_j\|^{2(q_j-1)} \|t_j\|^2 \geq 0 \quad \text{for all } (s,t) \in T^c_{(a,b)}(\mathscr{B}_1)$$

by Schwarz's inequality. Thus $O \cap \mathscr{H}$ is Levi pseudoconvex at $(a,b) \in O \cap \mathscr{B}_1 \subset \partial(O \cap \mathscr{H})$.

On the other hand, the corresponding objects at the point $\Phi(a,b) = (\tilde{a},\tilde{b})$ are given as follows: To simplify discussion, we change notation and write (a,b) in place of $(\tilde{a},\tilde{b})$. Then

$$T^c_{(a,b)}(\mathscr{B}_2) = \left\{ (s,t) \in \mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|}; \sum_{i=1}^{I} p_i \|a_i\|^{2(p_i-1)} \langle s_i, a_i \rangle - \sum_{j=1}^{J} q_j \|b_j\|^{2(q_j-1)} \langle t_j, b_j \rangle = 0 \right\}, \tag{3.3}$$

$$(3.4)\quad L_r((a,b);(s,t)) = \sum_{i=1}^{I} p_i(p_i-1)\|a_i\|^{2(p_i-2)}|\langle s_i, a_i\rangle|^2 + \sum_{i=1}^{I} p_i\|a_i\|^{2(p_i-1)}\|s_i\|^2 - \sum_{j=1}^{J} q_j(q_j-1)\|b_j\|^{2(q_j-2)}|\langle t_j, b_j\rangle|^2 - \sum_{j=1}^{J} q_j\|b_j\|^{2(q_j-1)}\|t_j\|^2 \quad \text{for all } (s,t) \in T^c_{(a,b)}(\mathscr{B}_2).$$

We have now two cases to consider:

Case 1. $|m| = 1$: For the defining function $\gamma(z,w) = \rho^q(w) - 1 = |w|^{2q} - 1$ for $\mathscr{H}$ on the open neighborhood O of the point (a,b), it is easily seen that, for every point $(z,w) \in O \cap \mathscr{B}_1$,

$$T^c_{(z,w)}(\mathscr{B}_1) = \mathbf{C}^{|\ell|} \times \{0\} \quad \text{and} \quad L_\gamma((z,w);(s,t)) = 0, \quad (s,t) \in T^c_{(z,w)}(\mathscr{B}_1),$$

that is, $O \cap \mathscr{B}_1$ is a Levi-flat real hypersurface in $\mathbf{C}^N$ in this case.

Once it is shown that $\tilde{O} \cap \mathscr{B}_2$ is not Levi-flat at $\Phi(a,b) = (\tilde{a}, \tilde{b}) \in \tilde{O} \cap \mathscr{B}_2$, we arrive at a contradiction, since $\Phi: O \to \tilde{O}$ is a biholomorphic mapping with $\Phi(O \cap \mathscr{B}_1) = \tilde{O} \cap \mathscr{B}_2$ and $O \cap \mathscr{B}_1$ is Levi-flat at $(a,b) \in O \cap \mathscr{B}_1$. Therefore we have only to prove that $\tilde{O} \cap \mathscr{B}_2$ is not Levi-flat at $(\tilde{a}, \tilde{b})$. To this end, we again use the notation (a,b) instead of $(\tilde{a}, \tilde{b})$ for a while.

Consider first the case $I = 1$. Then, putting $p = p_1$, $\ell = \ell_1$ and $r = q/p$, we have

$$\mathscr{H} = \{(z,w) \in \mathbf{C}^\ell \times \mathbf{C}; \|z\|^2 < |w|^{2r} < 1\} \quad (\text{as sets});$$

accordingly, we may assume that $p = 1$ from the beginning. Hence the defining function $r(z,w)$ for $\mathscr{H}$ on $\tilde{O}$ has the simple form $r(z,w) = \|z\|^2 - |w|^{2q}$. Note that $\ell \geq 2$ by our assumption $|\ell|\,|m| > 1$. Thus there exists a non-zero element $s \in \mathbf{C}^\ell$ such that $|\langle s, a\rangle| < \|s\|\,\|a\|$. Choose an element $t \in \mathbf{C}$ in such a way that $\langle s, a\rangle = q|b|^{2(q-1)}\bar{b}t$. It then follows from (3.3) and (3.4) that $(s,t) \in T^c_{(a,b)}(\mathscr{B}_2)$ and

$$L_r((a,b);(s,t)) = \{\|s\|^2\|a\|^2 - |\langle s, a\rangle|^2\}/|b|^{2q} > 0;$$

which implies that $\tilde{O} \cap \mathscr{B}_2$ is not Levi-flat at (a,b), as desired.

Consider next the case $I \geq 2$. In this case, we choose two elements $s \in \mathbf{C}^{|\ell|}$ and $t \in \mathbf{C}$ in such a way that

$$s = (s_1, s_2, \ldots, s_I) = (a_1, 0, \ldots, 0) \quad \text{and} \quad t = p_1\|a_1\|^{2p_1}/\{q|b|^{2(q-1)}\bar{b}\}.$$

Then it is obvious that (s,t) is a non-zero element of $T^c_{(a,b)}(\mathscr{B}_2)$ by (3.3). Moreover, since $\sum_{i=1}^{I} \|a_i\|^{2p_i} = |b|^{2q}$, we obtain by (3.4) that

$$L_r((a,b);(s,t)) = p_1^2\|a_1\|^{2p_1}(\|a_2\|^{2p_2} + \cdots + \|a_I\|^{2p_I})/|b|^{2q} > 0;$$

accordingly, $\tilde{O} \cap \mathscr{B}_2$ is not Levi-flat at (a, b), as required. Therefore we have shown that there does not exist a point $(a, b) \in \mathscr{B}_1$ such that $\Phi(a, b) \in \mathscr{B}_2$ in Case 1.

CASE 2. $|m| \geq 2$: If $m_1 \geq 2$, one can choose a non-zero element $t_1 \in \mathbf{C}^{m_1}$ in such a way that $\langle t_1, b_1 \rangle = 0$. Put $t = (t_1, 0, \dots, 0) \in \mathbf{C}^{|m|}$. Then $(0, t) \in T^c_{(a,b)}(\mathscr{B}_2)$ by (3.3) and

$$L_r((a, b); (0, t)) = -q_1 \|b_1\|^{2(q_1-1)} \|t_1\|^2 < 0$$

by (3.4). Thus $\tilde{O} \cap \mathscr{H}$ is not Levi pseudoconvex at the point $\Phi(a, b)$. However, this is a contradiction, since $\Phi : O \to \tilde{O}$ is a biholomorphic mapping with $\Phi(O \cap \mathscr{H}) = \tilde{O} \cap \mathscr{H}$ and $O \cap \mathscr{H}$ is Levi pseudoconvex at $(a, b) \in O \cap \mathscr{B}_1 \subset \partial(O \cap \mathscr{H})$, as shown before.

If $m_1 = 1$, then $m_2 \geq 1$ by our assumption $|m| \geq 2$. Hence there exists a non-trivial solution $(t_1, t_2^1) \in (\mathbf{C}^*)^2$ of the equation

$$q_1 |b_1|^{2(q_1-1)} \bar{b}_1 t_1 + q_2 \|b_2\|^{2(q_2-1)} \bar{b}_2^1 t_2^1 = 0.$$

Put $t = (t_1, t_2, 0, \dots, 0) \in \mathbf{C}^{|m|}$ with $t_2 = (t_2^1, 0, \dots, 0) \in \mathbf{C}^{m_2}$. Then $(0, t) \in T^c_{(a,b)}(\mathscr{B}_2)$ by (3.3) and

$$\begin{aligned} L_r((a, b); (0, t)) &= -q_1(q_1 - 1)|b_1|^{2(q_1-2)} |\bar{b}_1 t_1|^2 - q_1 |b_1|^{2(q_1-1)} |t_1|^2 \\ &\quad - q_2(q_2 - 1)\|b_2\|^{2(q_2-2)} |\bar{b}_2^1 t_2^1|^2 - q_2 \|b_2\|^{2(q_2-1)} |t_2^1|^2 \\ &= -q_1^2 |b_1|^{2(q_1-1)} |t_1|^2 - q_2^2 \|b_2\|^{2(q_2-2)} |b_2^1|^2 |t_2^1|^2 \\ &\quad - q_2 \|b_2\|^{2(q_2-2)} (\|b_2\|^2 - |b_2^1|^2) |t_2^1|^2 \\ &\leq -q_1^2 |b_1|^{2(q_1-1)} |t_1|^2 - q_2^2 \|b_2\|^{2(q_2-2)} |b_2^1|^2 |t_2^1|^2 < 0 \end{aligned}$$

by (3.4); which says that $\tilde{O} \cap \mathscr{H}$ is not Levi pseudoconvex at the point $\Phi(a, b)$, as desired. Therefore we arrive at the same contradiction as above. Eventually, we have shown the first assertion $\Phi(\mathscr{B}_1) \cap \mathscr{B}_2 = \emptyset$ in any cases.

To prove the second assertion, assume that there exists a point $(a, b) \in \mathscr{B}_2$ such that $\Phi(a, b) \in \mathscr{B}_1$. Then, interchanging the role of $\mathscr{B}_1$ and $\mathscr{B}_2$ and repeating exactly the same argument as in the proof of the first assertion, we obtain a contradiction; proving $\Phi(\mathscr{B}_2) \cap \mathscr{B}_1 = \emptyset$. In particular, we see that $\Phi(\mathscr{B}_2) \subset \{0\} \cup \mathscr{B}_2 \cup \mathscr{B}_3 = \overline{\mathscr{B}}_2$.

