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UNIVERSAL INEQUALITIES FOR EIGENVALUES OF A CLASS OF OPERATORS ON RIEMANNIAN MANIFOLD

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Abstract

In this paper, we investigate the boundary value problem of the following operator

$$\begin{cases} \Delta^2 u - a \operatorname{div} A\nabla u + Vu = \rho\lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial v}\right|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded domain in an n-dimensional complete Riemannian manifold M^n, A is a positive semidefinite symmetric (1,1)-tensor on M^n, V is a non-negative continuous function on Ω, v denotes the outwards unit normal vector field of $\partial\Omega$ and ρ is a weight function which is positive and continuous on Ω. By the Rayleigh-Ritz inequality, we obtain universal inequalities for the eigenvalues of these operators on bounded domain of complete manifolds isometrically immersed in a Euclidean space, and of complete manifolds admitting special functions which include the Hadamard manifolds with Ricci curvature bounded below, a class of warped product manifolds, the product of Euclidean spaces with any complete manifold and manifolds admitting eigenmaps to a sphere.

1. Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. The Dirichlet eigenvalue problem of the biharmonic operator:

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial v}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where v denotes the outwards unit normal vector field of $\partial\Omega$, and Δ^2 is the biharmonic operator on M. It is also called a clamped plate problem, which

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describes the characteristic vibrations of a clamped plate. An open question in estimates for eigenvalues of problem (1.1) is to give universal upper bounds of the $(k+1)$-th eigenvalue in terms of the first k eigenvalues.

To begin with, people were concerned about the case that Ω is a bounded domain in $\mathbf{R}^N$. In 1956, Payne, Pólya and Weinberger [13] obtained the following universal inequality:

$$\lambda_{k+1} - \lambda_k \le \frac{8(n+2)}{n^2 k} \sum_{i=1}^{k} \lambda_i. \tag{1.2}$$

By using an improved method, Hile and Protter [10] generalized the above results. In 1990, Hook [11] and Chen and Qian [2] independently proved

$$\frac{n^2 k^2}{8(n+2)} \le \sum_{i=1}^{k} \frac{\lambda_i^{1/2}}{\lambda_{k+1} - \lambda_i} \sum_{i=1}^{k} \lambda_i^{1/2}. \tag{1.3}$$

In 2006, Cheng and Yang [5] obtained the following sharper inequality

$$\lambda_{k+1} \le \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \left[\frac{8(n+2)}{n^2}\right]^{1/2} \frac{1}{k} \sum_{i=1}^{k} [\lambda_i(\lambda_{k+1} - \lambda_i)]^{1/2}. \tag{1.4}$$

It is natural to consider the estimates for eigenvalues of problem (1.1) on the other Riemannian manifolds. In 2007, Wang and Xia [18] gave

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{8(n+2)}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i \tag{1.5}$$

on an n-dimensional complete minimal submanifold in a Euclidean space, and

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{1}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)[n^2 + (2n+4)\lambda_i^{1/2}][n^2 + 4\lambda_i^{1/2}] \tag{1.6}$$

on an n-dimensional unit sphere.

In 2010, Cheng, Ichikawa and Mametsuka [6] proved that, for any complete Riemannian manifold M, there exists a universal bound of the $(k+1)$-th eigenvalue in terms of the first k eigenvalues of (1.1). They obtained the following remarkable inequality of eigenvalues

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \frac{1}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)[n^2 H_0^2 + (2n+4)\lambda_i^{1/2}](n^2 H_0^2 + 4\lambda_i^{1/2}), \tag{1.7}$$

where H_0^2 is a constant which only depends on the mean curvature of M. It is easy to find that (1.7) contains the inequalities (1.5) and (1.6).

In recent years, there are some researches on eigenvalue estimates for quadratic polynomial operator of laplacian:

$$\begin{cases} \Delta^2 u - p\Delta u + qu = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial v}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in an n-dimensional complete Riemannian manifold M, v denotes the outwards unit normal vector field of $\partial\Omega$, the constants $p, q \geq 0$, $\Delta = (\operatorname{div} \nabla)$ is the laplacian and Δ^2 is the biharmonic operator on M. In 2011, Sun and Qi [16] obtained universal eigenvalue inequalities of problem (1.8)

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ &\quad \leq \frac{1}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)((nH_0)^2 + (2n+4)E_i + np)((nH_0)^2 + 4E_i), \end{aligned} \tag{1.9}$$

where H_0 is a constant which only depends on the mean curvature of M and

$$E_i = \frac{1}{2}(-p + \sqrt{p^2 + 4(\lambda_i - q)}).$$

For other related results, one can see [4, 15, 17].

Based on the recent studies on the Dirichlet eigenvalues of elliptic operators in divergence form on Riemannian manifold, see [7, 8]. We find it necessary and also interesting to study the eigenvalues of the following problem:

Let $(M^n, \langle\, , \rangle)$ be an n-dimensional complete Riemannian manifold isometrically immersed in the Euclidean space $\mathbf{R}^N$ by Ψ, Ω is a bounded domain in M^n. In this paper, we are interested in extrinsic upper bounds for the eigenvalues of following operator defined on Ω:

$$\begin{cases} \Delta^2 u - a \operatorname{div} A\nabla u + Vu = \rho\lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial v}\right|_{\partial\Omega} = 0, \end{cases} \tag{1.10}$$

where A is a positive semidefinite symmetric (1,1)-tensor on M^n, a is a non-negative constant, $b \leq \bar{\lambda}_{A|\Omega} \leq c$ ($\bar{\lambda}_{A|\Omega}$ is the eigenvalues of A on Ω, b and c are constants, $b \geq 0$), $\operatorname{tr}(A)|_{\Omega} \leq$ constant d (i.e. A can also be viewed as a smooth symmetric and positive semidefine section of the bundle of all endomorphisms of $T(M^n)$), $\Delta = (\operatorname{div} \nabla)$ is the laplacian, ∇ is the gradient operator on M^n, V is non-negative continuous function on Ω, v denotes the outwards unit normal vector field of $\partial\Omega$ and ρ is a weight function which is positive and continuous on Ω.

One will find easily both problem (1.1) and (1.8) are special cases of problem (1.10). However, it is well known that problem (1.10) has a real and discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$, where each eigenvalue is repeated according to its multiplicity. Our results are stated as follows.

THEOREM 1.1. *Let $(M^n, \langle\, ,\, \rangle)$ be an n-dimensional complete Riemannian manifold isometrically immersed in the Euclidean space $\mathbf{R}^N$ by Ψ, Ω is a bounded domain in M^n. A is a positive semidefinite symmetric* (1,1)*-tensor on M^n and a is a non-negative constant. Assume that $b \le \bar{\lambda}_{A|\Omega} \le c$ ($\bar{\lambda}_{A|\Omega}$ is the eigenvalues of A on Ω, b and c are constants, $b \ge 0$) and that* $\operatorname{tr}(A)|_\Omega \le$ *constant d. Let ρ be a continuous function on Ω satisfying $\rho_1 \le \rho(x) \le \rho_2$, $\forall x \in \Omega$, for some positive constants ρ_1 and ρ_2. v denotes the outwards unit normal vector field of $\partial\Omega$. Let V be non-negative continuous function on Ω and denote by $\{\lambda_i\}_{i=1}^\infty$ the eigenvalues of problem*

$$\begin{cases} \Delta^2 u - a \operatorname{div} A\nabla u + Vu = \rho\lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial v}\right|_{\partial\Omega} = 0. \end{cases}$$

Then we have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \le \frac{\rho_2^2}{n^2\rho_1^3} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left(4\rho_1 E_i + n^2 \sup_\Omega |H|^2 \right) \times \left(n^2 \sup_\Omega |H|^2 + (2n+4)\rho_1 E_i + ad \right), \tag{1.11}$$

where $E_i = \dfrac{1}{2\rho_1}(-b + \sqrt{b^2 - 4(s - \lambda_i)\rho_1})$, H is the mean curvature vector of the immersion Ψ and $s = \inf_\Omega \dfrac{V}{\rho}$.

