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KODAI MATH. J.
38 (2015), 228–248

LATTICES IN CONTACT LIE GROUPS AND 5-DIMENSIONAL CONTACT SOLVMANIFOLDS

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Abstract

We investigate the existence and properties of uniform lattices in Lie groups and use these results to prove that, in dimension 5, there are exactly seven connected and simply connected contact Lie groups with uniform lattices, all of which are solvable. In particular, it is also shown that the special affine group has no uniform lattice.

1. Introduction

This paper investigates the geometry of compact contact manifolds that are uniformized by contact Lie groups, i.e., manifolds of the form $\Gamma\backslash G$ for some Lie group G with a left invariant contact structure and uniform lattice $\Gamma \subset G$. In particular, we restrict our attention to dimension five and describe which five-dimensional contact Lie groups admit uniform lattices. We prove that there are exactly seven connected and simply connected such Lie groups. Five of them are central extensions; the other two are semi-direct products. Furthermore, all seven are solvable. In contrast, there are only 4 connected and simply connected Lie groups with a lattice, that have a left invariant symplectic form [19].

This paper is organized as follows. In Section 2, we give the preliminaries for the work ahead. This includes both a review of several classical results and some original results regarding contact Lie groups. Fundamental to this paper are Theorem 2.10, which describes all five-dimensional contact Lie algebras, and the list in Subsection 2.3.2, which delineates the Lie algebras of all the five-dimensional unimodular contact Lie groups.

In Section 3, the main theorem of the paper (Theorem 3.1) is stated as well as an immediate corollary. This theorem is proven in Section 4. A major yet technical aspect of this proof is the list of certain structures on the Lie algebras

2000 *Mathematics Subject Classification*. 53D10, 53D35, 53C50, 53C25, 57R17.

Key words and phrases. lattice; uniform lattice; contact manifold; nilmanifold; solvmanifold; special affine group; Heisenberg group; invariant contact structure; boundary of contact type; Lie group; Lie algebra.

A part of this work was done while the author was supported by the IST Programme of the European Union (IST-2001-35443).

Received August 21, 2013; revised August 27, 2014.

of the Lie groups in Subsection 2.3.2. For ease of reading, this list has been relegated to Appendix I (Section 5).

The authors thank the referee of this paper for the many essential hints and suggestions. This paper would not have been possible without this help. The second author also thanks Dr. Patrick Chen at John Carroll University for several fruitful and clarifying conversations on the subject of this paper.

2. Preliminaries

2.1. Lattices on solvable Lie groups

A *lattice* of a Lie group G is a discrete subgroup Γ such that the manifold $\Gamma\backslash G$ has a finite volume. If $\Gamma\backslash G$ is compact, then Γ is called a *uniform lattice*. This is a well-studied field, and much of the following material has been derived from Chapter 2 of Part I in [23], much of which is itself an exposition of classical results by Mostow ([21]), Auslander ([1], [2]) and Raghunathan ([24]). More details as well as more results on this topic can be found within these various sources.

One of the most important results on lattices on general Lie groups was proved by Milnor in [20].

THEOREM 2.1. *If G is a Lie group with a uniform lattice, then its Lie algebra is unimodular.*

For nilpotent Lie groups, more precise results are known. In particular, a lattice on a nilpotent Lie group induces lattices on the central series of the Lie group.

THEOREM 2.2. *Let N be a simply connected nilpotent Lie group with lattice Γ. Let $\cdots \subset N_2 \subset N_1 \subset N_0 = N$ be the decreasing central series of N. Then, for each $j = 0, 1, 2, \ldots,$ $\Gamma \cap N_j$ is a lattice of N_j.*

And, most importantly, there is a well-known necessary and sufficient condition for the existance of a lattice on a given nilpotent Lie group.

THEOREM 2.3. *Let N be a nilpotent Lie group. Then N has a lattice if and only if its Lie algebra $\mathfrak{n}$ has a $\mathbf{Q}$-algebra $\mathfrak{n}_{\mathbf{Q}}$, that is, $\mathfrak{n}$ has a basis whose Lie structure constants are integers.*

For solvable Lie groups, several general results are well known. In order to describe these results, we first review some structure theory on solvable Lie groups. Let G be a simply-connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let N be the nilradical of G with corresponding Lie algebra $\mathfrak{n}$, i.e., N is the maximal nilpotent normal subgroup of G so that $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{g}$. This induces a short exact sequence

$$1 \to N \to G \to T \to 1,$$

where T is the Abelian group given by $T = N\backslash G$. G is called *splittable*, if this sequence splits, that is, there is a right inverse homomorphism of the projection $G \to T$. This condition is equivalent to the existence of a homomorphism $b : T \to Aut(N)$ such that G is isomorphic to the semi-direct product $N \rtimes_b T$.

The first known result regarding lattices of solvable groups indicates just how much more specialized subgroups lattices are for solvable Lie groups than they are for general Lie groups.

THEOREM 2.4. *A lattice on a solvable Lie group is a uniform lattice.*

Furthermore, we have Mostow's well-known result.

THEOREM 2.5 (Mostow [21]). *Let Γ be a lattice in a connected solvable Lie group G with nilradical N. Then $\Gamma \cap N$ is a lattice of N.*

In [27], Wang proved the general structure of a lattice of a solvable Lie group.

