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A NOTE ON THE ASYMPTOTIC BEHAVIOR OF CONFORMAL METRICS WITH NEGATIVE CURVATURES NEAR ISOLATED SINGULARITIES

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Abstract

The asymptotic behavior of conformal metrics with negative curvatures near an isolated singularity was described up to the second order derivatives by Kraus and Roth, 2008. We refine Kraus and Roth's result for the second order mixed derivatives and give estimates for higher order derivatives near an isolated singularity. We also compute the Minda-type limits for SK-metrics near the singularity. Combining these limits with Ahlfors' lemma, we provide two observations for SK-metrics.

1. Introduction

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$, $\mathbf{D}^* := \mathbf{D} \backslash \{0\}$, $\mathbf{D}_R := \{z \in \mathbf{C} : |z| < R\}$ and $\mathbf{D}_R^* := \mathbf{D}_R \backslash \{0\}$ for $R > 0$. If $G \subseteq \mathbf{C}$ is a domain, then every positive, upper semi-continuous function $\lambda : G \to (0, +\infty)$ on G induces a conformal metric $\lambda(z)|dz|$ (see [3, 4]), and $\lambda(z)$ is called the density of $\lambda(z)|dz|$. If $\lambda(z)|dz|$ is a regular conformal metric on G, i.e. $\lambda(z)$ is strictly positive and twice continuously differentiable on G, then the Gaussian curvature $\kappa_\lambda(z)$ of the density $\lambda(z)$ is defined by

$$\kappa_\lambda(z) = -\frac{\Delta \log \lambda(z)}{\lambda(z)^2}, \tag{1.1}$$

where Δ denotes the Laplace operator (see [11]).

Let $u(z) := \log \lambda(z)$. If $\kappa_\lambda(z) \equiv 0$, then $u(z)$ satisfies the Laplace equation $\Delta u = 0$, which means $u(z)$ is harmonic on G, so that the property of $u(z)$ can be studied by means of potential theory (see, e.g. [10]). If $\kappa_\lambda(z) \equiv -4$, and if $\lambda(z)|dz|$ is complete, then $\lambda(z)|dz|$ is the hyperbolic metric on G and (1.1) becomes the Liouville equation

$$\Delta u = 4e^{2u}. \tag{1.2}$$

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The asymptotic behavior of a solution $u(z)$ to (1.2) near a singularity was described by Nitsche [9]. If $\kappa_\lambda(z)$ is not a constant but a strictly negative, locally Hölder continuous function, (1.2) becomes the more general equation

$$\Delta u = -\kappa(z)e^{2u}. \tag{1.3}$$

Kraus and Roth [5] studied the asymptotic behavior of $u(z)$ near an isolated singularity. Each solution to (1.3) belongs to a class of subharmonic functions and it is corresponding to a special metric, called the SK-metric, according to Heins [3].

The existence and the uniqueness of the solutions to (1.3) are subject to suitable boundary conditions. In this article we are concerned with the asymptotic behavior of the solution to (1.3) near an isolated singularity, so it is sufficient to consider the behavior on $\mathbf{D}^*$, where the origin is an isolated singularity of order $\alpha \le 1$.

We denote

$$\partial^n = \frac{\partial^n}{\partial z^n}, \quad \bar{\partial}^n = \frac{\partial^n}{\partial \bar{z}^n}$$

for $n \ge 1$. The following theorem was proved by Kraus and Roth [5].

THEOREM A [5]. *Let $\kappa : \mathbf{D} \to \mathbf{R}$ be a (locally) Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbf{D}^* \to \mathbf{R}$ is a C^2-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbf{D}^*$, then u has an order $\alpha \in (-\infty, 1]$ at the origin. Define the remainder functions $v(z)$ and $w(z)$ by*

$$\begin{aligned} u(z) &= -\alpha \log|z| + v(z), && \text{if } \alpha < 1, \\ u(z) &= -\log|z| - \log\log(1/|z|) + w(z), && \text{if } \alpha = 1, \end{aligned}$$

according to the value of α. Then $v(z)$ and $w(z)$ are continuous in $\mathbf{D}$. Moreover, the first partial derivatives with respect to z and $\bar{z}$,

$$\partial v(z),\ \bar{\partial} v(z) \ \text{are continuous at } z = 0 \quad \text{if } \alpha < 1/2;$$

and

$$\begin{aligned} \partial v(z), \bar{\partial} v(z) &= O(1) && \text{if } \alpha = 1/2; \\ \partial v(z), \bar{\partial} v(z) &= O(|z|^{1-2\alpha}) && \text{if } 1/2 < \alpha < 1, \\ \partial w(z), \bar{\partial} w(z) &= O(|z|^{-1}(\log(1/|z|))^{-2}) && \text{if } \alpha = 1, \end{aligned}$$

when z approaches 0, with O being the Landau symbols. In addition, the second partial derivatives,

$$\partial^2 v(z),\ \partial\bar{\partial} v(z) \ \text{and} \ \bar{\partial}^2 v(z) \ \text{are continuous at } z = 0 \quad \text{if } \alpha \le 0;$$

and

$$\partial^2 v(z), \partial\bar{\partial} v(z), \bar{\partial}^2 v(z) = O(|z|^{-2\alpha}) \quad \text{if } 0 < \alpha < 1, \tag{1.4}$$

$$\partial^2 w(z), \partial\bar{\partial} w(z), \bar{\partial}^2 w(z) = O(|z|^{-2}(\log(1/|z|))^{-2}) \quad \text{if } \alpha = 1, \tag{1.5}$$

when z *tends to* 0.

Our first result gives estimates for higher order derivatives of $v(z)$, $w(z)$ near the singularity, and then improve the estimate of the mixed derivatives in Theorem A when the order $\alpha = 1$. The Hölder space $C^{m,\nu}(\mathbf{D}_R)$ consists of functions whose m-th order partial derivatives are locally Hölder continuous in $\mathbf{D}_R$ with exponent ν, $0 < \nu \le 1$, which is defined as a subspace of $C^m(\mathbf{D}_R)$.

Theorem 1.1. *Let* $\kappa(z)$, $u(z)$, $v(z)$, $w(z)$ *and* α *be the same as in Theorem* A. *If, in addition,* $\kappa \in C^{n-2,\nu}(\mathbf{D})$ *for an integer* $n \ge 2$, *and a real number* $0 < \nu \le 1$, *then for* $n_1, n_2 \ge 1$, $n_1 + n_2 = n$, *near the origin,* $v(z)$ *and* $w(z)$ *satisfy*

$$\partial^n v(z), \bar{\partial}^n v(z), \partial^{n_1}\bar{\partial}^{n_2} v(z) = O(|z|^{2-2\alpha-n}) \quad \text{if } 0 < \alpha < 1; \tag{1.6}$$

$$\partial^n w(z), \bar{\partial}^n w(z) = O(|z|^{-n}(\log(1/|z|))^{-2}) \quad \text{if } \alpha = 1, \tag{1.7}$$

$$\partial^{n_1}\bar{\partial}^{n_2} w(z) = O(|z|^{-n}(\log(1/|z|))^{-3}) \quad \text{if } \alpha = 1. \tag{1.8}$$

The higher order estimates in Theorem 1.1 are best. It can be verified by the generalized hyperbolic metric on the thrice-punctured sphere, see [12]. This article is a continuation of [12].

The hyperbolic metric is a complete metric with the negative constant Gaussian curvature, here we take the constant to be -4. Minda [8] investigated the behavior of the density of the hyperbolic metric in a neighborhood of a puncture of a plane domain using the uniformization theorem in 1997. Kraus and Roth extended Minda's limit for the case of variable curvature and obtained the following theorem for a conformal metric in 2008, where the limit for cusps is related to Minda's work.

Theorem B [5, 8]. *Let* $\lambda(z)|dz|$ *be a regular conformal metric on* $\mathbf{D}^*$ *with an isolated singularity at* $z = 0$. *Suppose that its curvature* $\kappa : \mathbf{D}^* \to \mathbf{R}$ *has a Hölder continuous extension to* $\mathbf{D}$ *such that* $\kappa(0) < 0$. *Then* $\log \lambda$ *has an order* $\alpha \le 1$ *at* $z = 0$ *and*

$$\lim_{z \to 0} |z| \log(1/|z|)\lambda(z) = \begin{cases} 0 & \text{if } \alpha < 1 \\ \dfrac{1}{\sqrt{-\kappa(0)}} & \text{if } \alpha = 1. \end{cases}$$

Our second result extends Theorem B and gives limits of Minda type.