THEOREM 1.2. *Under the same assumptions as Theorem* 1.1, *we obtain*

$$\lambda_{k+1} \le A_k + \sqrt{A_k^2 - B_k}, \tag{1.12}$$

where

$$A_k = \frac{1}{k}\left\{ \sum_{i=1}^k \lambda_i + \frac{\rho_2^2}{2n^2\rho_1^3} \sum_{i=1}^k \left(4\rho_1 E_i + n^2 \sup_\Omega |H|^2 \right) \times \left(n^2 \sup_\Omega |H|^2 + (2n+4)\rho_1 E_i + ad \right) \right\},$$

$$B_k = \frac{1}{k}\left\{ \sum_{i=1}^k \lambda_i^2 + \frac{\rho_2^2}{n^2\rho_1^3} \sum_{i=1}^k \lambda_i \left(4\rho_1 E_i + n^2 \sup_\Omega |H|^2 \right) \times \left(n^2 \sup_\Omega |H|^2 + (2n+4)\rho_1 E_i + ad \right) \right\},$$

$|H|$ is the mean curvature of the immersion Ψ, $E_i = \frac{1}{2\rho_1}(-b + \sqrt{b^2 - 4(s-\lambda_i)\rho_1})$ and $s = \inf_\Omega \frac{V}{\rho}$.

Putting $\operatorname{div} A\nabla = \Delta$, $a = p$, $V \equiv q$ *and* $\rho \equiv 1$ *in* (1.10), *we get* (1.9). *Namely, it is a corollary of Theorem* 1.1. *As applications of Theorems* 1.1 *and* 1.2, *we can also obtain some results for an n-dimensional complete minimal submanifold M^n in an Euclidean space $\mathbf{R}^N$, and an n-dimensional unit sphere $M^n = \mathbf{S}^n(1)$.*

Corollary 1.1. *Let $(M^n, \langle\,,\rangle)$ be an n-dimensional complete Riemannian manifold minimally isometrically immersed in the Euclidean space $\mathbf{R}^N$ by Ψ, Ω is a bounded domain in M^n. Assume that $b \le \bar{\lambda}_{A|\Omega} \le c$ ($\bar{\lambda}_{A|\Omega}$ is the eigenvalues of A on Ω, b and c are constants, $b \ge 0$). Let ρ be a continuous function on Ω satisfying $\rho_1 \le \rho(x) \le \rho_2$, $\forall x \in \Omega$, for some positive constants ρ_1 and ρ_2. v denotes the outwards unit normal vector field of $\partial\Omega$. Let V is non-negative continuous function on Ω. Then for the eigenvalues of problem* (1.10), *we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{\rho_2^2}{n^2\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)4E_i((2n+4)\rho_1 E_i + ad), \tag{1.13}$$

where $E_i = \frac{1}{2\rho_1}(-b + \sqrt{b^2 - 4(s-\lambda_i)\rho_1})$, $s = \inf_\Omega \frac{V}{\rho}$.

Corollary 1.2. *Under the same assumptions as Corollary* 1.1, *we obtain*

$$\lambda_{k+1} \le A_k + \sqrt{A_k^2 - B_k}, \tag{1.14}$$

where

$$A_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\lambda_i + \frac{\rho_2^2}{2n^2\rho_1^2}\sum_{i=1}^{k}4E_i((2n+4)\rho_1 E_i + ad)\right\},$$

$$B_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\lambda_i^2 + \frac{\rho_2^2}{n^2\rho_1^2}\sum_{i=1}^{k}\lambda_i 4E_i((2n+4)\rho_1 E_i + ad)\right\},$$

$$E_i = \frac{1}{2\rho_1}(-b + \sqrt{b^2 - 4(s-\lambda_i)\rho_1}), \quad s = \inf_\Omega \frac{V}{\rho}.$$

In Corollary 1.1 and 1.2, the mean curvature of the immersion Ψ equals 0 (i.e. $|H| = 0$), for Ψ is minimal. By Theorem 1.1 and Theorem 1.2, we can easily get the proofs.

Corollary 1.3. *Under the same assumptions as Theorem* 1.1, *assume that M^n is an n-dimensional unit sphere $\mathbf{S}^n(1)$, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \tag{1.15}$$
$$\leq \frac{\rho_2^2}{n^2\rho_1^3}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(4\rho_1 E_i+n^2)(n^2+(2n+4)\rho_1 E_i+ad),$$

where $E_i=\frac{1}{2\rho_1}(-b+\sqrt{b^2-4(s-\lambda_i)\rho_1})$, $s=\inf_\Omega \frac{V}{\rho}$.

Corollary 1.4. *Under the same assumptions as Corollary* 1.3, *we obtain*

$$\lambda_{k+1}\leq A_k+\sqrt{A_k^2-B_k}, \tag{1.16}$$

where

$$A_k=\frac{1}{k}\left(\sum_{i=1}^{k}\lambda_i+\frac{\rho_2^2}{2n^2\rho_1^3}\sum_{i=1}^{k}(4\rho_1 E_i+n^2)(n^2+(2n+4)\rho_1 E_i+ad)\right),$$
$$B_k=\frac{1}{k}\left(\sum_{i=1}^{k}\lambda_i^2+\frac{\rho_2^2}{n^2\rho_1^3}\sum_{i=1}^{k}\lambda_i(4\rho_1 E_i+n^2)(n^2+(2n+4)\rho_1 E_i+ad)\right),$$
$$E_i=\frac{1}{2\rho_1}(-b+\sqrt{b^2-4(s-\lambda_i)\rho_1}),\quad s=\inf_{\Omega}\frac{V}{\rho}.$$

Using $|H|=1$ and Theorem 1.1 and Theorem 1.2, we can also easily get the proof of Corollary 1.3 and Corollary 1.4.

Then, we prove universal inequalities for eigenvalues of these operators on complete manifolds admitting special functions. Our main result is as follows.

Theorem 1.3. *Let* $(M^n,\langle\,,\rangle)$ *be an n-dimensional complete Riemannian manifold and* Ω *is a bounded domain in* M^n. *A is a positive semidefinite symmetric* (1,1)-*tensor on* M^n *and a is a non-negative constant. Assume that* $b\leq\bar{\lambda}_{A|\Omega}\leq c$ ($\bar{\lambda}_{A|\Omega}$ *is the eigenvalues of A on* Ω, *b and c are constants*, $b\geq 0$). *Let* ρ *be a continuous function on* Ω *satisfying* $\rho_1\leq\rho(x)\leq\rho_2$, $\forall x\in\Omega$, *for some positive constants* ρ_1 *and* ρ_2. ν *denotes the outwards unit normal vector field of* $\partial\Omega$. *Let V is non-negative continuous function on* Ω *and denote by* $\{\lambda_i\}_{i=1}^{\infty}$ *the eigenvalues of problem*

$$\begin{cases}\Delta^2 u-a\,\mathrm{div}\,A\nabla u+Vu=\rho\lambda u & in\ \Omega,\\ u|_{\partial\Omega}=\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0.\end{cases}$$

1. *If there exists a* $C^4(\Omega)$ *function* $g:\Omega\to\mathbf{R}$ *such that*

$$|\nabla g|=1,\quad |\Delta g|\leq A_0,\quad on\ \Omega, \tag{1.17}$$

then we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{\rho_2^2}{\rho_1^3}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(8\rho_1 E_i + 2A_0^2)(10\rho_1 E_i + 2A_0^2 + ac). \tag{1.18}$$