Finally we claim that $\Phi(\mathscr{B}_1) \not\ni 0$. Indeed, assume to the contrary that there exists a point $(a, b) \in \mathscr{B}_1$ such that $\Phi(a, b) = 0$. Let $\hat{O}$ be an open neighborhood of $0 \in \mathbf{C}^N$ so small that $\hat{O} \cap \overline{\mathscr{B}}_1 = \emptyset$. Since Φ is continuous at (a, b) by Lemma 3, there is an open neighborhood U of (a, b) such that $\Phi(U) \subset \hat{O}$. Take a point $(\hat{a}, \hat{b}) \in U \cap \mathscr{B}_1$ with $J_\Phi(\hat{a}, \hat{b}) \neq 0$. Then there exists a small open neighborhood V of $(\hat{a}, \hat{b})$ such that $V \subset U$ and Φ induces a biholomorphic mapping, say again, $\Phi : V \to \Phi(V)$ with $\Phi(V \cap \mathscr{B}_1) = \Phi(V) \cap \partial\mathscr{H}$. Then, since $\Phi(V \cap \mathscr{B}_1)$ is now a non-empty open subset of $\hat{O} \cap \partial\mathscr{H}$, we have $\Phi(V \cap \mathscr{B}_1) \cap \mathscr{B}_2 \neq \emptyset$.

But, this contradicts the first assertion; proving our claim. Therefore, taking the first assertion into account, we conclude that $\Phi(\mathscr{B}_1) \subset \mathscr{B}_1 \cup \mathscr{B}_3 = \overline{\mathscr{B}}_1$ and hence $\Phi(\overline{\mathscr{B}}_1) \subset \overline{\mathscr{B}}_1$ by the continuity of Φ on $\overline{\mathscr{H}}\backslash\{0\}$. $\square$

Lemma 5. *Let us write $\Phi = (f, g)$ with respect to the coordinate system (z, w) in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$. Then $g : \mathscr{H} \to \mathbf{C}^{|m|}$ does not depend on the variables z; accordingly it has the form $g(z, w) = g(w)$ on $\mathscr{H}$.*

Proof. By the proof of Lemma 4, we can choose a point $(a, b) \in \mathscr{B}_1 \cap (\mathbf{C}^*)^N$ satisfying the following: $J_\Phi(a, b) \neq 0$, $(\tilde{a}, \tilde{b}) := \Phi(a, b) \in \mathscr{B}_1 \cap (\mathbf{C}^*)^N$ and there exist connected open neighborhoods O, $\tilde{O}$ of (a, b), $(\tilde{a}, \tilde{b})$, respectively, with $O \cup \tilde{O} \subset (\mathbf{C}^*)^N$ such that $O \cap \partial\mathscr{H} \subset \mathscr{B}_1$, $\tilde{O} \cap \partial\mathscr{H} \subset \mathscr{B}_1$ and Φ defines a biholomorphic mapping, say again, $\Phi : O \to \tilde{O}$ with $\Phi(O \cap \mathscr{H}) = \tilde{O} \cap \mathscr{H}$ and $\Phi(O \cap \mathscr{B}_1) = \tilde{O} \cap \mathscr{B}_1$. Let P_a (resp. P_b) be a polydisc in $\mathbf{C}^{|\ell|}$ (resp. $\mathbf{C}^{|m|}$) with center a (resp. b) so small that $P_{(a,b)} := P_a \times P_b$ has the compact closure in O. The proof is now divided into two cases as follows:

Case 1. $J = 1$: As a defining function for $\mathscr{B}_1$, one can choose $\rho(z, w) := \|w\|^2 - 1$ in this case. Taking a point $w \in P_b$ with $\|w\|^2 = 1$ arbitrarily, we put $g_w(z) := g(z, w)$, $z \in P_a$, and define $\hat{\rho}(\zeta') := \|g_w(z)\|^2$, $\zeta' = z \in P_a$. Then $\hat{\rho}(\zeta') = 1$ whenever $\|w\|^2 = 1$. Therefore, representing $g = (g_{|\ell|+1}, \dots, g_N)$ with respect to the coordinate system $\zeta'' = (\zeta_{|\ell|+1}, \dots, \zeta_N)$ in $\mathbf{C}^{|m|}$ and differentiating the both sides of the equation $\hat{\rho}(\zeta') = 1$ by ζ_k, $\bar{\zeta}_k$ $(1 \le k \le |\ell|)$, we obtain that, for every point $\zeta'' = w \in P_b$ with $\|\zeta''\|^2 = 1$,

$$\sum_{j=|\ell|+1}^{N} \left| \frac{\partial g_j}{\partial \zeta_k}(\zeta', \zeta'') \right|^2 = 0 \quad \text{for all } \zeta' \in P_a,\ 1 \le k \le |\ell|.$$

Hence, putting $H := \{(\zeta', \zeta'') \in P_{(a,b)}; \|\zeta''\|^2 = 1\}$, we have $\partial g_j(\zeta', \zeta'')/\partial\zeta_k = 0$ on H for every j, k. Since g is holomorphic on $P_{(a,b)}$ and H is a real-analytic hypersurface in $P_{(a,b)}$, it is obvious that every $\partial g_j(\zeta', \zeta'')/\partial\zeta_k = 0$ on $P_{(a,b)}$. Therefore $g(\zeta', \zeta'')$ does not depend on $z = \zeta'$ on $P_{(a,b)}$ and hence on $\mathscr{H}$ by analytic continuation, as desired.

Case 2. $J \ge 2$: In this case, taking a point $w \in P_b$ with $\rho^q(w) = 1$ arbitrarily, we set $g_w(z) = g(z, w)$, $z \in P_a$. Then, since $g_w(P_a) \subset (\mathbf{C}^*)^{|m|}$ by our choice of $\tilde{O}$, we can define a C^ω-smooth plurisubharmonic function $\hat{\rho}$ on P_a by setting $\hat{\rho}(z) := \rho^q(g_w(z))$, $z \in P_a$. It then follows that $\hat{\rho}(z) = 1$ on P_a, since

$$\Phi(P_a \times \{w\}) \subset \Phi(O \cap \mathscr{B}_1) \subset \{(u, v) \in \tilde{O}; \rho^q(v) = 1\}.$$

This combined with the strictly plurisubharmonicity of ρ^q on $(\mathbf{C}^*)^{|m|}$ implies that $g_w(z)$ is a constant mapping on P_a. As a result, defining the real-analytic hypersurface H in P_b by $H := \{w \in P_b; \rho^q(w) = 1\}$, we have shown that

$$g_w(z) = g(z, w) \text{ is constant on } P_a \text{ for any } w \in H. \tag{3.5}$$

Now, the holomorphic mapping g can be expanded uniquely as

$$g(z,w) = g(\zeta', \zeta'') = \sum_{v'} a_{v'}(\zeta'')(\zeta' - \zeta'_o)^{v'}, \quad (\zeta', \zeta'') \in P_{(a,b)},$$

which converges absolutely and uniformly on $P_{(a,b)}$, where $\zeta'_o = a$ and

$$a_{v'}(\zeta'') = (a^1_{v'}(\zeta''), \ldots, a^{|m|}_{v'}(\zeta''))$$

are $|m|$-tuples of holomorphic functions on P_b, and the summation is taken over all $v' = (v_1, \ldots, v_{|\ell|}) \in \mathbf{Z}^{|\ell|}$ with $v_1, \ldots, v_{|\ell|} \geq 0$. Then the assertion (3.5) tells us that

$$a_{v'}(\zeta'') = 0, \quad \zeta'' \in H, \quad \text{for } v' \neq 0.$$

Since $a_{v'}(\zeta'')$ are holomorphic on P_b and H is a real-analytic hypersurface in P_b, we have that $a_{v'}(\zeta'') = 0$ on P_b for $v' \neq 0$; consequently, $g(z,w) = a_0(\zeta'')$ does not depend on $z = \zeta'$ globally by analytic continuation.

Eventually, we have proved that $g(z,w)$ does not depend on z in any cases; thereby, completing the proof. □

4. Proofs of Theorems

Throughout this section, we denote by $\mathscr{E}^p_\ell$ the generalized complex ellipsoid in $\mathbf{C}^{|\ell|}$ as in Theorem 1 and write $\mathscr{E} = \mathscr{E}^p_\ell$. Also, $\mathscr{H}^{p,q}_{\ell,m}$ denotes the generalized Hartogs triangle in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$ as in Theorems 2, 3 and 4 with $|\ell|\,|m| > 1$ and we write $\mathscr{H} = \mathscr{H}^{p,q}_{\ell,m}$ for the sake of simplicity.

The proofs of our theorems will be carried out in the following four Subsections.

4.1. Proof of Theorem 1. Before undertaking the proof, we need a preparation. Let $p_1, \ldots, p_I \geq 1$ be the real numbers appearing in Theorem 1. Assuming that $I \geq 2$ and $\ell_2 = \cdots = \ell_s = 1$ $(2 \leq s \leq I)$ for a while, we consider the correspondence $\pi_{(1, p_2, \ldots, p_s, 1, \ldots, 1)}$ defined by

$$z \mapsto (z_1, (z_2)^{p_2}, \ldots, (z_s)^{p_s}, z_{s+1}, \ldots, z_I), \quad z = (z_1, \ldots, z_I) \in \mathbf{C}^{|\ell|}.$$

If all the p_i's are integers, this is a single-valued holomorphic mapping from $\mathbf{C}^{|\ell|}$ onto itself. However, if some of them are irrationals, then it provides an infinitely-many-valued holomorphic mapping from $\mathbf{C}^{\ell_1} \times (\mathbf{C}^*)^{s-1} \times \mathbf{C}^{\ell_{s+1}} \times \cdots \times \mathbf{C}^{\ell_I}$ onto itself. Thus, for later use, we need to introduce the concept of principal branch of $\pi_{(1, p_2, \ldots, p_s, 1, \ldots, 1)}$. For this purpose, let us fix an arbitrary point

$$z^o = (z^o_1, \ldots, z^o_I) \in \mathbf{C}^{|\ell|} \quad \text{with } z^o_2 \cdots z^o_s \neq 0.$$