THEOREM 1.2. *Let* $\kappa : \mathbf{D} \to \mathbf{R}$ *be of class* $C^{n-2,\nu}(\mathbf{D})$ *for an integer* $n \geq 2$, $0 < \nu \leq 1$ *with* $\kappa(0) < 0$. *If* $u : \mathbf{D}^* \to \mathbf{R}$ *is a* $C^{n,\nu}$*-solution to* $\Delta u = -\kappa(z)e^{2u}$ *in* $\mathbf{D}^*$, *then* u *has an order* $\alpha \in (-\infty, 1]$. *If the order of* u *is* $\alpha = 1$, *then for* $n_1, n_2 \geq 0$, $n_1 + n_2 \leq n$, *the limit*

$$l_{n_1,n_2} := \frac{1}{n_1!n_2!} \lim_{z\to 0} |z| \log(1/|z|) z^{n_1} \bar{z}^{n_2} \partial^{n_1} \bar{\partial}^{n_2} \lambda(z) \tag{1.9}$$

exists. Moreover, the numbers l_{n_1,n_2} *are given by*

$$l_{n_1,n_2} = \binom{-\frac{1}{2}}{n_1}\binom{-\frac{1}{2}}{n_2} \frac{1}{\sqrt{-\kappa(0)}},$$

where

$$\binom{\tau}{j} = \frac{\tau(\tau-1)\cdots(\tau-j+1)}{j!}$$

is the binomial coefficient.

In Section 2 we give a short survey of the hyperbolic metrics and the SK-metrics, and introduce some notations and definitions. The proof of Theorem 1.1 for corners and cusps is given in Section 3 by the use of potential theory. In Section 4 we first give a lemma for $u(z)$, then prove Theorem 1.2, and list several results of Minda-type for special metrics.

2. Preliminaries

Let X be a Riemann surface and Ω be a subdomain of X. For a point $p \in \Omega$, let z be local coordinates such that $z(p) = 0$. We say a conformal metric $\lambda(z)|dz|$ on the punctured domain $\Omega^* := \Omega \backslash \{p\}$ has a singularity of order $\alpha \leq 1$ at the point p, if, in local coordinates z,

$$\log \lambda(z) = \begin{cases} -\alpha \log|z| + O(1) & \text{if } \alpha < 1 \\ -\log|z| - \log\log(1/|z|) + O(1) & \text{if } \alpha = 1, \end{cases} \tag{2.1}$$

as $z(p) \to 0$. We call the point p a conical singularity (or corner) of order α if $\alpha < 1$ and a cusp if $\alpha = 1$. The generalized Gaussian curvature $\kappa_\lambda(z)$ of the density function $\lambda(z)$ is defined by

$$\kappa_\lambda(z) = -\frac{1}{\lambda(z)^2} \liminf_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} \log \lambda(z + re^{it})\, dt - \log \lambda(z)\right). \tag{2.2}$$

When $\lambda(z)$ is a regular function, the Gaussian curvature $\kappa_\lambda(z)$ is equivalent to definition (1.1) (see [12]). The Gaussian curvature defined by (2.2) is a conformal invariant. Suppose that $\lambda(z)|dz|$ is a conformal metric on a domain $G \in \mathbf{C}$ and $f : \Omega \to G$ is a holomorphic mapping of a Riemann surface Ω into

G. Then we can define the pullback $f^*\lambda(w)|dw|$ of $\lambda(z)|dz|$ by

$$f^*\lambda(w)|dw| := \lambda(f(w))|f'(w)|\,|dw|.$$

It is easy to see that $f^*\lambda(w)|dw|$ is a conformal metric on $\Omega\backslash\{$critical points of $f\}$ with Gaussian curvature $\kappa_{f^*\lambda}(w) = \kappa_\lambda(f(w))$. Using this conformal invariance, we can easily establish the relation between Riemann surfaces with conformal metrics. Here we can see that, on the punctured domain $\Omega\backslash\{$critical points of $f\}$, the critical points of f are the source of the singularities of negative integer orders.

We call an upper semi-continuous metric $\lambda(z)|dz|$ on a Riemann surface Ω an SK-metric if its Gaussian curvature is bounded above by -4 at every $z \in \Omega$. The hyperbolic metric on the unit disk $\mathbf{D}$ is given by

$$\lambda_{\mathbf{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}. \tag{2.3}$$

The following result is a fundamental theorem about SK-metrics by Ahlfors [1], also discussed by Heins [3], which claims that the hyperbolic metric $\lambda_{\mathbf{D}}(z)|dz|$ on the unit disk $\mathbf{D}$ is the unique maximal SK-metric on $\mathbf{D}$.

Theorem C [1]. *Let $\lambda_{\mathbf{D}}(z)|dz|$ be the hyperbolic metric on $\mathbf{D}$ given in* (2.3) *and $\lambda(z)|dz|$ be an SK-metric on $\mathbf{D}$. Then the inequality $\lambda(z) \le \lambda_{\mathbf{D}}(z)$ holds throughout the disk.*

On the punctured unit disk $\mathbf{D}^*$, the hyperbolic metric is expressed by

$$\lambda_{\mathbf{D}^*}(z)|dz| = \frac{|dz|}{2|z|\log(1/|z|)}$$

with the constant curvature -4. On the punctured disk $\mathbf{D}_R^*$, the hyperbolic metric with a conical singularity at the origin is given as follows, which is the conical version of Theorem C.

Theorem D [6, 11]. *For $R > 0$, let*

$$\lambda_{\alpha,R}(z) := \begin{cases} \dfrac{(1-\alpha)R^{1-\alpha}|z|^{-\alpha}}{R^{2(1-\alpha)} - |z|^{2(1-\alpha)}} = \dfrac{1-\alpha}{2|z|\sinh((1-\alpha)\log(R/|z|))} & \text{if } \alpha < 1, \\ \dfrac{1}{2|z|\log(R/|z|)} & \text{if } \alpha = 1 \end{cases}$$

for $z \in \mathbf{D}_R^*$. Then given an arbitrary SK-metric $\sigma(z)$ on $\mathbf{D}_R^*$ with a singularity at $z = 0$ of order α, we have $\sigma(z) \le \lambda_{\alpha,R}(z)$.

3. Proof of Theorem 1.1

We shall use potential theory as employed by Kraus and Roth [5]. Here we recall elementary facts without proof (see [2, 10] for details).

If equation (1.3) has a C^2-solution $u(z)$, then higher differentiability of $u(z)$ may follow from smoothness of $\kappa(z)$ according to the regularity of elliptic differential equations, see [2, p. 109]. For a bounded, integrable function $f(z)$ on a domain $\Omega \subseteq \mathbf{C}$, the integral

$$\frac{1}{2\pi}\int_{\Omega} L(z-\zeta)f(\zeta)\,d\sigma_\zeta$$

is called the logarithmic potential of f, where

$$L(z) = \log|z|,$$

$d\sigma_\zeta$ is the area element on the domain Ω. Write $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$ and let $0 < r \le 1$. The following lemma was mentioned in [5]. It is a consequence of the Riesz decomposition theorem, and can be obtained from Theorem 4.5.1 and Exercise 3.7.3 in [10].

Lemma E [5]. *Let u be a subharmonic function on $\mathbf{D}$. Suppose that $u \in C^2(\mathbf{D}^*)$ and Δu is integrable in $\mathbf{D}^*$ and*

$$\lim_{r\to 0}\frac{\sup_{|z|=r} u(z)}{\log(1/r)} = 0.$$

Then $u = h + \omega$ on $\mathbf{D}$, where h is a harmonic function on $\mathbf{D}$ and ω is the logarithmic potential of Δu.

To describe the higher order derivatives of the logarithmic potential, we use a multi-index $\boldsymbol{j} = (j_1, j_2)$, $|\boldsymbol{j}| = j_1 + j_2$, $j_1, j_2 = 0, 1, 2, \ldots$, so $(\zeta - z)^{\boldsymbol{j}} = (y_1 - x_1)^{j_1}(y_2 - x_2)^{j_2}$, $\boldsymbol{j}! = j_1!j_2!$. For $z = x_1 + ix_2$, denote

$$\frac{\partial}{\partial x_1} = \partial_1, \quad \frac{\partial}{\partial x_2} = \partial_2, \quad \partial^{\boldsymbol{j}} = \partial_1^{j_1}\partial_2^{j_2}.$$

For a given multi-index $\boldsymbol{j} = (j_1, j_2)$, we can choose $\boldsymbol{e}_\tau = (0,1)$ or $(1,0)$ for $\tau = 1, 2, \ldots$ such that $\boldsymbol{j} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_n$ with $n = |\boldsymbol{j}|$. For a function $f \in C^{|\boldsymbol{j}|}$, write $\zeta = y_1 + iy_2$, and define $P_n[f]$ by

$$P_n[f](\zeta, z) := \begin{cases} \displaystyle\sum_{|\boldsymbol{a}|\le n}\frac{(\zeta - z)^{\boldsymbol{a}}}{\boldsymbol{a}!}\partial^{\boldsymbol{a}} f(z) & \text{if } n \ge 1 \\ f(z) & \text{if } n = 0, \end{cases}$$

where $\boldsymbol{a}$ is a multi-index. For $m = 1, 2$, we obtain that

$$\frac{\partial}{\partial y_m}P_n[f](\zeta, z) = P_{n-1}[\partial_m f](\zeta, z),$$

see [11] for more details.

Now we introduce a function class $H_\lambda^{n,\nu}(M)$. For an integer n, two numbers ν, λ with $0 < \nu \le 1$ and a given M,

$$H_\lambda^{n,\nu}(M) := \left\{ f \in C^n(\mathbf{D}^*) : |\partial^{\boldsymbol{a}} f(z+w) - \partial^{\boldsymbol{a}} f(z)| \le M \frac{|w|^\nu}{|z|^{\lambda+n}} \right.$$
$$\left. \text{whenever } 2|w| < |z| < 1 \text{ for all } \mathbf{a} \text{ with } |\mathbf{a}| = n \right\}.$$

We can estimate the error term in the Taylor expansion for $f \in H_\lambda^{n,\nu}(M)$.