Specially, if there exists a $C^4(\Omega)$ *function* $\tilde{g}:\Omega\to\mathbf{R}$ *such that*

$$|\nabla\tilde{g}| = 1, \quad \Delta\tilde{g} = B_0, \quad on\ \Omega, \tag{1.19}$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{4\rho_2^2 E_i}{\rho_1} - B_0^2\right)\left(\frac{6E_i}{\rho_1} - \frac{B_0^2}{\rho_2} + ac\rho_1\right). \tag{1.20}$$

2. *If there exists* l $C^4(\Omega)$ *functions* $g_p:\Omega\to\mathbf{R}$ *such that*

$$\langle\nabla g_p, \nabla g_q\rangle = \delta_{pq}, \quad \Delta g_p = 0, \quad on\ \Omega, \quad p,q = 1,\dots,l, \tag{1.21}$$

then

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4\rho_2^2}{l^2\rho_1^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)E_i[(2l+4)\rho_1 E_i + acl]. \tag{1.22}$$

3. *If* M^n *admits an* $C^4(\Omega)$ *eigenmap* $g=(g_1,g_2,\dots,g_{m+1}):\Omega\to\mathbf{S}^m$ *corresponding to an eigenvalue* μ, *that is*, $\sum_{\alpha=1}^{m+1} g_\alpha^2 = 1$, $\Delta g_\alpha = -\mu g_\alpha$, $\alpha = 1,\dots,m+1$, *then we get*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{\rho_2^2}{4\mu^2\rho_1^3}\sum_{i=1}^{k}(6\mu\rho_1 E_i + \mu^2 + ac)(4\mu\rho_1 E_i + \mu^2). \tag{1.23}$$

In the above, A_0 *and* B_0 *are constants,* $\mathbf{S}^m$ *is the unit m-sphere,* $E_i = \frac{1}{2\rho_1}\left(-b + \sqrt{b^2 - 4(s-\lambda_i)\rho_1}\right)$ *and* $s = \inf_\Omega \frac{V}{\rho}$.

Now we give some examples of manifolds admitting special functions as in Theorem 1.3.

Example 1.1. Let M^n be an n-dimensional complete Riemannian manifold with Ricci curvature satisfying $Ric_M \ge -(n-1)c^2$, $c \ge 0$. Let $g \to \mathbf{R}$ be a smooth function with $|\nabla g| \equiv 1$. It has been proved by Sakai that $|\Delta g| \le (n-1)c$ on M^n (see theorem 3.5 in [14]). Assume further that M^n is a Hadamard manifold and let $\gamma:[0,+\infty)\to M^n$ be a geodesic ray, namely a unit speed geodesic with $d(\gamma(s),\gamma(s)) = t - s$ for any $t > s > 0$. The Busemann function b_γ corresponding to γ defined by

$$b_\gamma(x) = \lim_{t\to+\infty}(d(x,\gamma(t)) - t) \tag{1.24}$$

satisfies $|\nabla b_\gamma| \equiv 1$ (Cf. [1, 9]). We know from Sakai's theorem that $|\Delta b_\gamma| \leq (n-1)c$ on M^n. Thus, any Hadamard manifold with Ricci curvature bounded below by a negative constant admits functions satisfying (1.17).

In [7] M. do Carmo gives two examples as follow:

Example 1.2. Let (N, ds_N^2) be a complete Riemannian manifold and define a Riemannian manifold on $M = \mathbf{R} \times N$ by

$$\langle\, , \rangle = ds_M^2 = dt^2 + \eta^2(t)\, ds_N^2, \tag{1.25}$$

where η is a positive smooth function defined on $\mathbf{R}$ with $\eta(0) = 1$. Then, $(M, \langle\, , \rangle)$ is called a warped product and denoted by $M = \mathbf{R} \times_\eta N$. It is known that M is a complete Riemannian manifold. Set $\eta = e^{-t}$ and consider the warped product $M = \mathbf{R} \times_{e^{-t}} N$. Define $\psi : M \to \mathbf{R}$ by $\psi(x,t) = t$. Then, by some calculations (see Example 4.2 in [7]). We get

$$|\nabla\psi| = 1, \quad \Delta\psi = n-1. \tag{1.26}$$

That is, a warped product manifold $= \mathbf{R} \times_{e^{-t}} N$ admits functions satisfying (1.19).

Let $\mathbf{H}^n$ be the n-dimensional hyperbolic space with constant curvature -1. Using the upper half-space model, $\mathbf{H}^n$ is given by

$$\mathbf{R}_+^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_n > 0\} \tag{1.27}$$

with metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}. \tag{1.28}$$

One can check thet map $\Phi : \mathbf{R} \times_{e^{-t}} \mathbf{R}^{n-1} \to \mathbf{H}^n$ given by

$$\Phi(t,x) = (x, e^t) \tag{1.29}$$

is an isometry. Therefore, $\mathbf{H}^n$ admits a warped product model, $\mathbf{H}^n = \mathbf{R} \times_{e^{-t}} \mathbf{R}^{n-1}$.

Example 1.3. Let N be any complete Riemannian manifold. Let

$$M = \mathbf{R}^l \times N = \{(x_1, x_2, \ldots x_l, z) \,|\, (x_1, x_2, \ldots x_l) \in \mathbf{R}^l, z \in N\} \tag{1.30}$$

be the product of $\mathbf{R}^l$ and N endowed with the product metric. Consider the functions $\phi_q : M \to \mathbf{R}$, $q = 1, \ldots, l$, defined by

$$\phi_q(x_1, x_2, \ldots, x_l, z) = x_q. \tag{1.31}$$

It is easy to see that $\{\phi_q\}_{q=1}^l$ satisfy (1.21).

At last, we give

Example 1.4. Any compact homogeneous Riemannian manifold admits eigenmaps for the first positive eigenvalue of the Laplacian (Cf. [12]).

2. Proof of the Theorem 1.1 and the Theorem 1.2

Firstly, we give the following lemma.

LEMMA 2.1. *Under the same assumptions as Theorem* 1.1, *let* u_i *be i-th orthonormal eigenfunctions of problem* (1.10) *corresponding to eigenvalues* λ_i (*i.e.* $\int_\Omega \rho u_i u_j = \delta_{ij}$), $i = 1, \dots, k$. *Then we have*

$$\int_\Omega |\nabla u_i|^2 \le E_i, \tag{2.1}$$

where $E_i = \dfrac{1}{2\rho_1}(-b + \sqrt{b^2 - 4(s - \lambda_i)\rho_1})$, $s = \inf_\Omega \dfrac{V}{\rho}$.

Proof. Since

$$\int_\Omega |\nabla u_i|^2 = \int_\Omega -u_i(\Delta u_i) \le \left[\int_\Omega u_i^2 \int_\Omega (\Delta u_i)^2\right]^{1/2} \le \left[\frac{1}{\rho_1}\int_\Omega (\Delta u_i)^2\right]^{1/2}, \tag{2.2}$$

and

$$\begin{aligned} \lambda_i &= \int_\Omega u_i(\Delta^2 u_i - a \operatorname{div} A\nabla u_i + Vu_i) \\ &= \int_\Omega (\Delta u_i)^2 + a\int_\Omega \langle \nabla u_i, A\nabla u_i\rangle + \int_\Omega \frac{V}{\rho}\rho u_i^2. \end{aligned} \tag{2.3}$$

Substituting (2.2) into (2.3), and $\bar{\lambda}(A)|_\Omega \ge b$. Let $s = \inf_\Omega \dfrac{V}{\rho}$, so we have

$$\lambda_i \ge \rho_1\left(\int_\Omega |\nabla u_i|^2\right)^2 + b\int_\Omega |\nabla u_i|^2 + s, \tag{2.4}$$

$$0 \ge \rho_1\left(\int_\Omega |\nabla u_i|^2\right)^2 + b\int_\Omega |\nabla u_i|^2 + (s - \lambda_i). \tag{2.5}$$