Write each z^o_i $(2 \leq i \leq s)$ in the form

$$z^o_i = r^o_i \exp(\sqrt{-1}\theta^o_i) \quad \text{with } r^o_i > 0,\ 0 \leq \theta^o_i < 2\pi$$

and set

$$W_i(z_i^o) = \{z_i = r_i \exp(\sqrt{-1}\theta_i); r_i > 0, |\theta_i - \theta_i^o| < \pi\} = \mathbf{C}\setminus\{tz_i^o; t \le 0\};$$

$$W(z^o) = \mathbf{C}^{\ell_1} \times W_2(z_2^o) \times \cdots \times W_s(z_s^o) \times \mathbf{C}^{\ell_{s+1}} \times \cdots \times \mathbf{C}^{\ell_I};$$

$$\Pi_i(z_i) = (r_i)^{p_i} \exp(\sqrt{-1}p_i\theta_i), \quad z_i = r_i \exp(\sqrt{-1}\theta_i) \in W_i(z_i^o);$$

$$\Pi_{(1,p_2,\dots,p_s,1,\dots,1)}(z) = (z_1, \Pi_2(z_2), \dots, \Pi_s(z_s), z_{s+1}, \dots, z_I)$$

for $z = (z_1, \dots, z_I) \in W(z^o)$. Then $W(z^o)$ is a connected open dense subset of $\mathbf{C}^{|\ell|}$ containing z^o and $\Pi_{(1,p_2,\dots,p_s,1,\dots,1)}$ is a single-valued holomorphic mapping from $W(z^o)$ into $\mathbf{C}^{|\ell|}$. Moreover, it is injective on a small open neighborhood of z^o, since its Jacobian determinant does not vanish at z^o.

Definition. We call this mapping $\Pi_{(1,p_2,\dots,p_s,1,\dots,1)} : W(z^o) \to \mathbf{C}^{|\ell|}$ the *principal branch of* $\pi_{(1,p_2,\dots,p_s,1,\dots,1)}$ *on* $W(z^o)$.

Of course, in the case where $\ell_1 = 1$ as well as $\ell_2 = \cdots = \ell_s = 1$, one can define the *principal branch* $\Pi_{(p_1,p_2,\dots,p_s,1,\dots,1)} : W(z^o) \to \mathbf{C}^{|\ell|}$ *of* $\pi_{(p_1,p_2,\dots,p_s,1,\dots,1)}$ *on* $W(z^o)$ in exactly the same manner as above.

Now we are ready to prove Theorem 1. If $I = 1$, then $\mathscr{E}$ is the unit ball B^{ℓ_1} in $\mathbf{C}^{\ell_1}$ with $\ell_1 \ge 2$. Thus Theorem 1 is nothing but the main theorem of Alexander [1]. So, we assume that $I \ge 2$ in the following part. Accordingly, $\mathscr{E}$ is different from the unit ball and $p_i > 1$ for every $i = 2, \dots, I$. Moreover, in the cases where $\ell_i = 1$ for all $i = 1, \dots, I$ or $p_1 = 1$, $\ell_i = 1$ for $i = 2, \dots, I$, Theorem 1 is an immediate consequence of Dini-Primicerio [11]. Therefore, in order to complete the proof, we have to consider the following five cases:

Case (a). $p_1 = 1$ *and* $\ell_i \ge 2$ $(2 \le i \le I)$: In this case, $\mathscr{E}$ satisfies the condition (‡) in Section 2. On the other hand, by a result of Bell [8], our proper holomorphic mapping $f : \mathscr{E} \to \mathscr{E}$ extends to a holomorphic mapping defined on an open neighborhood D of $\bar{\mathscr{E}}$. Choose a C^ω-smooth strictly pseudoconvex boundary point z^o of $\mathscr{E}$. Then, since $J_f(z^o) \ne 0$ and f is unbranched at z^o (cf. [8]), one can find an open neighborhood V_{z^o} of z^o such that f gives rise to a biholomorphic mapping, say again f, from V_{z^o} onto $f(V_{z^o})$ with $f(V_{z^o} \cap \partial\mathscr{E}) = f(V_{z^o}) \cap \partial\mathscr{E}$. Shrinking V_{z^o} if necessary, we may assume that $O := V_{z^o} \cap \partial\mathscr{E}$ is a connected open subset of $\partial\mathscr{E}$ consisting of strictly pseudoconvex boundary points. Thus, if we define a connected open subset O' of $\partial\mathscr{E}$ by setting $O' := f(V_{z^o}) \cap \partial\mathscr{E}$, then O, O' and f satisfy all the requirements of Theorem B in Section 2; consequently, f is, in fact, a holomorphic automorphism of $\mathscr{E}$.

Case (b). $p_1 = 1$ *and* $\ell_i = 1$, $\ell_j \ge 2$ *for some* $2 \le i, j \le I$: In this case, we may rename the indices so that for some integer s with $2 \le s < I$, one has

$$\ell_2 = \cdots = \ell_s = 1, \quad \text{while } \ell_i \ge 2 \text{ for } s+1 \le i \le I.$$

Choose a point

$$z^o = (z_1^o, \dots, z_I^o) \in \partial\mathscr{E} \quad \text{with } |z_2^o| \cdots |z_s^o| \, \|z_{s+1}^o\| \cdots \|z_I^o\| \neq 0.$$

Then z^o is a C^ω-smooth strictly pseudoconvex boundary point of $\mathscr{E}$ and f is unbranched at z^o. Hence there exist a connected open neighborhood V_{z^o} of z^o and a connected open neighborhood V_{w^o} of $w^o := f(z^o)$ such that f gives rise to a biholomorphic mapping, say again f, from V_{z^o} onto V_{w^o}. In particular, w^o is also a C^ω-smooth strictly pseudoconvex boundary point of $\mathscr{E}$. Therefore, without loss of generality, we may assume that

$$V_{z^o} \cup V_{w^o} \subset \{z \in \mathbf{C}^{|\ell|}; |z_2| \cdots |z_s| \, \|z_{s+1}\| \cdots \|z_I\| \neq 0\}.$$

Consider here the principal branches

$$\Pi_{(1, p_2, \dots, p_s, 1, \dots, 1)} : W(z^o) \to \mathbf{C}^{|\ell|}, \quad \Pi_{(1, p_2, \dots, p_s, 1, \dots, 1)} : W(w^o) \to \mathbf{C}^{|\ell|}$$

and a generalized complex ellipsoid $\hat{\mathscr{E}}$ in $\mathbf{C}^{|\ell|}$ defined by

$$\hat{\mathscr{E}} = \{u \in \mathbf{C}^{|\ell|}; \|u_1\|^2 + \|u_{s+1}\|^{2p_{s+1}} + \cdots + \|u_I\|^{2p_I} < 1\},$$

where $u = (u_1, u_{s+1}, \dots, u_I) \in \mathbf{C}^{\ell_1+s-1} \times \mathbf{C}^{\ell_{s+1}} \times \cdots \times \mathbf{C}^{\ell_I} = \mathbf{C}^{|\ell|}$. Then, shrinking V_{z^o} if necessary, we may further assume that $V_{z^o} \subset W(z^o)$, $V_{w^o} \subset W(w^o)$ and both the restrictions

$$\Pi_{z^o} := \Pi_{(1, p_2, \dots, p_s, 1, \dots, 1)} | V_{z^o} : V_{z^o} \to \Pi_{z^o}(V_{z^o}) \quad \text{and}$$

$$\Pi_{w^o} := \Pi_{(1, p_2, \dots, p_s, 1, \dots, 1)} | V_{w^o} : V_{w^o} \to \Pi_{w^o}(V_{w^o})$$

are biholomorphic mappings. Since $|\Pi_i(z_i)|^2 = |z_i|^{2p_i}$ for $i = 2, \dots, s$, we now have

$$\Pi_{z^o}(V_{z^o} \cap \partial\mathscr{E}) = \Pi_{z^o}(V_{z^o}) \cap \partial\hat{\mathscr{E}} \quad \text{and} \quad \Pi_{w^o}(V_{w^o} \cap \partial\mathscr{E}) = \Pi_{w^o}(V_{w^o}) \cap \partial\hat{\mathscr{E}}.$$

Thus, putting $\hat{O}_{z^o} := \Pi_{z^o}(V_{z^o}) \cap \partial\hat{\mathscr{E}}$, $\hat{O}_{w^o} := \Pi_{w^o}(V_{w^o}) \cap \partial\hat{\mathscr{E}}$, we obtain a biholomorphic mapping

$$\hat{f} := \Pi_{w^o} \circ f \circ \Pi_{z^o}^{-1} : \Pi_{z^o}(V_{z^o}) \to \Pi_{w^o}(V_{w^o})$$

with $\hat{f}(\hat{O}_{z^o}) = \hat{O}_{w^o}$. Notice that the connected open subsets $\hat{O}_{z^o}$, $\hat{O}_{w^o}$ of $\partial\hat{\mathscr{E}}$ are contained in the strictly pseudoconvex part of $\partial\hat{\mathscr{E}}$ and $\hat{f}$ induces a CR-diffeomorphism from $\hat{O}_{z^o}$ onto $\hat{O}_{w^o}$. Also, note that $\hat{\mathscr{E}}$ satisfies the condition (‡) in Section 2. It then follows from Theorem B that $\hat{f}$ extends to a holomorphic automorphism, say again $\hat{f}$, of $\hat{\mathscr{E}}$. Thus we have

$$\hat{f}(\Pi_{z^o}(z)) = \Pi_{w^o}(f(z)) \quad \text{for all } z \in \mathscr{E} \cap W(z^o) \cap f^{-1}(W(w^o)) \tag{4.1}$$

by analytic continuation. Recall here that by Theorem A the holomorphic automorphism $\hat{f}$ has the form

$$\hat{f}(u) = (H(u_1), \gamma_{s+1}(u_1) A_{s+1} u_{\sigma(s+1)}, \dots, \gamma_I(u_1) A_I u_{\sigma(I)}),$$
$$u = (u_1, u_{s+1}, \dots, u_I) \in \hat{\mathscr{E}} \subset \mathbf{C}^{\ell_1+s-1} \times \mathbf{C}^{\ell_{s+1}} \times \cdots \times \mathbf{C}^{\ell_I} = \mathbf{C}^{|\ell|}$$