LEMMA 3.1. *Suppose that* $f \in H_\lambda^{n,\nu}(M)$. *Then for a fixed point* z, $0 < |z| < 1$,

$$|f(\zeta) - P_n[f](\zeta, z)| \le C \frac{|z-\zeta|^{n+\nu}}{|z|^{\lambda+n}}, \quad \textit{for } |\zeta - z| < \frac{|z|}{2}, \tag{3.1}$$

where C *is a positive constant depending only on* M *and* n.

Proof. Let $g(t) = f(z + t(\zeta - z))$, $0 \le t \le 1$. Then $g(1) = f(\zeta)$, $g(0) = f(z)$, and

$$g^{(k)}(t) = \sum_{|\boldsymbol{a}|=k} \frac{k!}{\boldsymbol{a}!} (\zeta - z)^{\boldsymbol{a}} \partial^{\boldsymbol{a}} f(z + t(\zeta - z)) \tag{3.2}$$

for $k \le n$. By Taylor's theorem, there exists $\xi \in [0,1]$ such that

$$f(\zeta) = g(1) = \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(n)}(\xi)}{n!}.$$

Substitute (3.2) and obtain that

$$\begin{aligned} f(\zeta) &= \sum_{k=1}^{n-1} \sum_{|\boldsymbol{a}|=k} \frac{(\zeta - z)^{\boldsymbol{a}} \partial^{\boldsymbol{a}} f(z)}{\boldsymbol{a}!} + \frac{g^{(n)}(\xi) - g^{(n)}(0) + g^{(n)}(0)}{n!} \\ &= \sum_{|\boldsymbol{a}| \le n} \frac{(\zeta - z)^{\boldsymbol{a}} \partial^{\boldsymbol{a}} f(z)}{\boldsymbol{a}!} + \frac{g^{(n)}(\xi) - g^{(n)}(0)}{n!} \\ &= P_n[f](\zeta, z) + \sum_{|\boldsymbol{a}|=n} \frac{(\zeta - z)^{\boldsymbol{a}}}{\boldsymbol{a}!} (\partial^{\boldsymbol{a}} f(z + \xi(\zeta - z)) - \partial^{\boldsymbol{a}} f(z)). \end{aligned}$$

Since $f \in H_\lambda^{n,\nu}(M)$,

$$|\partial^{\boldsymbol{a}} f(z + \xi(\zeta - z)) - \partial^{\boldsymbol{a}} f(z)| \le M \frac{|\zeta - z|^\nu}{|z|^{\lambda+n}}.$$

Thus

$$\left| \sum_{|\boldsymbol{a}|=n} \frac{(\zeta - z)^{\boldsymbol{a}}}{\boldsymbol{a}!} (\partial^{\boldsymbol{a}} f(z + \xi(\zeta - z)) - \partial^{\boldsymbol{a}} f(\zeta)) \right| \le C \frac{|\zeta - z|^{\nu+n}}{|z|^{\lambda+n}}$$

and (3.1) holds when $|\zeta - z| < |z|/2$. ∎

Using the multi-index, we can present the following result (see [2, p. 54] for the case $m = 0$ and [11] for others).

LEMMA F [2, 11]. *Let* $0 < r < R$, $f \in C^{n-2,\nu}(\mathbf{D}_r)$ *with* $0 < \nu \le 1$, *and let* $n \ge 2$, ω *be the logarithmic potential of* f. *Then* $\omega \in C^n(\mathbf{D}_r)$ *and for a multi-index* $\boldsymbol{j}$, $|\boldsymbol{j}| = n$,

$$\partial^{\boldsymbol{j}}\omega(z) = \frac{1}{2\pi}\int_{\mathbf{D}_R} \partial^{\boldsymbol{j}} L(z-\zeta)\cdot (f(\zeta) - P_{n-2}[f](\zeta,z))\,d\sigma_\zeta \tag{3.3}$$

$$-\frac{1}{2\pi}\sum_{\tau=1}^{n-1}\int_{\partial\mathbf{D}_R} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta,z)\cdot\langle N(\zeta),\boldsymbol{e}_{\tau+1}\rangle |d\zeta|,$$

where $\boldsymbol{\theta}_\tau := \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_\tau$, $\boldsymbol{\phi}_\tau := \boldsymbol{e}_{\tau+2} + \cdots + \boldsymbol{e}_n$ *for* $\tau = 1, \dots, n-1$ *and* $\boldsymbol{\phi}_{n-1} := (0,0)$. *Further,* $N(\zeta) = (N_1(\zeta), N_2(\zeta))$ *is the unit outward normal at the point* $\zeta \in \partial\mathbf{D}_R$ *with* $R > r$, $\langle\,,\rangle$ *is the inner product and the function* f *is extended to vanish outside of* $\mathbf{D}_r$.

Now Theorem 1.1 can be divided into two parts, the case $0 < \alpha < 1$ and the case $\alpha = 1$. We prove it separately. First we consider $v(z)$.

Proof of Theorem 1.1 *when* $0 < \alpha < 1$. Let

$$q(z) = -\kappa(z)e^{2v(z)}, \quad f(z) = q(z)|z|^{-2\alpha}.$$

We use induction on n to prove (1.6) and that $q \in H_\beta^{n-2,\nu}(M_n)$, where $\beta = \max\{0, 2\alpha - 2 + \nu\}$, M_n is a positive constant depending on n. When $n = 2$, (1.6) follows from Theorem A. For z and s with $0 < 2|s| < |z| < 1/2$, by Theorem A we have

$$|q(z+s) - q(z)| \le |\kappa(z+s) - \kappa(z)|\,|e^{2v(z+s)}| + |e^{2v(z+s)} - e^{2v(z)}|\,|\kappa(z)| \tag{3.4}$$

$$\le C_1|s|^\nu + C_2\frac{|s|}{|z|^{\beta'}} \le \frac{M_2\cdot|s|^\nu}{|z|^\beta},$$

where $\beta' = \max\{0, 2\alpha - 1\}$, M_2, C_1 and C_2 are positive constants. Thus $q \in H_\beta^{0,\nu}(M_2)$.

For $n \le K-1$, $K \ge 3$, we assume that (1.6) holds, and $q \in H_\beta^{n-2,\nu}(M)$. We consider $|\partial^{\boldsymbol{a}}q(z+s) - \partial^{\boldsymbol{a}}q(z)|$ for $\boldsymbol{a}$, $|\boldsymbol{a}| = K-2$, and (1.6) for $n = K$. Since (1.6) is true for $K-1$, the same technique as in (3.4) leads to

$$|\partial^{\boldsymbol{a}}q(z+s) - \partial^{\boldsymbol{a}}q(z)| \le \frac{M_K\cdot|s|^\nu}{|z|^{|\boldsymbol{a}|+\beta}}, \quad |a| = K-2. \tag{3.5}$$

We denote $\max\{M_2, \dots, M_K\}$ by M_K without loss of generality. Then (3.5) shows that $q \in H_\beta^{K-2,\nu}(M_K)$. It is easy to verify that, by induction,

$$\partial^{\boldsymbol{a}}f(z) = \sum_{|\boldsymbol{a}_1|+|\boldsymbol{a}_2|=|\boldsymbol{a}|} C_{\boldsymbol{a}_1,\boldsymbol{a}_2}\frac{z^{\boldsymbol{a}_1}\cdot\partial^{\boldsymbol{a}_2}q(z)}{|z|^{2\alpha+2|\boldsymbol{a}_1|}}$$

where $C_{\boldsymbol{a}_1,\boldsymbol{a}_2}$ is a constant. Let

$$g_{\boldsymbol{a}}(z) = \frac{z^{\boldsymbol{a}}}{|z|^{2\alpha+2|\boldsymbol{a}|}}, \quad 0 \le |\boldsymbol{a}| \le K-2.$$

Thus for z and s, $0 < |z| < 1/2$, $|s| < |z|/2$,

$$|g_{\boldsymbol{a}}(z+s) - g_{\boldsymbol{a}}(z)| \le \frac{C_{\boldsymbol{a}}|s|}{|z|^{2\alpha+|\boldsymbol{a}|+1}}.$$

We know the function $\partial^{\boldsymbol{b}_1}q(z)$ is a linear combination of the terms $-\partial^{\boldsymbol{a}_1}\kappa(z)\partial^{\boldsymbol{a}_2}(e^{2v(z)})$, $\boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a}$ and $|\boldsymbol{a}_1|, |\boldsymbol{a}_2| \ge 0$. Since $\kappa \in C^{K-2,\nu}(\mathbf{D})$, then $|\partial^{\boldsymbol{a}_1}\kappa(z)|$ is bounded above on $\mathbf{D}$. By Theorem A and the assumption of the induction, as $z \to 0$, $\partial^{\boldsymbol{a}_2}(e^{2v(z)})$ is continuous if $|\boldsymbol{a}_2| = 0$ or $|\boldsymbol{a}_2| = 1$, $0 < \alpha < 1/2$, and

$$\partial^{\boldsymbol{a}_2}(e^{2v(z)}) = \begin{cases} O(1) & \text{if } |\boldsymbol{a}_2| = 1,\ \alpha = 1/2, \\ O(|z|^{-2\alpha-|\boldsymbol{a}_2|+2}) & \text{otherwise.} \end{cases}$$