Thus, we have $b^2 - 4(s - \lambda_i) \ge 0$, and

$$\int_\Omega |\nabla u_i|^2 \le E_i = \frac{1}{2\rho_1}(-b + \sqrt{b^2 - 4(s - \lambda_i)\rho_1}). \tag{2.6}$$

LEMMA 2.2. *Under the same assumptions as Lemma* 2.1, *for any function* $f \in C^4(M^n)$, *we have*

$$\begin{aligned} &-2\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \int_\Omega f u_i\left(\langle \nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right) \\ &\quad \le \sum_{i=1}^k \delta_i(\lambda_{k+1} - \lambda_i)^2 v_i + \sum_{i=1}^k \frac{1}{\delta_i}(\lambda_{k+1} - \lambda_i)\int_\Omega \frac{1}{\rho}\left(\langle \nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right)^2, \end{aligned} \tag{2.7}$$

where δ_i $(i = 1, \dots, k)$ are any positive constants which satisfies $\{\delta_i\}$ is monotonically decreasing and

$$v_i = \int_\Omega [(2\langle \nabla f, \nabla u_i \rangle + u_i \Delta f)^2 - 2u_i \Delta u_i |\nabla f|^2 - 2afu_i \langle \nabla f, A\nabla u_i \rangle - afu_i^2 \operatorname{div} A\nabla f]. \tag{2.8}$$

Proof. Consider the trial functions

$$\psi_i = fu_i - \sum_{j=1}^{k} \gamma_{ij} u_j, \quad i = 1, \dots, k, \tag{2.9}$$

where

$$\gamma_{ij} = \int_\Omega \rho f u_i u_j = \gamma_{ji}. \tag{2.10}$$

Hence, we have

$$\int_\Omega \rho \psi_i u_j = 0 \tag{2.11}$$

and

$$\int_\Omega \rho \psi_i f u_i = \int_\Omega \rho \psi_i^2. \tag{2.12}$$

Then, it is easy to check

$$\begin{aligned} \Delta^2 \psi_i - a \operatorname{div} A\nabla \psi_i + V\psi_i &= \Delta^2 \left(fu_i - \sum_{j=1}^{k} \gamma_{ij} u_j \right) - a \operatorname{div} A\nabla \left(fu_i - \sum_{j=1}^{k} \gamma_{ij} u_j \right) \\ &\quad + V \left(fu_i - \sum_{j=1}^{k} \gamma_{ij} u_j \right) \\ &= \Delta^2 (fu_i) - a \operatorname{div} A\nabla (fu_i) + Vfu_i - \sum_{j=1}^{k} \gamma_{ij} \Delta^2 u_j \\ &\quad + a \sum_{j=1}^{k} \gamma_{ij} \operatorname{div} A\nabla u_j - V \sum_{j=1}^{k} \gamma_{ij} u_j \\ &= \Delta^2 (fu_i) - a \operatorname{div} A\nabla (fu_i) + Vfu_i - \sum_{j=1}^{k} \gamma_{ij} \lambda_j u_j \rho \end{aligned} \tag{2.13}$$

$$= f\Delta^2 u_i + 2\Delta f \Delta u_i + 2\langle \nabla f, \nabla \Delta u_i\rangle + u_i \Delta^2 f + 2\langle \nabla u_i, \nabla \Delta f\rangle$$
$$+ 2\Delta\langle \nabla f, \nabla u_i\rangle - a\,\mathrm{div}(fA\nabla u_i) - a\,\mathrm{div}(u_i A\nabla f) + Vfu_i - \sum_{j=1}^k \gamma_{ij}\lambda_j u_j \rho$$
$$= f\Delta^2 u_i + 2\Delta f \Delta u_i + 2\langle \nabla f, \nabla \Delta u_i\rangle + u_i \Delta^2 f + 2\langle \nabla u_i, \nabla \Delta f\rangle$$
$$+ 2\Delta\langle \nabla f, \nabla u_i\rangle - a\langle \nabla f, A\nabla u_i\rangle - af\,\mathrm{div}\,A\nabla u_i - a\langle \nabla u_i, A\nabla f\rangle$$
$$- au_i\,\mathrm{div}\,A\nabla f + Vfu_i - \sum_{j=1}^k \gamma_{ij}\lambda_j u_j \rho$$
$$= h_i + f\lambda_i u_i \rho - \sum_{j=1}^k \gamma_{ij}\lambda_j u_j \rho,$$

where

$$h_i = u_i\Delta^2 f + 2\langle \nabla f, \nabla \Delta u_i\rangle + 2\langle \nabla u_i, \nabla \Delta f\rangle + 2\Delta\langle \nabla f, \nabla u_i\rangle$$
$$+ 2\Delta f \Delta u_i - 2a\langle \nabla f, A\nabla u_i\rangle - au_i\,\mathrm{div}\,A\nabla f. \tag{2.14}$$

From (2.11), (2.12), (2.13) and (2.14), we get

$$\begin{aligned}\int_\Omega \psi_i(\Delta^2\psi_i - a\,\mathrm{div}\,A\nabla\psi_i + V\psi_i) &= \int_\Omega \psi_i\left(h_i + f\lambda_i u_i\rho - \sum_{j=1}^k \gamma_{ij}\lambda_j u_j\rho\right)\\ &= \int_\Omega fu_ih_i - \sum_{j=1}^k \int_\Omega \gamma_{ij}u_jh_i + \lambda_i\int_\Omega \psi_i^2\rho\\ &= \int_\Omega fu_ih_i - \sum_{j=1}^k \gamma_{ij}s_{ij} + \lambda_i\int_\Omega \psi_i^2\rho,\end{aligned} \tag{2.15}$$

where

$$s_{ij} = \int_\Omega h_i u_j. \tag{2.16}$$

Putting (2.15) into Rayleigh–Ritz inequality

$$\lambda_{k+1} \le \frac{\int_\Omega \psi_i(\Delta^2\psi_i - a\,\mathrm{div}\,A\nabla\psi_i + V\psi_i)}{\int_\Omega \psi_i^2\rho}, \tag{2.17}$$

it follows that

$$(\lambda_{k+1} - \lambda_i)\int_\Omega \psi_i^2\rho \le \int_\Omega fu_ih_i - \sum_{j=1}^k \gamma_{ij}s_{ij}. \tag{2.18}$$

Then, using Stokes' theorem, we obtain

$$\int_\Omega f u_i h_i = v_i, \tag{2.19}$$

where

$$v_i = \int_\Omega [(2\langle \nabla f, \nabla u_i\rangle + u_i \Delta f)^2 - 2u_i \Delta u_i |\nabla f|^2 - 2af u_i \langle \nabla f, A\nabla u_i\rangle - af u_i^2 \operatorname{div} A\nabla f].$$

Then, from the definitions of h_i and s_{ij}, we have

$$\begin{aligned} s_{ij} &= \int_\Omega h_i u_j \\ &= \int_\Omega u_j[\Delta^2(f u_i) - a \operatorname{div} A\nabla(f u_i) + V(f u_i) - f\lambda_i u_i \rho] \\ &= \int_\Omega f u_i(\Delta^2(u_j) - a \operatorname{div} A\nabla(u_j) + V(u_j)) - \lambda_i \int_\Omega f u_i u_j \rho \\ &= \int_\Omega f u_i \lambda_j u_j \rho - \lambda_i \int_\Omega f u_i u_j \rho \\ &= (\lambda_j - \lambda_i)\gamma_{ij}. \end{aligned} \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18), it gets

$$(\lambda_{k+1} - \lambda_i)\int_\Omega \psi_i^2 \rho \le v_i + \sum_{j=1}^k (\lambda_i - \lambda_j)\gamma_{ij}^2. \tag{2.21}$$