(think of u_i as column vectors), where $H \in \mathrm{Aut}(B^{\ell_1+s-1})$, γ_i's are nowhere vanishing holomorphic functions on B^{ℓ_1+s-1}, $A_i \in U(\ell_i)$ and σ is a permutation of $\{s+1, \dots, I\}$ satisfying the same conditions as in Theorem A. Now, representing $f = (f_1, \dots, f_I)$ with respect to the given coordinate system $z = (z_1, \dots, z_I)$ in $\mathbf{C}^{\ell_1} \times \cdots \times \mathbf{C}^{\ell_I} = \mathbf{C}^{|\ell|}$, we put

$$z' = (z_1, \dots, z_s), \quad z'' = (z_{s+1}, \dots, z_I); \quad f' = (f_1, \dots, f_s), \quad f'' = (f_{s+1}, \dots, f_I);$$

so that $z = (z', z'')$ and $f = (f', f'')$. Putting $\hat{u}_1 = (z_1, \Pi_2(z_2), \dots, \Pi_s(z_s))$, we then obtain by (4.1) that

$$\begin{aligned} &(f_1(z), \Pi_2(f_2(z)), \dots, \Pi_s(f_s(z))) = H(\hat{u}_1) \quad \text{and} \\ &f''(z) = (\gamma_{s+1}(\hat{u}_1) A_{s+1} z_{\sigma(s+1)}, \dots, \gamma_I(\hat{u}_1) A_I z_{\sigma(I)}) \end{aligned} \tag{4.2}$$

for all $z \in \mathscr{E} \cap W(z^o) \cap f^{-1}(W(w^o))$. Consequently, it follows from the first equation in (4.2) that $f'(z)$ does not depend on the variables z'' on the nonempty open subset $\mathscr{E} \cap W(z^o) \cap f^{-1}(W(w^o))$ of $\mathscr{E}$; and hence, $f'(z)$ has the form $f'(z) = f'(z')$ on $\mathscr{E}$ by analytic continuation. Moreover, notice that the set $\{z = (z', z'') \in W(z^o); z'' = 0\}$ is open dense in $\mathbf{C}^{\ell_1+s-1} \times \{0\} \equiv \mathbf{C}^{\ell_1+s-1}$, where we have put $0 = 0''$ for simplicity. Then by the second equation in (4.2) we have $f''(z) = 0$ for all points $z \in \mathscr{E}$ of the form $z = (z', 0)$. Therefore, if we put

$$\mathscr{E}^{[s]} = \{z' \in \mathbf{C}^{\ell_1+s-1}; \|z_1\|^2 + |z_2|^{2p_2} + \cdots + |z_s|^{2p_s} < 1\}$$

and define $f^{[s]} : \mathscr{E}^{[s]} \to \mathbf{C}^{\ell_1+s-1}$ by

$$f^{[s]}(z') = f'(z') = f'(z', 0) \quad \text{for } z' \in \mathscr{E}^{[s]},$$

then $\mathscr{E}^{[s]}$ is a generalized complex ellipsoid in $\mathbf{C}^{\ell_1+s-1}$ with $\ell_1 + s - 1 \geq 2$, $f^{[s]}(\mathscr{E}^{[s]}) = \mathscr{E}^{[s]}$, and $f^{[s]} : \mathscr{E}^{[s]} \to \mathscr{E}^{[s]}$ is a proper holomorphic mapping; so that $f^{[s]}$ has to be a holomorphic automorphism of $\mathscr{E}^{[s]}$ by Dini-Primicerio [11]. This combined with the fact (4.2) guarantees that the proper holomorphic mapping $f = (f', f'') = (f^{[s]}, f'')$ is injective on $\mathscr{E}$; and hence, it is necessarily a holomorphic automorphism of $\mathscr{E}$, as desired.

Case (c). $p_1 > 1$ *and* $\ell_i \geq 2$ $(2 \leq i \leq I)$: If $\ell_1 = 1$, in the proof of Case (b) we replace z^o, $\Pi_{(1, p_2, \dots, p_s, 1, \dots, 1)}$ and $\hat{\mathscr{E}}$ by a point

$$\tilde{z}^o = (\tilde{z}_1^o, \tilde{z}_2^o, \dots, \tilde{z}_I^o) \in \partial\mathscr{E} \quad \text{with } |\tilde{z}_1^o| \, \|\tilde{z}_2^o\| \cdots \|\tilde{z}_I^o\| \neq 0,$$

the principal branch $\Pi_{(p_1, 1, \dots, 1)} : W(\tilde{z}^o) \to \mathbf{C}^{|\ell|}$, and

$$\tilde{\mathscr{E}} = \{u \in \mathbf{C}^{|\ell|}; |u_1|^2 + \|u_2\|^{2p_2} + \cdots + \|u_I\|^{2p_I} < 1\},$$

where $u = (u_1, u_2, \dots, u_I) \in \mathbf{C} \times \mathbf{C}^{\ell_2} \times \cdots \times \mathbf{C}^{\ell_I} = \mathbf{C}^{|\ell|}$. Then, by repeating the same argument as in Case (b), we see that there exists a holomorphic automorphism

$$\tilde{f}(u) = (H(u_1), \gamma_2(u_1) A_2 u_{\sigma(2)}, \dots, \gamma_I(u_1) A_I u_{\sigma(I)})$$

of $\tilde{\mathscr{E}}$ such that

$$\Pi_1(f_1(z)) = H(\tilde{u}_1) \quad \text{with } \tilde{u}_1 := \Pi_1(z_1), \quad \text{and} \tag{4.3}$$
$$(f_2(z), \ldots, f_I(z)) = (\gamma_2(\tilde{u}_1)A_2 z_{\sigma(2)}, \ldots, \gamma_I(\tilde{u}_1)A_I z_{\sigma(I)})$$

for all $z \in \mathscr{E} \cap W(\tilde{z}^o) \cap f^{-1}(W(\tilde{w}^o))$, where $\tilde{w}^o := f(\tilde{z}^o)$. Thus $f_1(z)$ does not depend on the variables $(z_2, \ldots, z_I)$ and so it has the form $f_1(z) = f_1(z_1)$. Here, observe that the correspondence

$$\Phi : (z_1, z_2, \ldots, z_I) \mapsto (z_1, A_2 z_{\sigma(2)}, \ldots, A_I z_{\sigma(I)})$$

defines an automorphism of $\mathscr{E}$ and the proper holomorphic self-mapping $\Psi := \Phi^{-1} \circ f$ of $\mathscr{E}$ has the form

$$\Psi(z_1, z_2, \ldots, z_I) = (f_1(z_1), \gamma_2(\Pi_1(z_1))z_2, \ldots, \gamma_I(\Pi_1(z_1))z_I) \tag{4.4}$$

on the non-empty open subset $\mathscr{E} \cap W(\tilde{z}^o) \cap f^{-1}(W(\tilde{w}^o))$ of $\mathscr{E}$, since $\gamma_{\sigma(i)}(u_1) = \gamma_i(u_1)$ for $i = 2, \ldots, I$. Thus we may assume from the beginning that $f(z)$ has the form on the right-hand side of (4.4) on $\mathscr{E} \cap W(\tilde{z}^o) \cap f^{-1}(W(\tilde{w}^o))$. Under this assumption, we assert that f can be written in the form

$$f(z) = (f_1(z_1), \lambda_2(z_1)z_2, \ldots, \lambda_I(z_1)z_I) \quad \text{on } \mathscr{E}, \tag{4.5}$$

where λ_i's are nowhere vanishing holomorphic functions on Δ such that

$$\lambda_i(z_1) = \gamma_i(\Pi_1(z_1)), \quad z_1 \in \Delta \cap W_1(\tilde{z}_1^o), \quad 2 \le i \le I.$$

Indeed, this can be seen as follows. First of all, write $f_i = (f_i^1, \ldots, f_i^{\ell_i})$ with respect to the coordinate system $z_i = (z_i^1, \ldots, z_i^{\ell_i})$ in $\mathbf{C}^{\ell_i}$ for $i = 2, \ldots, I$. Being a holomorphic function on the complete Reinhardt domain $\mathscr{E}$, every component function f_i^α can now be expanded uniquely as

$$f_i^\alpha(z) = \sum_{k=0}^{\infty} P_k(z_1; z_2, \ldots, z_I), \quad z \in \mathscr{E},$$

which converges absolutely and uniformly on compact subsets of $\mathscr{E}$, where $P_k(z_1; z_2, \ldots, z_I)$ is a homogeneous polynomial of degree k in $(z_2, \ldots, z_I) = (z_2^1, \ldots, z_I^{\ell_I})$ whose coefficients are all holomorphic functions of z_1 defined on Δ. Then, the fact (4.4) tells us that, for every $k \neq 1$, we have $P_k(z_1; z_2, \ldots, z_I) = 0$ on $\mathscr{E}$ by analytic continuation. Clearly this implies that f can be described as in (4.5) by using some functions λ_i defined on Δ. Moreover, since f is proper, every λ_i cannot vanish at any point of Δ; proving our assertion.

Now, we put

$$\mathscr{E}^{[2]} = \{(z_1, z_2^1) \in \mathbf{C}^2; |z_1|^{2p_1} + |z_2^1|^{2p_2} < 1\}$$

and regard this as a complex submanifold of $\mathscr{E}$ in the canonical manner. Then $f(\mathscr{E}^{[2]}) = \mathscr{E}^{[2]}$ by (4.5) and the correspondence

$$f^{[2]} : (z_1, z_2^1) \mapsto (f_1(z_1), \lambda_2(z_1)z_2^1), \quad (z_1, z_2^1) \in \mathscr{E}^{[2]},$$

gives a proper holomorphic self-mapping of $\mathscr{E}^{[2]}$. It then follows from a result of Dini-Primicerio [11] that $f^{[2]}$ is a holomorphic automorphism of $\mathscr{E}^{[2]}$ and it is, in fact, a linear automorphism of $\mathscr{E}^{[2]}$. In particular, $f_1 \in \mathrm{Aut}(\Delta)$ and $f : \mathscr{E} \to \mathscr{E}$ is injective by (4.5); consequently, f is a holomorphic automorphism of $\mathscr{E}$.