Since $-2\alpha + 2 \ge 0$, $\partial^{\boldsymbol{a}}q(z) = O(|z|^{-|\boldsymbol{a}|})$ as $z \to 0$, with $0 \le |\boldsymbol{a}| \le K-2$. Then

$$\begin{aligned}
(3.6)\qquad & |\partial^{\boldsymbol{a}}f(z+s) - \partial^{\boldsymbol{a}}f(z)| \\
&= \left| \sum_{|\boldsymbol{a}_1|+|\boldsymbol{a}_2|=|\boldsymbol{a}|} C_{\boldsymbol{a}_1,\boldsymbol{a}_2}(g_{\boldsymbol{a}_1}(z+s)\partial^{\boldsymbol{a}_2}q(z+s) - g_{\boldsymbol{a}_1}(z)\partial^{\boldsymbol{a}_2}q(z)) \right| \\
&\le \sum_{|\boldsymbol{a}_1|+|\boldsymbol{a}_2|=|\boldsymbol{a}|} |C_{\boldsymbol{a}_1,\boldsymbol{a}_2}| \cdot \{g_{\boldsymbol{a}_1}(z+s)|\partial^{\boldsymbol{a}_2}q(z+s) - \partial^{\boldsymbol{a}_2}q(z)| \\
&\qquad\qquad + |\partial^{\boldsymbol{a}_2}q(z)|\,|g_{\boldsymbol{a}_1}(z+s) - g_{\boldsymbol{a}_1}(z)|\} \\
&\le \frac{M'_K \cdot |s|^{\nu}}{|z|^{2\alpha+\nu+|\boldsymbol{a}|}},
\end{aligned}$$

for some positive constant M'_K. Therefore $f \in H^{K-2,\nu}_{2\alpha+\nu}(M'_K)$.

For any multi-index $\boldsymbol{b}$, $|\boldsymbol{b}| \le K-2$, we know that $\partial^{\boldsymbol{b}}f(z)$ is a linear combination of the terms $\partial^{\boldsymbol{b}_1}q(z)\partial^{\boldsymbol{b}_2}(|z|^{-2\alpha})$, $\boldsymbol{b}_1 + \boldsymbol{b}_2 = \boldsymbol{b}$, $|\boldsymbol{b}_1|, |\boldsymbol{b}_2| \ge 0$, and $\partial^{\boldsymbol{b}_1}q(z) = O(|z|^{-|\boldsymbol{b}_1|})$ as $z \to 0$. Note that $\partial^{\boldsymbol{b}_2}(|z|^{-2\alpha}) = O(|z|^{-2\alpha-|\boldsymbol{b}_2|})$ as $z \to 0$, then we have

$$(3.7)\qquad |\partial^{\boldsymbol{b}}f(z)| = O(|z|^{-2\alpha-|\boldsymbol{b}|})$$

as $z \to 0$.

We choose R, $0 \le R \le 1$ and consider the case $0 < \alpha < 1$ on $\mathbf{D}_R$. Let $0 < z < R/2$, $r \in (|z|/3, |z|/2)$ and $\Omega_r := \{\zeta : |\zeta - z| < r\}$. We need to show

$$(3.8)\qquad \int_{\mathbf{D}_R \setminus \Omega_r} \frac{1}{|z-\zeta|^2} \frac{d\sigma_\zeta}{|\zeta|^{2\alpha}} \le \frac{C}{|z|^{2\alpha}}$$

for a positive constant C. Let $\zeta = \omega z$. Denote $D := \{\omega : |\omega| < R/|z|, |\omega - 1| > r/|z|\}$, $D_1 := \{\omega : |\omega| < 2, |\omega - 1| > 1/3\}$ and $D_2 := \{\omega : 2 \le |\omega| < R/|z|\}$. Obviously $D \subseteq D_1 \cup D_2$. Note that $|\omega - 1| \ge |\omega| - 1 \ge |\omega|/2$ is valid when $\omega \in D_2$. Then

$$\begin{aligned}\int_{\mathbf{D}_R\setminus\Omega_r} \frac{1}{|z-\zeta|^2}\frac{d\sigma_\zeta}{|\zeta|^{2\alpha}} &\le \int_D \frac{1}{|\omega z - z|^2}\frac{|z|^2\, d\sigma_\omega}{|\omega z|^{2\alpha}} \\ &\le \frac{1}{|z|^{2\alpha}}\left(\int_{D_1} + \int_{D_2}\right)\frac{1}{|\omega - 1|^2}\frac{d\sigma_\omega}{|\omega|^{2\alpha}} \le \frac{C_1}{|z|^{2\alpha}} + \int_{D_2}\frac{4 d\sigma_\omega}{|\omega|^{2\alpha+2}} \\ &\le \frac{C_1}{|z|^{2\alpha}} + \frac{4}{|z|^{2\alpha}}\int_0^{2\pi} d\theta \int_2^{R/|z|}\frac{dt}{t^{2\alpha+1}} \le \frac{C_1}{|z|^{2\alpha}} + \frac{C_2}{|z|^{2\alpha}}\left(2 - \frac{R}{|z|}\right) \le \frac{C}{|z|^{2\alpha}}.\end{aligned}$$

Thus (3.8) is true.

Since $\kappa \in C^{K-2,\nu}(\mathbf{D}_R)$, we have $u \in C^{K,\nu}(\mathbf{D}_R^*)$, $0 < \nu \le 1^\dagger$. Due to Riesz' decomposition theorem (see, e.g. [5]), we have

$$v(z) = h(z) + \frac{1}{2\pi}\int_{\mathbf{D}_R} L(z-\zeta)f(\zeta)\, d\sigma_\zeta \tag{3.9}$$

for a harmonic function h on $\mathbf{D}_R$ and $0 < z < R/2$. Now let $r \in (|z|/3, |z|/2)$ and $\Omega_r := \{\zeta : |\zeta - z| < r\}$. Then for a multi-index $\boldsymbol{j}$, $|\boldsymbol{j}| = K \ge 3$, (3.9) and (3.3) lead to

$$\begin{aligned}\partial^{\boldsymbol{j}} v(z) = \partial^{\boldsymbol{j}} h(z) &+ \frac{1}{2\pi}\left(\int_{\Omega_r} + \int_{\mathbf{D}_R\setminus\Omega_r}\right)\partial^{\boldsymbol{j}} L(z-\zeta)(f(\zeta) - P_{K-2}[f](\zeta, z))\, d\sigma_\zeta \\ &- \frac{1}{2\pi}\sum_{\tau=1}^{K-1}\int_{\partial\mathbf{D}_R}\partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta, z)\cdot\langle N(\zeta), \boldsymbol{e}_{\tau+1}\rangle|d\zeta|.\end{aligned} \tag{3.10}$$

for $z = x_1 + ix_2$ and a harmonic function h on $\mathbf{D}_R$, with the same symbols $\boldsymbol{\theta}_\tau$, $\boldsymbol{\phi}_\tau$ as in (3.3). We apply Green's identity as in [12] and obtain

$$\begin{aligned}&\int_{\mathbf{D}_R\setminus\Omega_r}\partial^{\boldsymbol{j}} L(z-\zeta)P_{K-2}[f](\zeta, z)\, d\sigma_\zeta \\ &\quad = -\frac{1}{2\pi}\sum_{\tau=1}^{K-1}\int_{\partial(\mathbf{D}_R\setminus\Omega_r)}\partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta, z)\cdot\langle N(\zeta), \boldsymbol{e}_{\tau+1}\rangle|d\zeta|\end{aligned} \tag{3.11}$$

---

†If $K = 2$, from Theorem A we have $u \in C^{2,\nu}(\mathbf{D}_R^*)$. If $K = 3$, $\kappa \in C^{1,\nu}(\mathbf{D}_R)$ and $-\kappa(z)e^{2u} \in C^{1,\nu}(\mathbf{D}_R^*)$. We consider the equation $\Delta U = -\kappa(z)e^{2u}$. By the standard regularity theorem (see, e.g. [2, Theorem 6.17]), $U \in C^{3,\nu}(\mathbf{D}_R^*)$. In our case, $u(z)$ is the solution of $\Delta u = -\kappa(z)e^{2u}$. Thus $u \in C^{3,\nu}(\mathbf{D}_R^*)$. By repeating this process, we have $u \in C^{K,\nu}(\mathbf{D}_R^*)$ if $C^{K,\nu}(\mathbf{D}_R^*)$ if $\kappa \in C^{K-2,\nu}(\mathbf{D}_R)$ for $K \ge 2$.