Then it follows from (2.9) that

$$\begin{aligned} -2\int_\Omega \psi_i\left(\langle A\nabla f, \nabla u_i\rangle + \frac{1}{2}u_i \Delta f\right) \\ = -2\int_\Omega f u_i\left(\langle \nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right) + 2\sum_{j=1}^k \gamma_{ij} t_{ij}, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} t_{ij} &= \int_\Omega u_j\left(\langle \nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right) \\ &= \int_\Omega u_j\left(\operatorname{div}(u_i\nabla f) - u_i\Delta f + \frac{1}{2}u_i\Delta f\right) \\ &= \int_\Omega \left(u_j \operatorname{div}(u_i \nabla f) - \frac{1}{2}u_i u_j \Delta f\right) \\ &= \int_\Omega \left(\operatorname{div}(u_j u_i \nabla f) - \langle \nabla u_j, u_i \nabla f\rangle - \frac{1}{2}u_i u_j \Delta f\right) \end{aligned} \tag{2.23}$$

$$= -\int_\Omega u_i\left(\langle\nabla f, \nabla u_j\rangle + \frac{1}{2}u_j\Delta f\right)$$
$$= -t_{ji}.$$

Multiplying $(\lambda_{k+1}-\lambda_i)$ in the both sides of (2.22), then taking sum on i from 1 to k, and using the following inequality

$$(2.24)\quad 2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)^2\gamma_{ij}t_{ij} = -2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)\gamma_{ij}t_{ij}$$
$$\geq \sum_{i,j=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2\gamma_{ij}^2 - \sum_{i,j=1}^{k}\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)t_{ij}^2,$$

where we take arbitrary positive constants δ_i $(i = 1, \dots, k)$ which satisfies $\{\delta_i\}$ is monotonically decreasing. Hence, we get

$$(2.25)\qquad -2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega \psi_i\left(\langle\nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right)$$
$$\geq -2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega f u_i\left(\langle\nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right)$$
$$-\sum_{i,j=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2\gamma_{ij}^2 - \sum_{i,j=1}^{k}\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)t_{ij}^2.$$

Then, utilizing (2.21), one has

$$(2.26)\qquad (\lambda_{k+1}-\lambda_i)^2\left[-2\int_\Omega \psi_i\left(\langle\nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right)\right]$$
$$= -2(\lambda_{k+1}-\lambda_i)^2\int_\Omega \sqrt{\rho}\psi_i\left[\frac{1}{\sqrt{\rho}}\left(\langle\nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right) - \sqrt{\rho}\sum_{j=1}^{k}t_{ij}u_j\right]$$
$$\leq \frac{(\lambda_{k+1}-\lambda_i)}{\delta_i}\int_\Omega\left[\frac{1}{\sqrt{\rho}}\left(\langle\nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right) - \sqrt{\rho}\sum_{j=1}^{k}t_{ij}u_j\right]^2$$
$$+\delta_i(\lambda_{k+1}-\lambda_i)^3\int_\Omega \rho\psi_i^2$$
$$\leq \frac{(\lambda_{k+1}-\lambda_i)}{\delta_i}\int_\Omega\left[\frac{1}{\rho}\left(\langle\nabla f, \nabla u_i\rangle + \frac{1}{2}u_i\Delta f\right)^2 - \sum_{j=1}^{k}t_{ij}^2\right]$$
$$+\delta_i(\lambda_{k+1}-\lambda_i)^2\left[v_i + \sum_{j=1}^{k}(\lambda_i-\lambda_j)\gamma_{ij}^2\right].$$

Taking summation on i from 1 to k in (2.26), we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left[-2\int_\Omega \psi_i\left(\langle\nabla f,\nabla u_i\rangle+\frac{1}{2}u_i\Delta f\right)\right]$$
$$\leq \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta_i}\int_\Omega\left[\frac{1}{\rho}\left(\langle\nabla f,\nabla u_i\rangle+\frac{1}{2}u_i\Delta f\right)^2-\sum_{j=1}^{k}t_{ij}^2\right]$$
$$+\sum_{i=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)^2\left[v_i+\sum_{j=1}^{k}(\lambda_i-\lambda_j)\gamma_{ij}^2\right]. \tag{2.27}$$

Since the positive constants δ_i $(i=1,\dots,k)$ satisfies that $\{\delta_i\}$ is monotonically decreasing, we can obtain

$$\sum_{i,j=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)^2(\lambda_i-\lambda_j)\gamma_{ij}^2 \leq -\sum_{i,j=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2\gamma_{ij}^2. \tag{2.28}$$

Then, putting (2.25) and (2.28) into (2.27), we have

$$-2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega fu_i\left(\langle\nabla f,\nabla u_i\rangle+\frac{1}{2}u_i\Delta f\right)$$
$$-\sum_{i,j=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2\gamma_{ij}^2-\sum_{i,j=1}^{k}\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)t_{ij}^2$$
$$\leq \sum_{i=1}^{k}\frac{(\lambda_{k+1}-\lambda_i)}{\delta_i}\int_\Omega\left[\frac{1}{\rho}\left(\langle\nabla f,\nabla u_i\rangle+\frac{1}{2}u_i\Delta f\right)^2-\sum_{j=1}^{k}t_{ij}^2\right]$$
$$+\sum_{i=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)^2v_i-\sum_{i,j=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)^2\gamma_{ij}^2. \tag{2.29}$$

Hence, we have

$$-2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega fu_i\left(\langle\nabla f,\nabla u_i\rangle+\frac{1}{2}u_i\Delta f\right)$$
$$\leq \sum_{i=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)^2v_i+\sum_{i=1}^{k}\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left(\langle\nabla f,\nabla u_i\rangle+\frac{1}{2}u_i\Delta f\right)^2.$$

Thus, Lemma 2.2 is proved to be true.

Now, by Lemma 2.1 and Lemma 2.2, we can give the proof of Theorem 1.1:

Proof of Theorem 1.1. Let $y^1, y^2, \dots, y^N$ be the standard coordinate functions of $\mathbf{R}^N$. Taking $f=y^\alpha$, $\alpha=1,\dots,N$, we get

$$
\begin{aligned}
(2.30) \qquad & -2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega y^\alpha u_i\left(\langle\nabla y^\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta y^\alpha\right)\\
&\le \sum_{i=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)^2 v_i^\alpha\\
&\quad+\sum_{i=1}^{k}\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)\int_\Omega\frac{1}{\rho}\left(\langle\nabla y^\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta y^\alpha\right)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
(2.31) \qquad v_i^\alpha = \int_\Omega &[(2\langle\nabla y^\alpha,\nabla u_i\rangle+u_i\Delta y^\alpha)^2-2u_i\Delta u_i|\nabla y^\alpha|^2\\
&-2ay^\alpha u_i\langle\nabla y^\alpha,A\nabla u_i\rangle-ay^\alpha u_i^2\operatorname{div}A\nabla y^\alpha].
\end{aligned}
$$