If $\ell_1 \geq 2$, then we have that $p_i > 1$ and $\ell_i \geq 2$ for all $i = 1, \dots, I$. Hence f is a holomorphic automorphism of $\mathscr{E}$ by Lemma 1.

Case (d). *$p_1 > 1$ and $\ell_i = 1$, $\ell_j \geq 2$ for some $2 \leq i, j \leq I$:* As in Case (b) we may assume that

$$\ell_i = 1 \ \ (2 \leq i \leq s) \quad \text{and} \quad \ell_j \geq 2 \ \ (s+1 \leq i \leq I)$$

for some integer s with $2 \leq s < I$.

If $\ell_1 = 1$, in the proof of Case (b) we replace z^o and $\Pi_{(1, p_2, \dots, p_s, 1, \dots, 1)}$ by a point

$$\tilde{z}^o = (\tilde{z}_1^o, \tilde{z}_2^o, \dots, \tilde{z}_I^o) \in \partial\mathscr{E} \quad \text{with } |\tilde{z}_1^o| \cdots |\tilde{z}_s^o| \, \|\tilde{z}_{s+1}^o\| \cdots \|\tilde{z}_I^o\| \neq 0$$

and the principal branch $\Pi_{(p_1, \dots, p_s, 1, \dots, 1)} : W(\tilde{z}^o) \to \mathbf{C}^{|\ell|}$. Then, by a small change of the proof in Case (b), one can see that f is a holomorphic automorphism of $\mathscr{E}$.

If $\ell_1 \geq 2$, then we consider a holomorphic automorphism $\varphi(z) = u$ of $\mathbf{C}^{|\ell|}$ induced by the change of coordinates

$$u = (u_1, \dots, u_{s-1}, u_s, u_{s+1}, \dots, u_I) = (z_2, \dots, z_s, z_1, z_{s+1}, \dots, z_I).$$

Then the image domain $\mathscr{E}^* = \varphi(\mathscr{E})$ is given by

$$\mathscr{E}^* = \{u \in \mathbf{C}^{|\ell|}; |u_1|^{2p_2} + \cdots + |u_{s-1}|^{2p_s} + \|u_s\|^{2p_1} + \|u_{s+1}\|^{2p_{s+1}} + \cdots + \|u_I\|^{2p_I} < 1\}.$$

Thus, the proof of showing $f \in \mathrm{Aut}(\mathscr{E})$ in the case $s = 2$ (resp. $s \geq 3$) can be reduced to that in the Case (c), $\ell_1 = 1$ (resp. Case (d), $\ell_1 = 1$, above).

Case (e). *$p_1 > 1$, $\ell_1 \geq 2$ and $\ell_i = 1$ $(2 \leq i \leq I)$:* In this case, after the change of coordinates

$$u = (u_1, \dots, u_{I-1}, u_I) = (z_2, \dots, z_I, z_1),$$

our $\mathscr{E}$ can be represented as

$$\mathscr{E} = \{u \in \mathbf{C}^{|\ell|}; |u_1|^{2p_2} + \cdots + |u_{I-1}|^{2p_I} + \|u_I\|^{2p_1} < 1\}$$

in the new coordinates $(u_1, \dots, u_I)$. Thus, in the case $I = 2$ (resp. $I \geq 3$), by the same argument as in the Case (c), $\ell_1 = 1$ (resp. Case (d)), we can check that f is a holomorphic automorphism of $\mathscr{E}$; proving the theorem in Case (e).

Eventually, we have proved that f is necessarily a holomorphic automorphism of $\mathscr{E}$ in any cases; thereby, completing the proof of Theorem 1. □

4.2. Proof of Theorem 2. It is obvious that the mapping Φ written in the form as in Theorem 2 is a holomorphic automorphism (and hence, proper holomorphic self-mapping) of $\mathscr{H}$. Conversly, take an arbitrary proper holomorphic self-mapping Φ of $\mathscr{H}$. Once it is shown that Φ is a holomorphic automorphism of $\mathscr{H}$, then Theorem 2 is an immediate consequence of our previous result [16; Theorem 2]. Therefore we have only to prove that Φ is a holomorphic automorphism of $\mathscr{H}$. To this end, write $\Phi = (\Phi_1, \dots, \Phi_N)$ with respect to the coordinate system $\zeta = (\zeta_1, \dots, \zeta_N)$ in $\mathbf{C}^N$. Since $|m| \geq 2$, we see that the Reinhardt domain $\mathscr{H}$ satisfies the condition that $\mathscr{H} \cap \{\zeta \in \mathbf{C}^N; \zeta_i = 0\} \neq \emptyset$ for each $1 \leq i \leq N$. Hence every component function Φ_i extends to a unique holomorphic function $\hat{\Phi}_i$ defined on $\mathscr{E}_\ell^p \times \mathscr{E}_m^q$ (cf. [21; p. 15]). Accordingly, we obtain a holomorphic extension $\hat{\Phi} := (\hat{\Phi}_1, \dots, \hat{\Phi}_N) : \mathscr{E}_\ell^p \times \mathscr{E}_m^q \to \mathbf{C}^N$ of Φ. Let us now represent again $\Phi = (f, g)$ and $f = (f_1, \dots, f_I)$, $g = (g_1, \dots, g_J)$ by coordinates $(z, w) = (z_1, \dots, z_I, w_1, \dots, w_J)$ in $\mathbf{C}^{|\ell|} \times \mathbf{C}^{|m|} = \mathbf{C}^N$ and denote by $\hat{f}$, $\hat{g}$ the holomorphic extensions of f, g to $\mathscr{E}_\ell^p \times \mathscr{E}_m^q$, respectively. Since $g(z, w)$ does not depend on the variables z by Lemma 5, $\hat{g}$ has the form $\hat{g}(z, w) = \hat{g}(w)$. Moreover, $\hat{g}(\bar{\mathscr{E}}_m^q) \subset \bar{\mathscr{E}}_m^q$, $\hat{g}(\partial \mathscr{E}_m^q) \subset \partial \mathscr{E}_m^q$ by Lemma 4 and $\hat{g}(0) \notin \partial \mathscr{E}_m^q$ by the maximum principle for the continuous plurisubharmonic function $\rho^q(\hat{g}(w))$ on $\mathscr{E}_m^q$. Thus $\hat{g}(\mathscr{E}_m^q) \subset \mathscr{E}_m^q$ and $\hat{g} : \mathscr{E}_m^q \to \mathscr{E}_m^q$ is a proper holomorphic mapping. Hence, by Theorem 1 $\hat{g}$ is a holomorphic automorphism of $\mathscr{E}_m^q$ with $\hat{g}(0) = 0$; and by Theorem A it can be written in the form

$$\hat{g}(w) = (B_1 w_{\tau(1)}, \dots, B_J w_{\tau(J)}), \quad w = (w_1, \dots, w_J) \in \mathscr{E}_m^q, \tag{4.6}$$

where $B_j \in U(m_j)$ and τ is a permutation of $\{1, \dots, J\}$ such that $\tau(j) = t$ if and only if $(m_j, q_j) = (m_t, q_t)$.

Now we wish to prove that Φ is, in fact, a holomorphic automorphism of $\mathscr{H}$. To this end, let us introduce a holomorphic automorphism Ψ of $\mathscr{H}$ defined by $\Psi(z, w) := (z, \hat{g}^{-1}(w))$. Then, replacing Φ by $\Psi \circ \Phi$ if necessary, we may assume that Φ has the form $\Phi(z, w) = (f(z, w), w)$ on $\mathscr{H}$. Therefore, if we set

$$\mathscr{E}_w = \{z \in \mathbf{C}^{|\ell|}; \rho^p(z) < \rho^q(w)\} \quad \text{and} \quad f_w(z) = f(z, w), \; z \in \mathscr{E}_w,$$

for an arbitrarily given point $w \in \mathscr{E}_m^q \backslash \{0\}$, then it is obvious that f_w induces a proper holomorphic self-mapping of $\mathscr{E}_w$. On the other hand, putting

$$r_i = 1/(\rho^q(w))^{1/(2p_i)} \quad (1 \leq i \leq I),$$

we have a biholomorphic mapping $\Lambda : \mathscr{E}_w \to \mathscr{E}_\ell^p$ defined by

$$\Lambda(z) = (r_1 z_1, \dots, r_I z_I), \quad z = (z_1, \dots, z_I) \in \mathscr{E}_w.$$

Recall that $\mathscr{E}_\ell^p$ is the unit ball $B^{|\ell|}$ or a generalized complex ellipsoid in $\mathbf{C}^{|\ell|}$ with $|\ell| \geq 2$, $\mathbf{R} \ni p_i \geq 1$ $(1 \leq i \leq I)$ according to $I = 1$ or $I \geq 2$. Then, being a proper holomorphic self-mapping of $\mathscr{E}_\ell^p$, the composite mapping $\Lambda \circ f_w \circ \Lambda^{-1} : \mathscr{E}_\ell^p \to \mathscr{E}_\ell^p$ must be a holomorphic automorphism of $\mathscr{E}_\ell^p$ by Alexander [1] or Theorem 1. In particular, we see that $f_w : \mathscr{E}_w \to \mathscr{E}_w$ is injective for any $w \in \mathscr{E}_m^q \backslash \{0\}$; accordingly, $\Phi(z, w) = (f_w(z), w)$ itself is injective on $\mathscr{H}$. Therefore we conclude that Φ is actually a holomorphic automorphism of $\mathscr{H}$, as desired. □