Then (3.10) becomes

$$\begin{aligned}(3.12)\qquad \partial^{\boldsymbol{j}} v(z) = \partial^{\boldsymbol{j}} h(z) &+ \frac{1}{2\pi}\int_{\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta)(f(\zeta) - P_{K-2}[f](\zeta,z))\,d\sigma_\zeta \\ &+ \frac{1}{2\pi}\int_{\mathbf{D}_R\backslash\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta)\,d\sigma_\zeta \\ &- \frac{1}{2\pi}\sum_{\tau=1}^{K-1}\int_{\partial\Omega_r} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta,z)\cdot\langle N(\zeta),\boldsymbol{e}_{\tau+1}\rangle |d\zeta|\end{aligned}$$

It is known that [2, p. 17]

$$(3.13)\qquad |\partial^{\boldsymbol{j}} L(z-\zeta)| \le \frac{|\boldsymbol{j}|!}{|z-\zeta|^{|\boldsymbol{j}|}}.$$

Since $f \in H^{K-2,v}_{2\alpha+v}(M'_K)$, by Lemma 3.1, for $z \in \mathbf{D}^*$ and $\zeta \in \Omega_r$,

$$\begin{aligned}&\left|\int_{\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta)(f(\zeta) - P_{K-2}[f](\zeta,z))\,d\sigma_\zeta\right| \\ &\quad\le \int_{\Omega_r} \frac{K!}{|\zeta-z|^K}\frac{M'_k|\zeta-z|^{K-2+v}}{|z|^{2\alpha+v+K-2}}\,d\sigma_\zeta \\ &\quad\le C\int_{\Omega_r}\frac{d\sigma_\zeta}{|\zeta-z|^{2-v}}\frac{1}{|z|^{2\alpha+v+K-2}} \le \frac{C_1}{|z|^{2\alpha+K-2}}.\end{aligned}$$

When $\zeta \in \mathbf{D}_R\backslash\Omega_r$, we have $|z|/3 \le |\zeta - z| \le 2 + |z| \le 3$. Then by (3.7) for $\zeta \in \mathbf{D}_R\backslash\Omega_r$, $|(\zeta-z)^{\boldsymbol{b}}\partial^{\boldsymbol{b}} f(z)| = O(|z|^{-2\alpha})$ as $z \to 0$. That means $|P_{K-2}[f](\zeta,z)| = O(|z|^{-2\alpha})$ for $\zeta \in \mathbf{D}_R\backslash\Omega_r$, as $z \to 0$. Thus for z, $0 < z < R/2$, from (3.8),

$$\begin{aligned}\left|\int_{\mathbf{D}_R\backslash\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta)\,d\sigma_\zeta\right| &= \left|\int_{\mathbf{D}_R\backslash\Omega_r}\frac{K!}{|\zeta-z|^K}\frac{q(\zeta)}{|\zeta|^{2\alpha}}\,d\sigma_\zeta\right| \\ &\le \frac{C}{|z|^{K-2}}\int_{\mathbf{D}_R\backslash\Omega_r}\frac{1}{|\zeta-z|^2}\frac{d\sigma_\zeta}{|\zeta|^{2\alpha}} \le \frac{C_3}{|z|^{2\alpha+K-2}}.\end{aligned}$$

To estimate $\partial^{n_1}\bar\partial^{n_2} v(z)$, we note that $L(z-\zeta)$ is a harmonic function with respect to z when $z \ne \zeta$, so in the expression of $\partial^{n_1}\bar\partial^{n_2} v(z)$, $n_1, n_2 \ge 1$, $n_1 + n_2 = n$, the first two integrals in (3.12) cancel, we have only the last term in (3.12). Writing $\zeta = z + re^{i\theta}$ and taking $\boldsymbol{e}_{\tau+1} = (0,1)$ without loss of generality, we have

$$\begin{aligned}(3.14)\qquad \left|\int_{\partial\mathbf{D}_R} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot\langle N(\zeta),\boldsymbol{e}_{\tau+1}\rangle|d\zeta|\right| &\le \int_{\partial\mathbf{D}_R}\frac{|\boldsymbol{\theta}_\tau|!}{|\zeta-z|^{|\boldsymbol{\theta}_\tau|}}\cdot|\sin\theta|\,|d\zeta| \\ &= \int_0^{2\pi}\frac{|\boldsymbol{\theta}_\tau|!}{r^{|\boldsymbol{\theta}_\tau|}}|\sin\theta| r\,d\theta \le \frac{C}{|z|^{|\boldsymbol{\theta}_\tau|-1}}.\end{aligned}$$

Since

$$P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta, z) = \sum_{|\boldsymbol{a}| \le \tau-1} \frac{(\zeta - z)^{\boldsymbol{a}}}{\boldsymbol{a}!} \partial^{\boldsymbol{a}+\boldsymbol{\phi}_\tau} f(z), \tag{3.15}$$

then for $\zeta = z + re^{i\theta}$, by using (3.7), it yields

$$\begin{aligned} |P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta, z)| &\le \sum_{|\boldsymbol{a}| \le \tau-1} \frac{|\zeta - z|^{|\boldsymbol{a}|}}{\boldsymbol{a}!} |\partial^{\boldsymbol{a}+\boldsymbol{\phi}_\tau} f(z)| \\ &\le \sum_{|\boldsymbol{a}| \le \tau-1} \frac{|\zeta - z|^{|\boldsymbol{a}|}}{\boldsymbol{a}!|z|^{2\alpha+|\boldsymbol{a}|+|\boldsymbol{\phi}_\tau|}} \le \frac{C}{|z|^{2\alpha+|\boldsymbol{\phi}_\tau|}}. \end{aligned}$$

Combining the above formula with (3.14), we obtain

$$\left| \int_{\partial \mathbf{D}_R} \partial^{\boldsymbol{\theta}_\tau} L(z - \zeta) \cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta, z) \cdot \langle N(\zeta), \boldsymbol{e}_{\tau+1} \rangle |d\zeta| \right| = O\left(\frac{1}{|z|^{2\alpha+K-2}} \right) \tag{3.16}$$

as $z \to 0$, because $|\boldsymbol{\theta}_\tau| + |\boldsymbol{\phi}_\tau| = K - 1$ for $\tau = 1, 2, \ldots, K-1$. Therefore (1.6) holds for $n = K$. By induction, $q \in H_\beta^{n-2,\nu}(M_n)$, and (1.6) holds for all n, $n \ge 2$. □

The proof for $w(z)$ in Theorem 1.1 is based on the following lemma.

Lemma G [5]. *Let $\kappa : \mathbf{D} \to \mathbf{R}$ be a continuous function with $\kappa(0) < 0$ and*

$$\kappa(z) = \kappa(0) + O\left(\frac{1}{(\log(1/|z|))^2} \right)$$

as $z \to 0$. If $u : \mathbf{D}^* \to \mathbf{R}$ is a solution to $\Delta u = -\kappa(z) e^{2u}$ with $u(z) = -\log|z| - \log\log(1/|z|) + w(z)$ where $w(z) = O(1)$ for $z \to 0$, then there exists $0 < \rho < 1$ such that

$$|-\kappa(z) e^{2w(z)} - 1| \le \frac{C}{\log(1/|z|)}, \quad z \in \mathbf{D}_\rho^*, \tag{3.17}$$

for a constant $C > 0$.

Proof of Theorem 1.1 *when $\alpha = 1$.* Let

$$q(z) = -\kappa(z) e^{2w(z)} - 1, \quad f(z) = \frac{q(z)}{|z|^2 (\log(1/|z|))^2}. \tag{3.18}$$

We use induction to prove (1.7), (1.8) and that

$$|\partial^{\boldsymbol{a}} q(z+s) - \partial^{\boldsymbol{a}} q(z)| \le \frac{C \cdot |s|}{|z|^{|\boldsymbol{a}|+1} \log(1/|z|)}, \quad 1 \le |\boldsymbol{a}| \le n-2 \tag{3.19}$$

holds for a positive constant C, when $z \in \mathbf{D}_{\tilde{\rho}}^*$, $|s| < |z|/2$, where $\tilde{\rho}$ is selected such that Lemma G holds in $\mathbf{D}_{\tilde{\rho}}^*$.