On the other hand, we have the following equalities:

$$
\sum_{\alpha=1}^{N}|\nabla y^\alpha|^2=n,\quad \sum_{\alpha=1}^{N}\langle\nabla y^\alpha,\nabla u_i\rangle^2=|\nabla u_i|^2,
$$

$$
\sum_{\alpha=1}^{N}(\Delta y^\alpha)^2=n^2|H|^2,\quad \sum_{\alpha=1}^{N}(\langle\nabla y^\alpha,\nabla u_i\rangle\Delta y^\alpha)=0,
$$

see [3, Lemma 2.1], where $|H|$ is the mean curvature. Then, from Lemma 2.1, we obtain

$$
\begin{aligned}
(2.32) \qquad \sum_{\alpha=1}^{N}v_i^\alpha &= \int_\Omega\left[\sum_{\alpha=1}^{N}(2\langle\nabla y^\alpha,\nabla u_i\rangle+u_i\Delta y^\alpha)^2-2u_i\Delta u_i\sum_{\alpha=1}^{N}|\nabla y^\alpha|^2\right]\\
&\quad-2\sum_{\alpha=1}^{N}\int_\Omega ay^\alpha u_i\langle\nabla y^\alpha,A\nabla u_i\rangle-\sum_{\alpha=1}^{N}\int_\Omega ay^\alpha u_i^2\operatorname{div}A\nabla y^\alpha\\
&=n^2\int_\Omega|H|^2u_i^2+4\int_\Omega|\nabla u_i|^2-2n\int_\Omega u_i\Delta u_i\\
&\quad-2\sum_{\alpha=1}^{N}\int_\Omega ay^\alpha u_i\langle\nabla y^\alpha,A\nabla u_i\rangle-\sum_{\alpha=1}^{N}\int_\Omega ay^\alpha u_i^2\operatorname{div}A\nabla y^\alpha\\
&=n^2\int_\Omega|H|^2u_i^2+4\int_\Omega|\nabla u_i|^2-2n\int_\Omega u_i\Delta u_i\\
&\quad-\sum_{\alpha=1}^{N}\int_\Omega a[y^\alpha\langle\nabla y^\alpha,A\nabla u_i^2\rangle+y^\alpha u_i^2\operatorname{div}A\nabla y^\alpha]
\end{aligned}
$$

$$
\begin{aligned}
&= n^2\int_\Omega |H|^2u_i^2 + 4\int_\Omega |\nabla u_i|^2 - 2n\int_\Omega u_i\Delta u_i\\
&\quad -\sum_{\alpha=1}^N\int_\Omega a[y^\alpha\langle A\nabla y^\alpha, \nabla u_i^2\rangle + y^\alpha u_i^2 \operatorname{div} A\nabla y^\alpha]\\
&= n^2\int_\Omega |H|^2u_i^2 + 4\int_\Omega |\nabla u_i|^2 - 2n\int_\Omega u_i\Delta u_i\\
&\quad -\sum_{\alpha=1}^N\int_\Omega a y^\alpha(\operatorname{div}(u_i^2 A\nabla y^\alpha))\\
&= n^2\int_\Omega |H|^2u_i^2 + (2n+4)\int_\Omega |\nabla u_i|^2 - \sum_{\alpha=1}^N\int_\Omega a(-u_i^2\langle\nabla y^\alpha, A\nabla y^\alpha\rangle)\\
&= n^2\int_\Omega |H|^2\rho u_i^2 + (2n+4)\int_\Omega |\nabla u_i|^2 + \int_\Omega a u_i^2(\operatorname{tr}(A))\\
&= n^2\int_\Omega \frac{1}{\rho}|H|^2\rho u_i^2 + (2n+4)\int_\Omega |\nabla u_i|^2 + \int_\Omega \frac{a}{\rho}\rho u_i^2(\operatorname{tr}(A))\\
&\le \frac{n^2}{\rho_1}\sup_\Omega |H|^2 + (2n+4)E_i + \frac{ad}{\rho_1},
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega\sum_{\alpha=1}^N\left(\langle\nabla y^\alpha, \nabla u_i\rangle + \frac{1}{2}u_i\Delta y^\alpha\right)^2 &= \int_\Omega\frac{1}{\rho}|\nabla u_i|^2 + \frac{1}{4}n^2\int_\Omega\frac{1}{\rho}u_i^2|H|^2\\
&\le \frac{E_i}{\rho_1} + \frac{n^2}{4\rho_1^2}\sup_\Omega |H|^2,
\end{aligned}
\tag{2.33}
$$

and

$$
-2\sum_{\alpha=1}^N\int_\Omega y^\alpha u_i\left(\langle\nabla y^\alpha, \nabla u_i\rangle + \frac{1}{2}u_i\Delta y^\alpha\right) = \int_\Omega u_i^2\sum_{\alpha=1}^N|\nabla y^\alpha|^2 \ge \frac{n}{\rho_2}.
\tag{2.34}
$$

Substituting (2.32), (2.33) and (2.34) into (2.30) yields

$$
\begin{aligned}
\frac{n}{\rho_2}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2 &\le \sum_{i=1}^k\delta_i(\lambda_{k+1}-\lambda_i)^2\left(\frac{n^2}{\rho_1}\sup_\Omega |H|^2 + (2n+4)E_i + \frac{ad}{\rho_1}\right)\\
&\quad + \sum_{i=1}^k\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)\left(\frac{E_i}{\rho_1} + \frac{n^2}{4\rho_1^2}\sup_\Omega |H|^2\right).
\end{aligned}
\tag{2.35}
$$

Now, we can define

$$
\delta_i = \frac{\delta}{\frac{n^2}{\rho_1}\sup_\Omega |H|^2 + (2n+4)E_i + \frac{ad}{\rho_1}},
\tag{2.36}
$$

where δ is an arbitrary positive constant, and it is easy to find $\{\delta_i\}$ is monotonically decreasing. Thus, we have

$$\frac{n}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2 + \sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\left(\frac{E_i}{\rho_1}+\frac{n^2}{4\rho_1^2}\sup_{\Omega}|H|^2\right) \times\left(\frac{n^2}{\rho_1}\sup_{\Omega}|H|^2+(2n+4)E_i+\frac{ad}{\rho_1}\right). \tag{2.37}$$

Then taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{4E_i}{\rho_1}+\frac{n^2}{\rho_1^2}\sup_{\Omega}|H|^2\right)\left(\frac{n^2}{\rho_1}\sup_{\Omega}|H|^2+(2n+4)E_i+\frac{ad}{\rho_1}\right)}{4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2}\right\}^{1/2} \tag{2.38}$$

in (2.37), we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{\rho_2^2}{n^2\rho_1^3}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\rho_1E_i+n^2\sup_{\Omega}|H|^2\right) \times\left(n^2\sup_{\Omega}|H|^2+(2n+4)\rho_1E_i+ad\right). \tag{2.39}$$

It completes the proof of Theorem 1.1.

Proof of Theorem 1.2.
Since

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{\rho_2^2}{n^2\rho_1^3}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\rho_1E_i+n^2\sup_{\Omega}|H|^2\right) \times\left(n^2\sup_{\Omega}|H|^2+(2n+4)\rho_1E_i+ad\right), \tag{2.40}$$

we have

$$\sum_{i=1}^{k}(\lambda_{k+1}^2-2\lambda_{k+1}\lambda_i+\lambda_i^2) \le \frac{\rho_2^2}{n^2\rho_1^3}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\rho_1E_i+n^2\sup_{\Omega}|H|^2\right) \times\left(n^2\sup_{\Omega}|H|^2+(2n+4)\rho_1E_i+ad\right). \tag{2.41}$$

By the direct calculations, one get

$$k\lambda_{k+1}^2 - 2kA_k\lambda_{k+1} + kB_k \leq 0, \tag{2.42}$$

where

$$A_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\lambda_i + \frac{\rho_2^2}{2n^2\rho_1^3}\sum_{i=1}^{k}\left(4\rho_1 E_i + n^2 \sup_{\Omega}|H|^2\right)\right.$$
$$\left.\times\left(n^2 \sup_{\Omega}|H|^2 + (2n+4)\rho_1 E_i + ad\right)\right\},$$

$$B_k = \frac{1}{k}\left\{\sum_{i=1}^{k}\lambda_i^2 + \frac{\rho_2^2}{n^2\rho_1^3}\sum_{i=1}^{k}\lambda_i\left(4\rho_1 E_i + n^2 \sup_{\Omega}|H|^2\right)\right.$$
$$\left.\times\left(n^2 \sup_{\Omega}|H|^2 + (2n+4)\rho_1 E_i + ad\right)\right\}.$$

Hence, we have

$$\lambda_{k+1} \leq A_k + \sqrt{A_k^2 - B_k}. \tag{2.43}$$

3. Proof of the Theorem 1.3

Proof of Theorem 1.3. Similar calculations as in the lemma 2.1 give

$$\int_{\Omega}|\nabla u_i|^2 \leq E_i, \tag{3.1}$$

where $E_i = \frac{1}{2\rho_1}(-b + \sqrt{b^2 - 4(s-\lambda_i)\rho_1}),\ s = \inf_{\Omega}\frac{V}{\rho}$.