4.3. Proof of Theorem 3. Clearly, the mapping Φ having the form as in Theorem 3 is a holomorphic automorphism (and hence, proper holomorphic self-mapping) of $\mathscr{H}$. Therefore, taking an arbitrary proper holomorphic self-mapping Φ of $\mathscr{H}$, we would like to prove that Φ can be written in the form as in Theorem 3. For this purpose, we begin with noting the following: Since $|m| \geq 2$, by the same reasoning as in the proof of Theorem 2, every holomorphic function $h(\zeta)$ on $\mathscr{H}$ extends uniquely to a holomorphic function $\hat{h}(\zeta)$ on $\Delta \times \mathscr{E}_m^q$, where Δ is the unit disc in $\mathbf{C}$. Since $q_j \geq 1$ $(1 \leq j \leq J)$, $\Delta \times \mathscr{E}_m^q$ is a geometrically convex domain in $\mathbf{C}^N$; and hence, it is a pseudoconvex domain. Thus $\Delta \times \mathscr{E}_m^q$ is just the envelope of holomorphy of $\mathscr{H}$; accordingly, $|\hat{h}(\zeta)| \leq K$ on $\Delta \times \mathscr{E}_m^q$ if $|h(\zeta)| \leq K$ on $\mathscr{H}$ (cf. [21; p. 93]). In particular, our proper holomorphic mapping $\Phi = (\Phi_1, \ldots, \Phi_N) = (f, g)$ extends to a unique holomorphic mapping $\hat{\Phi} := (\hat{\Phi}_1, \ldots, \hat{\Phi}_N) = (\hat{f}, \hat{g})$ from $\Delta \times \mathscr{E}_m^q$ to $\mathbf{C}^N$ with $|\hat{\Phi}_j(\zeta)| \leq 1$ on $\Delta \times \mathscr{E}_m^q$ for every $j = 1, \ldots, N$. Moreover, since $\hat{g}$ has the form $\hat{g}(z, w) = \hat{g}(w)$ by Lemma 5, in exactly the same way as in the proof of Theorem 2, one can prove that $\hat{g}$ is a holomorphic automorphism of $\mathscr{E}_m^q$ of the form (4.6); and so $\hat{\Phi}$ is a holomorphic self-mapping of $\Delta \times \mathscr{E}_m^q$ with $\hat{\Phi}(0,0) = (0,0)$, as seen by taking the limit $(z, w) \to (0,0)$ through $\mathscr{H}$. In particular, we have $\hat{f}(0,0) = 0$. Anyway, in order to prove Theorem 3, we may again assume that Φ has the form $\Phi(z, w) = (f(z, w), w)$ on $\mathscr{H}$.

Under the situation above, the only thing which has to be proved now is that $f(z, w)$ can be written in the form $f(z, w) = Az$ on $\mathscr{H}$, where $A \in \mathbf{C}$ with $|A| = 1$. To verify this, we need a few preparation. First of all, since our $\Phi(z, w) = (f(z, w), w)$ is holomorphic on some open neighborhood of $\overline{\mathscr{H}} \backslash \{0\}$ by Lemma 3, one can choose a small $\varepsilon > 0$ in such a way that Φ is holomorphic on the Reinhardt domain Γ_ε defined by

$$\Gamma_\varepsilon = \{(z, w) \in \mathbf{C} \times \mathbf{C}^{|m|}; |z| < 1 + \varepsilon, 1 - \varepsilon < \rho^q(w) < 1 + \varepsilon\} \supset \overline{\mathscr{B}}_2.$$

Since $|m| \geq 2$, Γ_ε also satisfies the condition that $\Gamma_\varepsilon \cap \{\zeta \in \mathbf{C}^N; \zeta_i = 0\} \neq \emptyset$ for each $1 \leq i \leq N$; and hence, Φ extends to a unique holomorphic mapping $\tilde{\Phi} : O_\varepsilon \to \mathbf{C}^N$, where O_ε is the bounded Reinhardt domain in $\mathbf{C} \times \mathbf{C}^{|m|}$ given by

$$O_\varepsilon = \{(z, w) \in \mathbf{C} \times \mathbf{C}^{|m|}; |z| < 1 + \varepsilon, \rho^q(w) < 1 + \varepsilon\} \supset \overline{\Delta \times \mathscr{E}_m^q}.$$

Therefore we may assume that our extension $\hat{\Phi}(z, w) = (\hat{f}(z, w), w)$ is holomorphic on O_ε. Then, being a holomorphic function on the Reinhardt domain O_ε containing the origin $0 = (0,0)$ in $\mathbf{C} \times \mathbf{C}^{|m|} = \mathbf{C}^N$, $\hat{f}$ can be expanded uniquely as a power series

$$\hat{f}(z, w) = \hat{f}(\zeta) = \sum_\nu A_\nu \zeta^\nu, \quad A_\nu = \frac{1}{\nu!} \frac{\partial^{|\nu|} \hat{f}(0)}{\partial \zeta_1^{\nu_1} \cdots \partial \zeta_N^{\nu_N}}, \tag{4.7}$$

which converges absolutely and uniformly on compact subsets of O_ε (in particular, on $\overline{\Delta \times \mathscr{E}_m^q}$), where the summation is taken over all $\nu = (\nu_1, \ldots, \nu_N) \in \mathbf{Z}^N$ with $\nu_1, \ldots, \nu_N \geq 0$.

Now, recall that $\Phi(\mathscr{B}_2) \subset \bar{\mathscr{B}}_2$ by Lemma 4; and so $\hat{\Phi}(\bar{\mathscr{B}}_2) \subset \bar{\mathscr{B}}_2$. Accordingly

$$|\hat{f}(z,w)|^{2p} = \rho^q(w) \quad \text{whenever } |z|^{2p} = \rho^q(w) \leq 1;$$

and so

$$(4.8) \quad |\hat{f}((\rho^q(w))^{1/2p}, w_1 \exp(\sqrt{-1}\theta_1), \dots, w_J \exp(\sqrt{-1}\theta_J))|^2 = (\rho^q(w))^{1/p}$$

for any $(z,w) \in \bar{\mathscr{B}}_2$ and $\theta_j = (\theta_j^1, \dots, \theta_j^{m_j}) \in \mathbf{R}^{m_j}$, where we have put

$$w_j \exp(\sqrt{-1}\theta_j) = (w_j^1 \exp(\sqrt{-1}\theta_j^1), \dots, w_j^{m_j} \exp(\sqrt{-1}\theta_j^{m_j}))$$

for $j = 1, \dots, J$. Notice that this equation (4.8) holds also for any point $w \in \mathbf{C}^{|m|}$ with $\rho^q(w) \leq 1$, because one can always find a point $z \in \mathbf{C}$ such that $(z,w) \in \bar{\mathscr{B}}_2$. Therefore, writting $A_\nu = A_{a\alpha}$ for $\nu = (a,\alpha) \in \mathbf{Z} \times \mathbf{Z}^{|m|}$ in (4.7), we obtain that

$$(\rho^q(w))^{1/p} = \sum_{a,b,\alpha} A_{a\alpha}\bar{A}_{b\alpha}(\rho^q(w))^{(a+b)/2p}|w_1^{\alpha_1}|^2 \cdots |w_J^{\alpha_J}|^2,$$

which converges absolutely and uniformly on $\overline{\mathscr{E}_m^q}$, where

$$\alpha = (\alpha_1, \dots, \alpha_J) \quad \text{with } \alpha_j = (\alpha_j^1, \dots, \alpha_j^{m_j}),$$
$$w_j^{\alpha_j} = (w_j^1)^{\alpha_j^1} \cdots (w_j^{m_j})^{\alpha_j^{m_j}} \quad \text{for } 1 \leq j \leq J,$$

and the summation is taken over all $0 \leq a, b \in \mathbf{Z}$, $\alpha = (\alpha_1, \dots, \alpha_J) \in \mathbf{Z}^{|m|}$ with $\alpha_j^k \geq 0$ $(1 \leq j \leq J, 1 \leq k \leq m_j)$. Hence, considering the special case where

$$w = (w_1, w_2, \dots, w_J) = (w_1, 0, \dots, 0) \quad \text{with } w_1 = (\xi, 0, \dots, 0),\ \xi \in \mathbf{C};$$
$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_J) = (\alpha_1, 0, \dots, 0) \quad \text{with } \alpha_1 = (\lambda, 0, \dots, 0),\ \lambda \in \mathbf{Z}$$

and writting $A_{a\alpha} = c_{a\lambda}$, we obtain that, for any $\xi \in \mathbf{C}$ with $|\xi| \leq 1$,

$$(4.9) \quad |\xi|^{2q_1/p} = \sum_{\lambda \geq 1} |c_{0\lambda}|^2 |\xi|^{2\lambda} + \sum_{\mu \geq 1} 2\,\mathrm{Re}(c_{1\mu}\bar{c}_{0\mu})|\xi|^{q_1/p+2\mu} + |c_{10}|^2|\xi|^{2q_1/p}$$
$$+ \sum_{a+b=k \geq 3} c_{a0}\bar{c}_{b0}|\xi|^{kq_1/p} + \sum_{\lambda \geq 1, a+b=k \geq 2} c_{a\lambda}\bar{c}_{b\lambda}|\xi|^{kq_1/p+2\lambda},$$

since $c_{00} = \hat{f}(0) = 0$. Thus

$$(4.10) \quad \lim_{\xi \to 0} (\text{the right-hand side of } (4.9))/|\xi|^{2q_1/p} = 1.$$

Note that if we define the holomorphic function $h(z,\xi)$ by

$$h(z,\xi) = \hat{f}(z,\xi,0,\dots,0) \quad \text{on } \{(z,\xi) \in \mathbf{C}^2; |z| < 1+\varepsilon, |\xi| < 1+\varepsilon\},$$

then the Taylor expansion of $h(z,\xi)$ is given by $h(z,\xi) = \sum_{a,\lambda} c_{a\lambda} z^a \xi^\lambda$, which converges absolutely and uniformly on $\overline{\Delta^2}$. Moreover it should be remarked that, since $|h(z,\xi)| \le 1$ on $\overline{\Delta^2}$, Gutzmer's inequality assures us that

$$(4.11)\qquad \sum_{a,\lambda=0}^{\infty} |c_{a\lambda}|^2 r^{2a}\rho^{2\lambda} \le 1, \quad 0 \le r,\rho \le 1; \quad \text{and so} \quad \sum_{a,\lambda=0}^{\infty} |c_{a\lambda}|^2 \le 1.$$

Now we assert that

$$(4.12)\qquad \begin{aligned} &c_{a\lambda} = 0 \quad \text{for all } (a,\lambda) \ne (1,0), \quad \text{and} \\ &h(z,\xi) = c_{10}z \quad \text{with } |c_{10}| = |\partial \hat{f}(0)/\partial z| = 1. \end{aligned}$$

For the verification of this, we have two cases to consider:

1) $q_1/p \notin \mathbf{N}$: Notice that $2\lambda \ne 2q_1/p$ and $q_1/p + 2\mu \ne 2q_1/p$ for any $\lambda,\mu \in \mathbf{N}$ in this case. Hence, it follows from (4.9) and (4.10) that $|c_{10}|^2 = 1$. This combined with the inequality (4.11) yields at once that $c_{a\lambda} = 0$ for all $(a,\lambda) \ne (1,0)$ and so $h(z,\xi) = c_{10}z$ with $|c_{10}| = 1$, as asserted.