When $n = 2$, (1.5) includes (1.7). Now let $R < 1/e^2$ and $\rho := \min\{R/2, \tilde{\rho}\}$, let $r \in (|z|/3, |z|/2)$ and $\Omega_r = \{\zeta : |\zeta - z| < r\}$. Kraus and Roth gave the following expression for $\partial_l \partial_j w(z)$ in [5], $l, j = 1, 2$ and $z \in \mathbf{D}_\rho^*$,

$$\begin{aligned}\partial_l \partial_j w(z) = \partial_l \partial_j h(z) &+ \frac{1}{2\pi} \int_{\mathbf{D}_\rho \setminus \Omega_r} \partial_l \partial_j L(z - \zeta) f(\zeta)\, d\sigma_\zeta \\ &+ \frac{1}{2\pi} \int_{\Omega_r} \partial_l \partial_j L(z - \zeta)(f(\zeta) - f(z))\, d\sigma_\zeta \\ &- \frac{1}{2\pi} f(z) \int_{\partial\Omega_r} \partial_j L(z - \zeta) N_l(\zeta) |d\zeta|,\end{aligned}$$

where $N(\zeta) = (N_1(\zeta), N_2(\zeta))$ is the unit outward normal at the point $\zeta \in \partial\Omega_r$. As for $\partial\bar{\partial} w(z)$, the first two integrals are canceled. From (3.14) we have

$$\left| \int_{\partial\Omega_r} \partial_j L(z - \zeta) N_l(\zeta) |d\zeta| \right| \le C,$$

then by Lemma G,

$$\left| f(z) \int_{\partial\Omega_r} \partial_j L(z - \zeta) N_l(\zeta) |d\zeta| \right| \le |f(z)| C_1 \le \frac{C_1}{|z|^2 (\log(1/|z|))^3}.$$

Therefore $\partial\bar{\partial} w(z) = O(|z|^{-2} (\log(1/|z|))^{-3})$ as $z \to 0$. Thus (1.7), (1.8) hold when $n = 2$. Since $\Delta u = -\kappa(z) \cdot \exp\{-2 \log|z| - 2 \log\log(1/|z|) + 2w(z)\}$, by calculation we know that there exists $0 < \rho < 1$ such that

$$|-\kappa(z) e^{2w(z)} - 1| \ge \frac{C}{\log(1/|z|)}, \quad z \in \mathbf{D}_\rho^*,$$

for a constant $C > 0$. Combined with Lemma G, we have

$$\frac{-A}{L(z)} = \frac{A}{\log(1/|z|)} \le |q(z)| \le \frac{B}{\log(1/|z|)} = \frac{-B}{L(z)}, \quad z \in \mathbf{D}_\rho^*, \tag{3.20}$$

where A, B are positive constants. For z and s, $z \in \mathbf{D}_\rho^*$, $|s| < |z|/2$,

$$\begin{aligned} |q(z+s) - q(z)| &\le C \left| \frac{1}{L(z+s)} - \frac{1}{L(z)} \right| \\ &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{L(z+ts)}\, \mathrm{d}t \right| = \left| \int_0^1 \nabla\left(\frac{1}{L(z+ts)} \right) \cdot s\, \mathrm{d}t \right| \\ &\le |s| \cdot \int_0^1 \nabla\left(\frac{1}{L(z+ts)} \right) \mathrm{d}t \le |s| \sup_{0 \le t \le 1} \left| \nabla\left(\frac{1}{L(z+ts)} \right) \right| \\ &\le \frac{M_2' \cdot |s|}{|z| (\log(1/|z|))^2}. \end{aligned} \tag{3.21}$$

Thus (3.19) is true when $n = 2$. Now suppose that (1.7), (1.8) and (3.19) hold for $n \leq K - 1$, $K \geq 3$, and consider them for $n = K$. It can be verified by induction that,

$$|\partial^{\boldsymbol{b}} f(z)| \leq \frac{C(\rho)}{|z|^{|\boldsymbol{b}|+2}(\log(1/|z|))^3}, \quad z \in \mathbf{D}_\rho^* \tag{3.22}$$

and the same technique as in (3.21) leads to

$$|\partial^{\boldsymbol{a}} q(z+s) - \partial^{\boldsymbol{a}} q(z)| \leq \frac{M_K' \cdot |s|}{|z|^{|\boldsymbol{a}|+1} \log(1/|z|)}, \quad \text{when } |\boldsymbol{a}| = K - 2.$$

Thus (3.19) is true, and, similar to (3.6),

$$|\partial^{\boldsymbol{a}} f(z+s) - \partial^{\boldsymbol{a}} f(z)| \leq \frac{C \cdot |s|}{|z|^{|\boldsymbol{a}|+1}(\log(1/|z|))^3},$$

then,

$$|f(\zeta) - P_{K-2}[f](\zeta, z)| \leq \frac{C \cdot |z - \zeta|^{K-1}}{|z|^{K+1}(\log(1/|z|))^3}, \tag{3.23}$$

when $|\zeta - z| < |z|/2$, $z \in \mathbf{D}_\rho^*$.

Since $\kappa \in C^{K-2,\nu}(\mathbf{D})$, $u \in C^{K,\nu}(\mathbf{D}^*)$. We will show that

$$w(z) = h(z) + \frac{1}{2\pi} \int_{\mathbf{D}_R} L(z - \zeta) f(\zeta)\, d\sigma_\zeta \tag{3.24}$$

for $z \in \mathbf{D}_\rho^*$, where h is harmonic on $\mathbf{D}_\rho$. Let

$$t(z) := -\log\log(1/|z|), \quad p(z) := w(z) + t(z) = u(z) + \log|z|$$

for $z \in \mathbf{D}_\rho^*$. Since on $\mathbf{D}_\rho^*$,

$$\Delta p(z) = -\kappa(z) e^{2u} = \frac{-\kappa(z) e^{2w(z)}}{|z|^2(\log(1/|z|))^2} > 0,$$

$p(z)$ is subharmonic on $\mathbf{D}_\rho^*$ and $\lim_{z \to 0} p(z) = -\infty$, $p(z)$ is subharmonic on $\mathbf{D}_\rho$. As $\Delta p(z)$ is integrable over $\mathbf{D}_\rho$, by Lemma E,

$$p(z) = h_p(z) + \frac{1}{2\pi} \int_{\mathbf{D}_R} L(z - \zeta) \frac{-\kappa(\zeta) e^{2w(\zeta)}}{|\zeta|^2(\log(1/|\zeta|))^2}\, d\sigma_\zeta, \quad z \in \mathbf{D}_\rho^*,$$

where $h_p(z)$ is harmonic on $\mathbf{D}_\rho^*$. For $t(z)$, we also have

$$t(z) = h_t(z) + \frac{1}{2\pi} \int_{\mathbf{D}_R} L(z - \zeta) \frac{1}{|\zeta|^2(\log(1/|\zeta|))^2}\, d\sigma_\zeta, \quad z \in \mathbf{D}_\rho^*,$$

where $h_t(z)$ is harmonic on $\mathbf{D}_\rho^*$. Letting $w(z) = p(z) - t(z)$ gives (3.24) with $h(z) = h_p(z) - h_t(z)$.

From (3.3) and (3.11), for a multi-index $\boldsymbol{j}$, $|\boldsymbol{j}| = K \geq 3$, we have

$$\begin{aligned}\partial^{\boldsymbol{j}} w(z) = \partial^{\boldsymbol{j}} h(z) &+ \frac{1}{2\pi}\int_{\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta)(f(\zeta) - P_{K-2}[f](\zeta,z))\, d\sigma_\zeta \\ &+ \frac{1}{2\pi}\int_{\mathbf{D}_R\setminus\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta)\, d\sigma_\zeta \\ &- \frac{1}{2\pi}\sum_{\tau=1}^{K-1}\int_{\partial\Omega_r} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta,z)\cdot\langle N(\zeta), \boldsymbol{e}_{\tau+1}\rangle |d\zeta|\end{aligned} \tag{3.25}$$

for a harmonic function h on $\mathbf{D}_\rho$. From (3.23), for $z \in \mathbf{D}_\rho^*$ and $\zeta \in \Omega_r$,

$$\begin{aligned}&\left|\int_{\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta)(f(\zeta) - P_{K-2}[f](\zeta,z))\, d\sigma_\zeta\right| \\ &\qquad \leq \int_{\Omega_r} \frac{K!}{|\zeta - z|^K}\frac{C|z-\zeta|^{K-1}}{|z|^{K+1}(\log(1/|z|))^3}\, d\sigma_\zeta \leq \frac{C}{|z|^K(\log(1/|z|))^3}.\end{aligned} \tag{3.26}$$

When $\zeta \in \mathbf{D}_R\setminus\Omega_r$, we have $|z|/3 \leq |\zeta - z| \leq R + |z| \leq (3R)/2$. By (3.22), for $z \in \mathbf{D}_\rho^*$, $\zeta \in \mathbf{D}_R\setminus\Omega_r$,

$$|(\zeta - z)^{\boldsymbol{b}}\partial^{\boldsymbol{b}} f(z)| \leq \frac{C_5}{|z|^2(\log(1/|z|))^3},$$

and then

$$|P_{K-2}[f](\zeta,z)| \leq \frac{C_6}{|z|^2(\log(1/|z|))^3}.$$

Similar to (3.8), we have

$$\int_{\mathbf{D}_R\setminus\Omega_r} \frac{1}{|\zeta - z|^2}\frac{1}{|\zeta|^2(\log(1/|\zeta|))^2}\, d\sigma_\zeta \leq \frac{C}{|z|^K(\log(1/|z|))^2} \tag{3.27}$$

for a positive constant C. Thus for $\zeta \in \mathbf{D}_R\setminus\Omega_r$, from (3.27),

$$\begin{aligned}\left|\int_{\mathbf{D}_R\setminus\Omega_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta)\, d\sigma_\zeta\right| &= \left|\int_{\mathbf{D}_R\setminus\Omega_r} \frac{K!}{|\zeta - z|^K}\frac{q(\zeta)}{|\zeta|^2(\log(1/|\zeta|))^2}\, d\sigma_\zeta\right| \\ &\leq \int_{\mathbf{D}_R\setminus\Omega_r} \frac{K!}{|\zeta - z|^K}\frac{C_6}{|\zeta|^2(\log(1/|\zeta|))^2}\, d\sigma_\zeta \\ &\leq \frac{C_7}{|z|^{K-2}}\int_{\mathbf{D}_R\setminus\Omega_r} \frac{1}{|\zeta - z|^2}\frac{1}{|\zeta|^2(\log(1/|\zeta|))^2}\, d\sigma_\zeta \\ &\leq \frac{C_8}{|z|^K(\log(1/|z|))^2}.\end{aligned}$$