1. Taking $f = g$ on Ω in (2.7), using (1.17), (3.1), $\langle\nabla g, A\nabla g\rangle \leq c|\nabla g|^2$ and Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_{\Omega}u_i^2 \\
&\leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_{\Omega}u_i^2|\nabla g|^2 \\
&\leq -2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_{\Omega}gu_i\left(\langle\nabla g,\nabla u_i\rangle + \frac{1}{2}u_i\Delta g\right) \\
&\leq \sum_{i=1}^{k}\delta_i(\lambda_{k+1}-\lambda_i)^2 v_i + \sum_{i=1}^{k}\frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)\int_{\Omega}\frac{1}{\rho}\left(\langle\nabla g,\nabla u_i\rangle + \frac{1}{2}u_i\Delta g\right)^2,
\end{aligned} \tag{3.2}$$

and

$$v_i = \int_\Omega [(2\langle \nabla g, \nabla u_i \rangle + u_i \Delta g)^2 - 2u_i \Delta u_i |\nabla g|^2 - 2agu_i \langle \nabla g, A\nabla u_i \rangle - agu_i^2 \operatorname{div} A\nabla g] = \int_\Omega [(2\langle \nabla g, \nabla u_i \rangle + u_i \Delta g)^2 - 2u_i \Delta u_i] + a\int_\Omega u_i^2 \langle \nabla g, A\nabla g \rangle \leq \int_\Omega 8\langle \nabla g, \nabla u_i \rangle^2 + 2u_i^2 (\Delta g)^2 + 2\int_\Omega |\nabla u_i|^2 + a\int_\Omega \frac{1}{\rho}\rho u_i^2 \langle \nabla g, A\nabla g \rangle \leq \int_\Omega 8|\nabla u_i|^2 + 2|\nabla u_i|^2 + \int_\Omega 2u_i^2 A_0^2 + \frac{ac}{\rho_1} \leq 10E_i + \frac{2A_0^2}{\rho_1} + \frac{ac}{\rho_1}. \tag{3.3}$$

We also have

$$\int_\Omega \frac{1}{\rho}\left(\langle \nabla g, \nabla u_i \rangle + \frac{1}{2}u_i \Delta g\right)^2 \leq \int_\Omega \frac{1}{\rho}\left(2\langle \nabla g, \nabla u_i \rangle^2 + \frac{1}{2}u_i^2 (\Delta g)^2\right) \leq \int_\Omega \frac{1}{\rho} 2|\nabla u_i|^2 + A_0^2 \int_\Omega \frac{1}{2\rho} u_i^2 \leq \frac{2E_i}{\rho_1} + \frac{A_0^2}{2\rho_1^2}. \tag{3.4}$$

Thus, one have

$$\frac{1}{\rho_2}\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^k \delta_i (\lambda_{k+1} - \lambda_i)^2 \left(10E_i + \frac{2A_0^2}{\rho_1} + \frac{ac}{\rho_1}\right) + \sum_{i=1}^k \frac{1}{\delta_i}(\lambda_{k+1} - \lambda_i)\left(\frac{2E_i}{\rho_1} + \frac{A_0^2}{2\rho_1^2}\right). \tag{3.5}$$

Putting

$$\delta_i = \frac{\delta}{10E_i + \frac{2A_0^2}{\rho_1} + \frac{ac}{\rho_1}} \tag{3.6}$$

in (3.5), where δ is an arbitrary positive constant, and it is easy to find $\{\delta_i\}$ is monotonically decreasing.

Thus, one can get

$$\frac{1}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2 + \sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\left(\frac{2E_i}{\rho_1}+\frac{A_0^2}{2\rho_1^2}\right)\left(10E_i+\frac{2A_0^2}{\rho_1}+\frac{ac}{\rho_1}\right). \tag{3.7}$$

Taking

$$\delta = \left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\frac{8E_i}{\rho_1}+\frac{2A_0^2}{\rho_1^2}\right)\left(10E_i+\frac{2A_0^2}{\rho_1}+\frac{ac}{\rho_1}\right)}{4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2}\right\}^{1/2}, \tag{3.8}$$

we have (1.18)

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{\rho_2^2}{\rho_1^3}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(8\rho_1E_i+2A_0^2)(10\rho_1E_i+2A_0^2+ac).$$

Specially, we take $f=\tilde{g}$ on Ω in (2.7) and use (1.19), we get

$$\begin{aligned} v_i &= \int_\Omega [(2\langle\nabla\tilde{g},\nabla u_i\rangle + u_i\Delta\tilde{g})^2 - 2u_i\Delta u_i|\nabla\tilde{g}|^2 \\ &\qquad - 2a\tilde{g}u_i\langle\nabla\tilde{g}, A\nabla u_i\rangle - a\tilde{g}u_i^2 \operatorname{div} A\nabla\tilde{g}] \\ &= \int_\Omega (6|\nabla u_i|^2 + 4u_iB_0\langle\nabla\tilde{g},\nabla u_i\rangle + u_i^2B_0^2) + a\int_\Omega u_i^2\langle\nabla\tilde{g}, A\nabla\tilde{g}\rangle \\ &\leq \int_\Omega (6|\nabla u_i|^2 - 2u_i^2B_0\Delta\tilde{g} + u_i^2B_0^2) + a\int_\Omega u_i^2\langle\nabla\tilde{g}, A\nabla\tilde{g}\rangle \\ &\leq 6E_i - B_0^2\int_\Omega u_i^2 + \frac{ac}{\rho_1} \\ &\leq 6E_i - \frac{B_0^2}{\rho_1} + \frac{ac}{\rho_1}, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\int_\Omega \frac{1}{\rho}\left(\langle\nabla\tilde{g},\nabla u_i\rangle + \frac{1}{2}u_i\Delta\tilde{g}\right)^2 \\ &\qquad \leq \int_\Omega \frac{1}{\rho}\left(\langle\nabla\tilde{g},\nabla u_i\rangle^2 + u_i\Delta\tilde{g}\langle\nabla\tilde{g},\nabla u_i\rangle + \frac{1}{4}u_i^2(\Delta\tilde{g})^2\right) \end{aligned} \tag{3.10}$$

$$\leq \int_\Omega \frac{1}{\rho}|\nabla u_i|^2 - \int_\Omega \frac{1}{4\rho}u_i^2(\Delta\tilde{g})^2$$
$$\leq \frac{E_i}{\rho_1} - \frac{B_0^2}{4\rho_1^2}.$$

By similar calculations as in (3.2), we have (1.20).