2) $q_1/p \in \mathbf{N}$: If $q_1/(2p) \notin \mathbf{N}$, then the term of $|\xi|^{2q_1/p}$ does not appear in the second summation on the right-hand side of (4.9). Hence, by (4.9) and (4.10) we obtain that $|c_{10}|^2 + |c_{0\lambda_o}|^2 = 1$ with $\lambda_o = q_1/p$; and so $c_{a\lambda} = 0$ for all $(a,\lambda) \ne (1,0),(0,\lambda_o)$ by (4.11).

If $q_1/(2p) \in \mathbf{N}$, then we put $\mu_o = q_1/(2p)$. Note that the terms of $|\xi|^{2\lambda}$ $(\lambda \le \mu_o)$ do not appear in the second summation, since $q_1/p + 2\mu \ge q_1/p + 2$ for any $\mu \in \mathbf{N}$. Then $|c_{0\lambda}|^2 = 0$ for all $\lambda \le \mu_o$ by (4.9) and (4.10); consequently, $2\,\mathrm{Re}(c_{1\mu_o}\bar{c}_{0\mu_o})|\xi|^{2q_1/p} = 0$ and the second summation does not contain the term of $|\xi|^{2q_1/p}$. Thus, by the same reasoning as above, we obtain that $|c_{10}|^2 + |c_{0\lambda_o}|^2 = 1$ and $c_{a\lambda} = 0$ for all $(a,\lambda) \ne (1,0),(0,\lambda_o)$. Therefore, in any cases, h can be written in the form

$$h(z,\xi) = c_{10}z + c_{0\lambda_o}\xi^{\lambda_o} \quad \text{with } |c_{10}|^2 + |c_{0\lambda_o}|^2 = 1.$$

Recall that $|c_{10}z + c_{0\lambda_o}\xi^{\lambda_o}| = |h(z,\xi)| \le 1$ for any $(z,\xi) \in \overline{\Delta^2}$. Clearly this can only happen when $|c_{10}| + |c_{0\lambda_o}| \le 1$; and so $|c_{10}|\,|c_{0\lambda_o}| = 0$. Here assume that $c_{10} = 0$. Then

$$\hat{\Phi}(z,w_1^1,0,\dots,0) = (c_{0\lambda_o}(w_1^1)^{\lambda_o}, w_1^1, 0,\dots,0)$$

does not depend on the variables z. But, this is absurd, because if we put

$$\mathscr{H}^{[2]} = \{(z,w_1^1) \in \mathbf{C}^2; |z|^{2p} < |w_1^1|^{2q_1} < 1\},$$

which is regarded as a complex submanifold of $\mathscr{H}$ in the canonical manner, and consider the correspondence $\Phi^{[2]} : (z,w_1^1) \mapsto \hat{\Phi}(z,w_1^1,0,\dots,0)$, then $\Phi^{[2]}$ induces a proper holomorphic self-mapping of $\mathscr{H}^{[2]}$. Therefore $c_{0\lambda_o} = 0$, $|c_{10}| = 1$ and $h(z,\xi) = c_{10}z$. As a result, we have verified our assertion (4.12) in any cases.

Finally, we shall complete the proof by showing that $f(z,w)$ has the form required in the theorem. For this purpose, recall that $\hat{\Phi}$ is a holomorphic self-mapping of the bounded Reinhardt domain $\Delta \times \mathscr{E}_m^q$ with $\hat{\Phi}(0,0) = (0,0)$. In addition to this, we have

$$|J_{\hat{\Phi}}(0,0)| = |\partial \hat{f}(0,0)/\partial z| = |c_{10}| = 1$$

by (4.12). Consequently, by well-known theorems of H. Cartan, $\hat{\Phi}$ is a holomorphic automorphism of $\Delta \times \mathscr{E}_m^q$ and it is, in fact, linear (cf. [14; pp. 268–270]). Moreover, by considering the holomorphic automorphism $\Lambda := \Psi^{-1} \circ \hat{\Phi}$ of $\Delta \times \mathscr{E}_m^q$, where Ψ is a holomorphic automorphism of $\Delta \times \mathscr{E}_m^q$ defined by $\Psi(z, w) := (c_{10} z, w)$, it is easily seen that Λ is a linear automorphism of $\Delta \times \mathscr{E}_m^q$ having the form

$$\Lambda(\zeta) = (\zeta' + M\zeta'', \zeta''), \quad \zeta = (\zeta', \zeta'') = (z, w) \in \Delta \times \mathscr{E}_m^q,$$

(think of ζ as column vectors), where M is a certain $1 \times |m|$ matrix. Thus, denoting by Λ^n the n-th iteration of Λ, we have

$$\Lambda^n(\zeta) = (\zeta' + nM\zeta'', \zeta''), \quad \zeta \in \Delta \times \mathscr{E}_m^q, \quad n = 1, 2, \ldots.$$

Hence M has to be the zero matrix, that is, Λ is the identity transformation of $\Delta \times \mathscr{E}_m^q$, since $\{\Lambda^n\}_{n=1}^{\infty}$ is contained in the isotropy subgroup K_0 of $\mathrm{Aut}(\Delta \times \mathscr{E}_m^q)$ at the origin $0 = (0, 0) \in \Delta \times \mathscr{E}_m^q$ and K_0 is compact, as is well-known.

Eventually, we have shown that $\hat{\Phi}$ has the form required in Theorem 3; thereby completing the proof. □

4.4. Proof of Theorem 4. By routine computations we can check that the transformation Φ appearing in Theorem 4 induces a proper holomorphic self-mapping of $\mathscr{H}$ in any cases (cf. [13; p. 212]). Conversly, we take an arbitrary proper holomorphic mapping $\Phi : \mathscr{H} \to \mathscr{H}$ and write $\Phi = (f, g)$ with respect to the coordinate system (z, w) in $\mathbf{C}^{|\ell|} \times \mathbf{C}$. Then g does not depend on the variables z by Lemma 5; and so it has the form $g(z, w) = g(w)$. Since g is a holomorphic function defined on some open neighborhood of $\overline{\Delta} \backslash \{0\}$ with $g(\partial\Delta) \subset \partial\Delta$ by Lemma 4 and since g is bounded on Δ^*, g now extends to a holomorphic function $\hat{g}$ defined on some open neighborhood of $\overline{\Delta}$ with $\hat{g}(\overline{\Delta}) \subset \overline{\Delta}$. Moreover, $\hat{g}(0) \notin \partial\Delta$ by the maximum principle. Accordingly, $\hat{g}$ gives rise to a proper holomorphic self-mapping of Δ and it is a finite Blaschke product. Since $\hat{g} = g$ on Δ^*, it is easily checked that $\hat{g}(w_o) = 0$ only when $w_o = 0$. Thus $\hat{g}$ must be of the form

$$\hat{g}(w) = Bw^k \quad \text{for some } k \in \mathbf{N},\ B \in \mathbf{C} \text{ with } |B| = 1.$$

Therefore, taking the composite mapping $\Psi \circ \Phi$ instead of Φ if necessary, where Ψ is the automorphism of $\mathscr{H}$ defined by $\Psi(z, w) = (z, B^{-1}w)$, we may assume that Φ has the form $\Phi(z, w) = (f(z, w), w^k)$ on $\mathscr{H}$. We have two cases to consider:

Case I. $I = 1$: In this case, putting $r = q/p$, we have

$$\begin{aligned} \mathscr{H}_{\ell_1, 1}^{p, q} &= \{(z, w) \in \mathbf{C}^{\ell_1} \times \mathbf{C}; \|z\|^{2p} < |w|^{2q} < 1\} \\ &= \{(z, w) \in \mathbf{C}^{\ell_1} \times \mathbf{C}; \|z\|^{2} < |w|^{2r} < 1\} = \mathscr{H}_{\ell_1, 1}^{1, r}. \end{aligned}$$

Taking this into account, we shall divide the proof into two subcases as follows:

CASE (I.1). $r \in \mathbf{N}$: We have a biholomorphic mapping $\Lambda : \mathscr{H} \to B^{\ell_1} \times \Delta^*$ defined by

$$\Lambda(z, w) = (z/w^r, w), \quad (z, w) \in \mathscr{H}.$$

Thus the composite mapping

$$\Psi := \Lambda \circ \Phi \circ \Lambda^{-1} : B^{\ell_1} \times \Delta^* \to B^{\ell_1} \times \Delta^*$$

gives a proper holomorphic self-mapping of $B^{\ell_1} \times \Delta^*$. Recall that $\ell_1 \geq 2$. Then Ψ can be written in the form

$$\Psi(\xi, \eta) = (H(\xi), G(\eta)), \quad (\xi, \eta) \in B^{\ell_1} \times \Delta^*,$$

by making use of some proper holomorphic mappings $H : B^{\ell_1} \to B^{\ell_1}$ and $G : \Delta^* \to \Delta^*$ (cf. [21; p. 77]). Therefore, by the main theorem of Alexander [1], H is a holomorphic automorphism of B^{ℓ_1} and Φ can be described as

$$\Phi(z, w) = (w^{kr} H(z/w^r), w^k), \quad (z, w) \in \mathscr{H};$$

which proves our assertion in (I.1) of Theorem 4.