Note that $0 \le |\boldsymbol{\phi}_\tau| \le K-2$, for $\zeta \in \partial\Omega_r$, from (3.15) and (3.22) we have

$$
\begin{aligned}
(3.28)\qquad |P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta, z)| &\le \sum_{|\boldsymbol{a}| \le \tau-1} \frac{|\zeta - z|^{|\boldsymbol{a}|}}{\boldsymbol{a}!} |\partial^{\boldsymbol{a}+\boldsymbol{\phi}_\tau} f(z)| \\
&\le \sum_{|\boldsymbol{a}| \le \tau-1} \frac{r^{|\boldsymbol{a}|}}{\boldsymbol{a}!} \frac{C(\rho)}{|z|^{|\boldsymbol{a}|+|\boldsymbol{\phi}_\tau|+2}(\log(1/|z|))^3} \\
&\le \frac{C_9}{|z|^{|\boldsymbol{\phi}_\tau|+2}(\log(1/|z|))^3}.
\end{aligned}
$$

Then for $\zeta = z + re^{i\theta}$,

$$
\begin{aligned}
&\left| \int_{\partial \mathbf{D}_R} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta) \cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](\zeta, z) \cdot \langle N(\zeta), \boldsymbol{e}_{\tau+1} \rangle |d\zeta| \right| \\
&\qquad \le \sum_{\tau=1}^{K-1} \int_0^{2\pi} \frac{C}{|z-\zeta|^{|\boldsymbol{\theta}_\tau|}} \frac{r\,d\theta}{|z|^{|\boldsymbol{\phi}_\tau|+2}(\log(1/|z|))^3} = O\left(\frac{1}{|z|^K (\log(1/|z|))^3} \right)
\end{aligned}
$$

as $z \to 0$. So (1.7), (1.8) hold for $n = K$. Thus we obtain (1.7) for $n = K$. Since the first three terms in (3.25) vanish for mixed partial derivatives, (3.28) yields (1.8) for $n = K$. By induction (1.7) and (1.8) hold for all n, $n \ge 2$. $\square$

The second order derivative of $w(z)$ in Theorem A is contained in Theorem 1.1. However, for the mixed partial derivative, the estimate (1.8) is more accurate than (1.5).

Corollary 3.2. *Let $\kappa : \mathbf{D} \to \mathbf{R}$ be a locally Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbf{D}^* \to \mathbf{R}$ is a C^2-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbf{D}^*$ with the order $\alpha = 1$ at the point $z = 0$, then for the remainder function $w(z)$ as in Theorem* A, *the second partial derivatives satisfy*

$$\partial\bar{\partial} w(z) = O(|z|^{-2}(\log(1/|z|))^{-3}).$$

Remark 3.2. The sharpness of Theorem 1.1 can be verified by the hyperbolic metric given by Theorem D when $0 < \alpha < 1$. However, the remainder function $w(z) = 0$ when $\alpha = 1$, that means it is trivial for our estimate. So we use the generalized hyperbolic metric $\lambda_{\alpha,\beta,\gamma}(z)|dz|$ on the thrice-punctured sphere $\mathbf{P} \backslash \{z_1, z_2, z_3\}$ with singularities of order $\alpha, \beta, \gamma \le 1$ at z_1, z_2, z_3 to show that our result is sharp (see [12]), where $\alpha + \beta + \gamma > 2$ (see [6] for the formula of $\lambda_{\alpha,\beta,\gamma}(z)|dz|$). Theorems 3.3 and 4.2 in [12] verify that Theorems A and 1.1 are sharp (see [12]).

4. Proof of Theorem 1.2 and related results

At first we need the following result for the function u. Considering Theorems A and 1.1, if we add the assumption that κ is $(n-2)$-th order (locally) Hölder continuous, we can obtain the limits for higher order derivatives of $u(z)$.

THEOREM 4.1. *Let $\kappa : \mathbf{D} \to \mathbf{R}$ be of class $C^{n-2,\nu}(\mathbf{D})$ for an integer $n \geq 2$, $0 < \nu \leq 1$ with $\kappa(0) < 0$. If $u : \mathbf{D}^* \to \mathbf{R}$ is a $C^{n,\nu}$-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbf{D}^*$, then u has an order $\alpha \in (-\infty, 1]$ and for $n_1, n_2 \geq 1$, $n_1 + n_2 \leq n$,*

(i) $\lim_{z\to 0} z^n \partial^n u(z) = \dfrac{\alpha}{2}(-1)^n(n-1)! = \lim_{z\to 0} \bar{z}^n \bar{\partial}^n u(z)$,

(ii) $\lim_{z\to 0} z^{n_1}\bar{z}^{n_2}\partial^{n_1}\bar{\partial}^{n_2} u(z) = 0$.

Proof. When $0 < \alpha < 1$, $u(z) = -\alpha \log|z| + v(z)$. Theorems 1.1 implies that

$$\lim_{z\to 0} z^n \partial^n v(z) = 0, \quad \lim_{z\to 0} z^{n_1}\bar{z}^{n_2}\partial^{n_1}\bar{\partial}^{n_2} v(z) = 0$$

for $n_1, n_2, n \geq 1$. Since

$$\partial^n \log|z| = \frac{(-1)^{n-1}(n-1)!}{2z^n}, \quad \partial^{n_1}\bar{\partial}^{n_2} \log|z| = 0, \tag{4.1}$$

so

$$\lim_{z\to 0} z^n \partial^n u(z) = -\alpha \lim_{z\to 0} z^n \partial^n \log|z| + \lim_{z\to 0} z^n \partial^n v(z) = \frac{\alpha}{2}(-1)^n(n-1)!,$$

$$\lim_{z\to 0} z^{n_1}\bar{z}^{n_2}\partial^{n_1}\bar{\partial}^{n_2} u(z) = 0.$$

When $\alpha = 1$, $u(z) = -\log|z| - \log\log(1/|z|) + w(z)$. We have

$$\lim_{z\to 0} z^n \partial^n w(z) = 0, \quad \lim_{z\to 0} z^{n_1}\bar{z}^{n_2}\partial^{n_1}\bar{\partial}^{n_2} w(z) = 0$$

for $n_1, n_2, n \geq 1$, from Theorems 1.1. By induction,

$$\partial^n \log\log(1/|z|) = \sum_{j=1}^{n} \frac{C_j^{(n)}}{z^n(\log(1/|z|))^j}$$

with constant $C_j^{(n)}$ for $1 \leq j \leq n$. If we fix n_2, then

$$\partial^{n_1}\bar{\partial}^{n_2} \log\log(1/|z|) = \sum_{j=1}^{n_2} \frac{C_j^{(n_1,n_2)}}{z^{n_1}\bar{z}^{n_2}(\log(1/|z|))^{j+1}}$$

with constant $C_j^{(n_1,n_2)}$ for $1 \leq j \leq n_2$. So

$$\lim_{z\to 0} z^n \partial^n \log\log(1/|z|) = 0, \quad \lim_{z\to 0} z^{n_1}\bar{z}^{n_2}\partial^{n_1}\bar{\partial}^{n_2} \log\log(1/|z|) = 0$$

for $n_1, n_2, n \geq 1$. Combining with (4.1) leads to

$$\lim_{z\to 0} z^n \partial^n u(z) = -\alpha \lim_{z\to 0} z^n \partial^n \log|z| + \lim_{z\to 0} z^n \partial^n v(z) = \alpha \frac{(-1)^n (n-1)!}{2},$$

$$\lim_{z\to 0} z^{n_1} \bar{z}^{n_2} \partial^{n_1} \bar{\partial}^{n_2} u(z) = 0. \qquad \square$$

From the proof above, we can obtain a stronger limit for the mixed derivative of $u(z)$ when the order $\alpha = 1$,

$$\lim_{z\to 0} z^{n_1} \bar{z}^{n_2} (\log(1/|z|))^2 \partial^{n_1} \bar{\partial}^{n_2} u(z) = C_1^{(n_1,n_2)} = \frac{(-1)^{n_1+n_2-1}}{4} (n_1-1)!(n_2-1)!,$$

which is given in [12].

On the basis of Theorem 4.1, we can provide Theorem 1.2 as a higher order estimate for a conformal metric with the negative curvature near the origin when $\alpha = 1$.