2. Taking $f = g_p$ on Ω in (2.7), one get

$$\frac{1}{\rho_2}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2 \leq \sum_{i=1}^k \delta_i(\lambda_{k+1}-\lambda_i)^2 v_i^p + \sum_{i=1}^k \frac{1}{\delta_i}(\lambda_{k+1}-\lambda_i)\int_\Omega \frac{1}{\rho}\left(\langle\nabla g_p, \nabla u_i\rangle + \frac{1}{2}u_i\Delta g_p\right)^2, \tag{3.11}$$

and

$$\begin{aligned} v_i^p &= \int_\Omega [(2\langle\nabla g_p, \nabla u_i\rangle + u_i\Delta g_p)^2 - 2u_i\Delta u_i|\nabla g_p|^2 \\ &\quad - 2ag_pu_i\langle\nabla g_p, A\nabla u_i\rangle - ag_pu_i^2 \operatorname{div} A\nabla g_p] \\ &= \int_\Omega 4\langle\nabla g_p, \nabla u_i\rangle^2 - \int_\Omega 2u_i\Delta u_i + a\int_\Omega u_i^2\langle\nabla g_p, A\nabla g_p\rangle \\ &\leq \int_\Omega 4\langle\nabla g_p, \nabla u_i\rangle^2 + 2\int_\Omega |\nabla u_i|^2 + a\int_\Omega \frac{1}{\rho}\rho u_i^2\langle\nabla g_p, A\nabla g_p\rangle \\ &\leq \int_\Omega 4\langle\nabla g_p, \nabla u_i\rangle^2 + 2E_i + \frac{ac}{\rho_1}, \end{aligned} \tag{3.12}$$

we also have

$$\int_\Omega \frac{1}{\rho}\left(\langle\nabla g_p, \nabla u_i\rangle + \frac{1}{2}u_i\Delta g_p\right)^2 \leq \int_\Omega \frac{1}{\rho}\langle\nabla g_p, \nabla u_i\rangle^2. \tag{3.13}$$

Let $\delta_i = \delta$ $(i = 1, \ldots, k)$, δ is an arbitrary positive constant. δ_i is also monotonically decreasing. Thus, we know from (3.11), (3.12) and (3.13) that

$$\frac{1}{\rho_2}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2 \leq \sum_{i=1}^k \delta(\lambda_{k+1}-\lambda_i)^2\left(\int_\Omega 4\langle\nabla g_p, \nabla u_i\rangle^2 + 2E_i + \frac{ac}{\rho_1}\right) + \sum_{i=1}^k \frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\int_\Omega \frac{1}{\rho}\langle\nabla g_p, \nabla u_i\rangle^2. \tag{3.14}$$

From (1.21), we know that

$$\sum_{p=1}^{l} \langle \nabla g_p, \nabla u_i \rangle^2 \le |\nabla u_i|^2.$$

Hence, we obtain by summing over p in (3.14) that

$$\begin{aligned}
(3.15)\quad \frac{l}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 &\le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\sum_{p=1}^{l}\left(\int_\Omega 4\langle \nabla g_p, \nabla u_i\rangle^2 + 2E_i + \frac{ac}{\rho_1}\right)\\
&\quad + \sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\sum_{p=1}^{l}\int_\Omega \frac{1}{\rho}\langle \nabla g_p, \nabla u_i\rangle^2\\
&\le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\left(\int_\Omega 4|\nabla u_i|^2 + 2lE_i + \frac{acl}{\rho_1}\right)\\
&\quad + \sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\int_\Omega \frac{1}{\rho}|\nabla u_i|^2\\
&\le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\left(4E_i + 2lE_i + \frac{acl}{\rho_1}\right)\\
&\quad + \sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\frac{E_i}{\rho_1}.
\end{aligned}$$

Taking

$$(3.16)\qquad \delta = \left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)E_i}{4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2[(4+2l)\rho_1E_i + acl]}\right\}^{1/2},$$

we obtain (1.22).

3. Taking the Laplacian of the equation

$$\sum_{\alpha=1}^{m+1} g_\alpha^2 = 1,$$

and using the fact that

$$\Delta g_\alpha = -\mu g_\alpha, \quad \alpha = 1, \dots, m+1,$$

we get

$$\sum_{\alpha=1}^{m+1} |\nabla g_\alpha|^2 = \mu.$$

Then, by taking take $f = g_\alpha$ on Ω in (2.7), letting $\delta_i =$ arbitrary positive constant δ $(i = 1, \dots, k)$ and sunmming over α, we have

$$
\begin{aligned}
(3.17)\qquad & \frac{\mu}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{m+1}\int_\Omega u_i^2|\nabla g_\alpha|^2 \\
&\le -2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{m+1}\int_\Omega g_\alpha u_i\left(\langle\nabla g_\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta g_\alpha\right) \\
&\le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{m+1}v_i^\alpha \\
&\quad+\sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\sum_{\alpha=1}^{m+1}\int_\Omega\frac{1}{\rho}\left(\langle\nabla g_\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta g_\alpha\right)^2.
\end{aligned}
$$

We also have

$$
\begin{aligned}
(3.18)\qquad \sum_{\alpha=1}^{m+1}v_i^\alpha &= \sum_{\alpha=1}^{m+1}\int_\Omega[(2\langle\nabla g_\alpha,\nabla u_i\rangle+u_i\Delta g_\alpha)^2-2u_i\Delta u_i|\nabla g_\alpha|^2 \\
&\qquad -2ag_\alpha u_i\langle\nabla g_\alpha, A\nabla u_i\rangle-ag_\alpha u_i^2\operatorname{div}A\nabla g_\alpha] \\
&= \sum_{\alpha=1}^{m+1}\int_\Omega(2\langle\nabla g_\alpha,\nabla u_i\rangle-u_i\mu g_\alpha)^2-\int_\Omega 2\mu u_i\Delta u_i \\
&\quad + a\sum_{\alpha=1}^{m+1}\int_\Omega u_i^2\langle\nabla g_\alpha, A\nabla g_\alpha\rangle \\
&\le \int_\Omega 4\sum_{\alpha=1}^{m+1}\langle\nabla g_\alpha,\nabla u_i\rangle^2+\int_\Omega u_i^2\mu^2+2\mu\int_\Omega|\nabla u_i|^2 \\
&\quad + a\sum_{\alpha=1}^{m+1}\int_\Omega u_i^2\langle\nabla g_\alpha, A\nabla g_\alpha\rangle \\
&\le \int_\Omega 4\sum_{\alpha=1}^{m+1}|\nabla g_\alpha|^2|\nabla u_i|^2+\frac{\mu^2}{\rho_1}+2\mu E_i+\frac{ac}{\rho_1} \\
&\le 4\mu E_i+\frac{\mu^2}{\rho_1}+2\mu E_i+\frac{ac}{\rho_1}=6\mu E_i+\frac{\mu^2}{\rho_1}+\frac{ac}{\rho_1},
\end{aligned}
$$

and

$$\sum_{\alpha=1}^{m+1}\int_\Omega \frac{1}{\rho}\left(\langle\nabla g_\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\Delta g_\alpha\right)^2=\sum_{\alpha=1}^{m+1}\int_\Omega \frac{1}{\rho}\left(\langle\nabla g_\alpha,\nabla u_i\rangle+\frac{1}{2}u_i\mu g_\alpha\right)^2 \tag{3.19}$$
$$\le \int_\Omega \frac{1}{\rho}\left(\sum_{\alpha=1}^{m+1}\langle\nabla g_\alpha,\nabla u_i\rangle^2+\frac{u_i^2\mu^2}{4}\right)$$
$$\le \frac{\mu E_i}{\rho_1}+\frac{\mu^2}{4\rho_1^2}.$$

Thus, we get

$$\frac{\mu}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^2\left(6\mu E_i+\frac{\mu^2}{\rho_1}+\frac{ac}{\rho_1}\right) \tag{3.20}$$
$$+\sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)\left(\frac{\mu E_i}{\rho_1}+\frac{\mu^2}{4\rho_1^2}\right).$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)(4\mu\rho_1E_i+\mu^2)}{4\rho_1\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2(6\mu\rho_1E_i+\mu+ac)}\right\}^{1/2}, \tag{3.21}$$

we obtain (1.23). This completes the proof of Theorem 1.3.

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