CASE (I.2). $r \notin \mathbf{N}$: We set

$$\mathscr{E}_w = \{z \in \mathbf{C}^{\ell_1}; \|z\|^2 < |w|^{2r}\}, \quad f_w(z) = f(z, w), \ z \in \mathscr{E}_w,$$

for an arbitrarily given point $w \in \Delta^*$. Then f_w induces a proper holomorphic mapping from $\mathscr{E}_w$ onto $\mathscr{E}_{w^k}$. On the other hand, we have a biholomorphic mapping $\Lambda_w : \mathscr{E}_w \to B^{\ell_1}$ defined by

$$\Lambda_w(z) = z/w^r, \quad z \in \mathscr{E}_w,$$

where w^r stands for the branch of the power function w^r such that $1^r = 1$ when we consider it as a function of w. Hence the composite mapping

$$\Psi_w := \Lambda_{w^k} \circ f_w \circ \Lambda_w^{-1} : B^{\ell_1} \to B^{\ell_1}$$

is a proper holomorphic self-mapping of B^{ℓ_1} with $\ell_1 \geq 2$; consequently, it follows again from the main theorem of Alexander [1] that Ψ_w is a holomorphic automorphism of B^{ℓ_1}. Moreover, since Ψ_w depends holomorphically on w, Ψ_w does not depend on the choice of w by the proof of [2; Theorem 2]. Therefore f_w can be written in the form

$$f_w(z) = w^{kr} H(z/w^r), \quad z \in \mathscr{E}_w,$$

by using some element $H \in \mathrm{Aut}(B^{\ell_1})$. Once it is shown that $H(0) = 0$, H must be a unitary transformation, i.e., H has the form $H(\xi) = A\xi$ on B^{ℓ_1} with some $A \in U(\ell_1)$. Then

$$f(z, w) = f_w(z) = w^{(k-1)r} Az \quad \text{on } \mathscr{H}.$$

Moreover, since $f(z, w)$ is a single-valued holomorphic function on $\mathscr{H}$, it is easily seen that $(k-1)r \in \mathbf{Z}$; proving our assertion in (I.2) of Theorem 4. Therefore we have only to verify that $H(0) = 0$. To this end, we assume that $H(0) \neq 0$. Then, since

$$f(0, \exp(\sqrt{-1}\theta)w_o) = \exp(\sqrt{-1}kr\theta)w_o^{kr}H(0), \quad -\pi < \theta < \pi,$$

where w_o is a fixed real number with $0 < w_o < 1$, and

$$\lim_{\theta \downarrow -\pi} f(0, \exp(\sqrt{-1}\theta)w_o) = f(0, -w_o) = \lim_{\theta \uparrow \pi} f(0, \exp(\sqrt{-1}\theta)w_o),$$

it follows at once that $kr \in \mathbf{N}$. Moreover, choose a point $z_o \in \mathscr{E}_w$, $z_o \neq 0$, in such a way that $H(z_o/w^r) \neq 0$ for all $1/2 \leq |w| < 1$ and consider the function $f(z_o, \exp(\sqrt{-1}\theta)w_o)$ of $\theta \in (-\pi, \pi)$, where w_o is a real number such that $(z_o, w_o) \in \mathscr{H}^{1,r}_{\ell_1,1}$ and $1/2 \leq w_o < 1$. Then, noting the facts that $kr \in \mathbf{N}$ and $H \in \mathrm{Aut}(B^{\ell_1})$, we obtain that

$$z_o/\{\exp(\sqrt{-1}\pi r)w_o^r\} = z_o/\{\exp(-\sqrt{-1}\pi r)w_o^r\}$$

by taking the limit $\theta \to \pm\pi$ as above; so that $r \in \mathbf{N}$. But, this contradicts our assumption $r \notin \mathbf{N}$. Thus $H(0) = 0$ and Φ has to be of the form required in (I.2) of Theorem 4.

Case II. $I \geq 2$: In this case, if we set

$$\mathscr{E}_w = \{z \in \mathbf{C}^{|\ell|}; \rho^p(z) < |w|^{2q}\}, \quad f_w(z) = f(z, w), \ z \in \mathscr{E}_w,$$

for an arbitrarily given point $w \in \Delta^*$, then f_w induces a proper holomorphic mapping from $\mathscr{E}_w$ onto $\mathscr{E}_{w^k}$. On the other hand, we have a biholomorphic mapping $\Lambda_w : \mathscr{E}_w \to \mathscr{E}^p_\ell$ defined by

$$\Lambda_w(z) = (z_1/w^{q/p_1}, \dots, z_I/w^{q/p_I}), \quad z = (z_1, \dots, z_I) \in \mathscr{E}_w.$$

Thus the composite mapping

$$\Psi_w := \Lambda_{w^k} \circ f_w \circ \Lambda_w^{-1} : \mathscr{E}^p_\ell \to \mathscr{E}^p_\ell$$

is a proper holomorphic self-mapping of the generalized complex ellipsoid $\mathscr{E}^p_\ell$ with $1 \leq p_i \in \mathbf{R}$ $(1 \leq i \leq I)$; consequently, Ψ_w is a holomorphic automorphism of $\mathscr{E}^p_\ell$ by Theorem 1. Moreover, by the same reasoning as in Case (I.2), Ψ_w does not depend on w. Therefore, according to Theorem A, we shall consider two cases where $p_1 = 1$ and $p_1 \neq 1$ separately.

Consider first the case where $p_1 = 1$. Then, applying Theorem A, Case I to the holomorphic automorphism $\Psi := \Psi_w$ of $\mathscr{E}^p_\ell$, we can see that f_w has the form

$$\begin{aligned} f_w(z) = (&w^{kq}H(z_1/w^q), w^{(k-1)q/p_2}\gamma_2(z_1/w^q)A_2 z_{\sigma(2)}, \dots, \\ &w^{(k-1)q/p_I}\gamma_I(z_1/w^q)A_I z_{\sigma(I)}), \end{aligned} \tag{4.13}$$

since $p_{\sigma(i)} = p_i$ $(2 \leq i \leq I)$, where $H \in \mathrm{Aut}(B^{\ell_1})$, $A_i \in U(\ell_i)$, σ is a permutation of $\{2, \dots, I\}$ and γ_i's are nowhere vanishing holomorphic functions on B^{ℓ_1} given as in Theorem A, Case I. Hence we obtain the following:

Case (II.1). $p_1 = 1$, $q \in \mathbf{N}$: In this case, $w^{kq}H(z_1/w^q)$ and $\gamma_i(z_1/w^q)$ are single-valued holomorphic functions on $\mathscr{H}$ as well as $f(z,w)$. Therefore we have $(k-1)q/p_i \in \mathbf{Z}$ for all $i = 2, \dots, I$; proving our assertion (II.1) of Theorem 4.

Case (II.2). $p_1 = 1$, $q \notin \mathbf{N}$: In this case, we put

$$\mathscr{H}^{[2]} = \{(z_1, w) \in \mathbf{C}^{\ell_1} \times \mathbf{C}; \|z_1\|^2 < |w|^{2q} < 1\}$$

and regard this as a complex submanifold of $\mathscr{H}$ in the canonical manner. Then $\Phi(\mathscr{H}^{[2]}) = \mathscr{H}^{[2]}$ by (4.13) and the restriction $\Phi|\mathscr{H}^{[2]} : \mathscr{H}^{[2]} \to \mathscr{H}^{[2]}$ gives a proper holomorphic mapping. Consequently, by the proof of (I.2) above, $H \in \mathrm{Aut}(B^{\ell_1})$ appearing in (4.13) has to satisfy the condition $H(0) = 0$ and it reduces to a unitary transformation $H(\xi) = A\xi$ on B^{ℓ_1} given by some $A \in U(\ell_1)$. Notice that every function $\gamma_i(\xi) = 1$ on B^{ℓ_1} in this case. Thus we conclude that f_w has the form

$$f_w(z) = (w^{(k-1)q}Az_1, w^{(k-1)q/p_2}A_2 z_{\sigma(2)}, \dots, w^{(k-1)q/p_I}A_I z_{\sigma(I)})$$

with $(k-1)q/p_i \in \mathbf{Z}$ for all $i = 1, \dots, I$; thereby, Φ has the form required in (II.2) of Theorem 4.

Consider next the case where $p_1 \neq 1$. Then, applying Theorem A, Case II to the holomorphic automorphism Ψ, we can see that f_w has the form

$$f_w(z) = (w^{(k-1)q/p_1}A_1 z_{\sigma(1)}, \dots, w^{(k-1)q/p_I}A_I z_{\sigma(I)}),$$

since $p_{\sigma(i)} = p_i$ for every $i = 1, \dots, I$, where $A_i \in U(\ell_i)$ and σ is a permutation of $\{1, \dots, I\}$ as in Theorem A, Case II. Moreover, since $f(z,w)$ is a single-valued holomorphic function on $\mathscr{H}$, it is obvious that $(k-1)q/p_i \in \mathbf{Z}$ for all $i = 1, \dots, I$; which proves our assersion (II.3) of Theorem 4.

Finally, by recalling our previous results [16], [17] on the structure of holomorphic automorphism groups of generalized Hartogs triangles, it is easy to see that the proper holomorphic self-mapping Φ of $\mathscr{H}$ appearing in Theorem 4 is a holomorphic automorphism of $\mathscr{H}$ if and only if $k = 1$ in any cases.

Therefore the proof of Theorem 4 is now completed. □

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