Proof of Theorem 1.2. We write $\lambda(z) = e^{u(z)}$, then $\partial\lambda(z) = \lambda(z)\partial u(z)$, and

$$\partial^n \lambda(z) = \sum_{j=0}^{n-1} \binom{n-1}{j} \partial^{n-j} u(z) \partial^j \lambda(z)$$

by induction, where $\partial^0 \lambda(z) = \bar{\partial}^0 \lambda(z) = \lambda(z)$, so

$$l_{n_1,0} = \frac{1}{n_1!} \lim_{z\to 0} \sum_{j=0}^{n_1-1} \binom{n_1-1}{j} z^{n_1-j} \partial^{n_1-j} u(z) \cdot |z| \log(1/|z|) z^j \partial^j \lambda(z).$$

Theorem B gives that $l_{0,0} = 1/\sqrt{-\kappa(0)}$. From the existence of $\lim_{z\to 0} z^{n_1-j} \partial^{n_1-j} u(z)$ and $l_{0,0}$, we know that $l_{n_1,0}$ exists. The limit (ii) in Theorem 4.1 enables us further to write l_{n_1,n_2} as a sum of the terms only containing pure derivatives of $u(z)$,

$$l_{n_1,n_2} = \frac{1}{n_1! n_2!} \lim_{z\to 0} \sum_{j=0}^{n_1-1} \binom{n_1-1}{j} z^{n_1-j} \partial^{n_1-j} u(z) |z| \log(1/|z|) \bar{z}^{n_2} z^j \bar{\partial}^{n_2} \partial^j \lambda(z), \tag{4.2}$$

thus the existence of $l_{n_1,0}$ guarantees l_{n_1,n_2} exists.

Now we consider $l_{1,0}$. By Theorem 4.1, $\lim_{z\to 0} z\partial u(z) = -1/2$. In combination with Theorem B, we have

$$\begin{aligned} \lim_{z\to 0} z|z| \log(1/|z|) \partial\lambda(z) &= \lim_{z\to 0} z|z| \log(1/|z|) \partial u(z) \lambda(z) \\ &= \lim_{z\to 0} |z| \log(1/|z|) \lambda(z) \cdot z\partial u(z) = -\frac{1}{2\sqrt{-\kappa(0)}}. \end{aligned}$$

That means $l_{1,0}$ is a real number, so $l_{1,0} = \overline{l_{0,1}} = l_{0,1}$. Since

$$\bar{\partial}^{n_1} \lambda(z) = \sum_{j=0}^{n_1-1} \binom{n_1-1}{j} \bar{\partial}^{n_1-j} u(z) \bar{\partial}^j \lambda(z), \tag{4.3}$$

then $l_{n_1,0} = l_{0,n_1}$ by induction. From (4.2), (4.3), and (i) of Theorem 4.1, we have

$$\begin{aligned} l_{n_1,n_2} &= \sum_{j=0}^{n_1-1} \lim_{z\to 0} \frac{1}{n_1!n_2!} \frac{(n_1-1)!}{j!(n_1-1-j)!} z^{n_1-j}\partial^{n_1-j}u(z)\cdot|z|\log(1/|z|)\bar{z}^{n_2}z^j\bar{\partial}^{n_2}\partial^j\lambda(z) \\ &= \frac{1}{n_1}\sum_{j=0}^{n_1-1}\frac{1}{n_2!}\frac{1}{j!(n_1-1-j)!}\lim_{z\to 0} z^{n_1-j}\partial^{n_1-j}u(z)\cdot\lim_{z\to 0}|z|\log(1/|z|)\bar{z}^{n_2}z^j\bar{\partial}^{n_2}\partial^j\lambda(z) \\ &= \frac{1}{n_1}\sum_{j=0}^{n_1-1}\frac{(-1)^{n_1-j}}{2}\frac{1}{n_2!j!}\lim_{z\to 0}|z|\log(1/|z|)\bar{z}^{n_2}z^j\bar{\partial}^{n_2}\partial^j\lambda(z) = \frac{1}{2n_1}\sum_{j=0}^{n_1-1}(-1)^{n_1-j}l_{j,n_2}. \end{aligned}$$

Then

$$n_1\cdot l_{n_1,n_2} = \frac{1}{2}\sum_{j=0}^{n_1-2}(-1)^{n_1-j}l_{j,n_2} - \frac{1}{2}l_{n_1-1,n_2} = -(n_1-1)l_{n_1-1,n_2} - \frac{1}{2}l_{n_1-1,n_2}.$$

Since $l_{0,n_1} = l_{n_1,0}$,

$$\begin{aligned} l_{n_1,n_2} &= \frac{-\frac{1}{2}-n_1+1}{n_1}l_{n_1-1,n_2} = \cdots = \binom{-\frac{1}{2}}{n_1}l_{0,n_2} \\ &= \binom{-\frac{1}{2}}{n_1}l_{n_2,0} = \binom{-\frac{1}{2}}{n_1}\binom{-\frac{1}{2}}{n_2}l_{0,0}. \end{aligned}$$

Thus (i) is valid and (ii) follows form (i). □

For any regular conformal metric with negative curvature function $k(z)$ near the singularity, we have the following corollary of Theorem 1.2.

Corollary 4.2. *Let $\lambda(z)|dz|$ be a regular conformal metric on $\mathbf{D}^*$. Suppose that the curvature $\kappa:\mathbf{D}^*\to\mathbf{R}$ has a Hölder continuous extension to $\mathbf{D}$ such that $\kappa(0)<0$ and the order of $\log\lambda$ is $\alpha=1$ at $z=0$. Then*

(i) $\lim_{z\to 0} z|z|\log(1/|z|)\partial\lambda(z) = -\dfrac{1}{2\sqrt{-\kappa(0)}}$,

(ii) $\lim_{z\to 0} z^2|z|\log(1/|z|)\partial^2\lambda(z) = \dfrac{3}{4\sqrt{-\kappa(0)}}$,

(iii) $\lim_{z\to 0}|z|^3\log(1/|z|)\partial\bar{\partial}\lambda(z) = \dfrac{1}{4\sqrt{-\kappa(0)}}$.

Theorem 1.2 implies that, in the case $\alpha=1$, the limit (1.9) can be described by the use of the curvature κ. On the other hand, when $\alpha<1$, the analogous limit

$$l := \lim_{z\to 0}|z|^\alpha\lambda(z) \tag{4.4}$$

also exists under the assumption of Theorem A. However, it cannot be described only in terms of the curvature of $\lambda(z)$. The following two examples show that, l defined by (4.4) depends on the global shape of the domain.

Example 1. If $\lambda(z)|dz| := \lambda_{\alpha,R}(z)|dz|$ is the hyperbolic metric given in Theorem D on $\mathbf{D}_R^*$ with the order $\alpha \in (0,1)$ at the origin, then $l = (1-\alpha)/R^{1-\alpha}$.

Example 2. Let $0 < \alpha, \beta < 1$ and $0 < \gamma \le 1$ such that $\alpha + \beta + \gamma > 2$ and $\lambda(z)|dz|$ be the generalized hyperbolic metric on the thrice-punctured Riemann sphere $\hat{\mathbf{C}} \backslash \{0, 1, \infty\}$ with conical singularities of orders α, β, γ at 0, 1, ∞, respectively. Then

$$l = \frac{\delta}{1-\delta^2}(1-\alpha), \tag{4.5}$$

where

$$\delta = \frac{\Gamma(c)}{\Gamma(2-c)} \left(\frac{\Gamma(1-a)\Gamma(1-b)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \right)^{1/2}$$

with

$$a = \frac{\alpha+\beta-\gamma}{2}, \quad b = \frac{\alpha+\beta+\gamma-2}{2}, \quad c = \alpha.$$

The constant $\dfrac{\delta}{1-\delta^2}(1-\alpha)$ in the right hand side of (4.5) is due to Kraus, Roth and Sugawa. They did not give it explicitly, but it is easy to obtain it from [6, Corollary 4.4].

From Theorem A and the maximum principle Theorem D, the limit l in (4.4) exists at the origin for an SK-metric. Moreover, if $\kappa(z)$ and $u(z)$ satisfy the assumption of Theorem 4.1, then $u(z)$ is of class C^2 in a punctured neighborhood of $z = 0$. If we define

$$l_{n_1,n_2} := \frac{1}{n_1! n_2!} \lim_{z \to 0} |z|^{\alpha} z^{n_1} \bar{z}^{n_2} \partial^{n_1} \bar{\partial}^{n_2} \lambda_{\alpha,R}(z), \tag{4.6}$$

then the following result gives l_{n_1,n_2} in terms of the recurrence relation similar to Theorem 1.2.

THEOREM 4.3. *Let $\kappa : \mathbf{D} \to \mathbf{R}$ be of class $C^{n-2,\nu}(\mathbf{D})$ for an integer $n \ge 2$, $0 < \nu \le 1$ and $\kappa(0) < 0$. If $u : \mathbf{D}^* \to \mathbf{R}$ is a $C^{n,\nu}$-solution to $\Delta u = -\kappa(z) e^{2u}$ in $\mathbf{D}^*$, then u has order $\alpha \in (-\infty, 1]$ and for $n_1, n_2 \ge 0$, $n_1 + n_2 \le n$, the limit l_{n_1,n_2} defined by* (4.6) *exists and satisfies*

$$l_{n_1,n_2} = \begin{pmatrix} -\frac{\alpha}{2} \\ n_1 \end{pmatrix} \begin{pmatrix} -\frac{\alpha}{2} \\ n_2 \end{pmatrix} l,$$

where l is defined by (4.4).

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