THEOREM 2.6 (Wang [27]). *A group Γ is isomorphic to a discrete subgroup in a simply-connected Lie group if and only if there is a lattice Δ of a simply-connected nilpotent Lie group and non-negative integer k such that*

$$0 \to \Delta \to \Gamma \to \mathbf{Z}^k \to 0$$

is a short, exact sequence.

In particular, this implies that, if $G = N \rtimes_b T$ is a simply-connected splittable solvable Lie group with nilradical N and Γ is a lattice of G, then Γ is isomorphic to $\Delta \rtimes_b T_{\mathbf{Z}}$ where Δ is a lattice of N and $T_{\mathbf{Z}}$ a lattice of T such that $b(T_{\mathbf{Z}}) \subset Aut(\Delta)$.

2.2. Heisenberg groups

Besides $\mathbf{R}^m$ under addition, the most encountered Lie group in the work below will be the Heisenberg groups in three and five dimensions. In general, the *$(2n+1)$-dimensional Heisenberg group $\mathscr{H}eis^{2n+1}$* is the subgroup of $Sl(n+2, \mathbf{R})$ given by

$$\mathscr{H}eis^{2n+1} = \left\{ \sigma = \begin{pmatrix} 1 & y & x \\ 0 & I_n & z^t \\ 0 & 0 & 1 \end{pmatrix} : y, z \in \mathbf{R}^n,\ x \in \mathbf{R} \right\},$$

where the column vector z^t is the transpose of the vector $z = (z_1, \ldots, z_n)$ and I_n is the identity map of $\mathbf{R}^n$. Equivalently, $\mathscr{H}eis^{2n+1}$ can be considered as the central extension of the symplectic Lie group $\mathbf{R}^{2n}$ under addition with the standard symplectic form ω_1.

The Lie algebra of $\mathscr{H}eis^{2n+1}$ is given by

$$\mathfrak{h}_{2n+1} = \left\{ X = \begin{pmatrix} 0 & b & a \\ 0 & 0 & c^t \\ 0 & 0 & 0 \end{pmatrix} : b, c \in \mathbf{R}^n,\ a \in \mathbf{R} \right\}.$$

For $i, j \in \{1, \dots, n+2\}$, let $e_{i,j}$ be the $(n+2) \times (n+2)$ matrix, all of whose entries are zero except the ij-th entry which is equal to 1. We set $e_1 := e_{1,n+2}$, $e_k := e_{1,k}$ and $e_{n+k} := e_{k,n+2}$ for $k = 2, \dots, n+1$. Then $\{e_1, \dots, e_{2n+1}\}$ is a basis of $\mathfrak{h}_{2n+1}$ with exactly n nontrivial Lie brackets relations, namely, $[e_k, e_{n+k}] = e_1$ for all $k = 2, \dots, n+1$. If we let $(e_1^*, \dots, e_{2n+1}^*)$ stand for the dual basis of $(e_1, \dots, e_{2n+1})$, then e_1^* is a contact form on $\mathfrak{h}_{2n+1}$. In terms of the original coordinates on $\mathscr{H}eis^{2n+1}$, the left invariant vector fields are given by $e_1^+ = \dfrac{\partial}{\partial x}$, $e_k^+ = \dfrac{\partial}{\partial y_{k-1}}$, $e_{n+k}^+ = \dfrac{\partial}{\partial z_{k-1}} + y_{k-1}\dfrac{\partial}{\partial y}$ for $k = 2, \dots, n+1$. The left invariant contact form on $\mathscr{H}eis^{2n+1}$ corresponding to e_1^* is $e_1^{*,+} = dx - \sum_{i=1}^n y_i \, dz_i$.

The exponential map $\exp : \mathfrak{h}_{2n+1} \to \mathscr{H}eis^{2n+1}$ is a diffeomorphism, and we denote its inverse by log. Specifically, these mappings are given by

$$\exp \begin{pmatrix} 0 & a & c \\ 0 & 0_n & b^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a & c + \frac{1}{2}ab^t \\ 0 & I_n & b^t \\ 0 & 0 & 1 \end{pmatrix},$$

$$\log \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x & z - \frac{1}{2}xy^t \\ 0 & 0_n & y^t \\ 0 & 0 & 0 \end{pmatrix}.$$

We focus in on the case where $n = 1$. Let N be the Lie group given by $N = \mathscr{H}eis^3 \times \mathbf{R}$. Its Lie algebra is given by $\mathfrak{n} = \mathfrak{h}_3 \oplus \mathbf{R}$. Furthermore N has a left invariant nondegenerate closed 2-form, hence defining a left invariant symplectic structure $\omega = dx \wedge dz + dw \wedge dy$, where w is the coordinate in $\mathbf{R}$. It corresponds to the symplectic form $\omega_2 = e_1^* \wedge e_3^* + e_4^* \wedge e_2^*$, on $\mathfrak{h}_3 \oplus \mathbf{R}$, where $\mathfrak{h}_3 = \langle e_1, e_2, e_3 \rangle_{\mathbf{R}}$ as above.

Since lattices of $\mathscr{H}eis^3$ exist (see, for example, [14]), lattices on N exist. In fact, if Γ is a lattice of N, then

$$[\Gamma, \Gamma] \subset \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} : x \in \mathbf{R} \right\} \subset N.$$

In particular, $[\Gamma, \Gamma]$ is a subgroup of Γ. So, there is $x_0 \in \mathbf{R}^+$ such that

$$[\Gamma, \Gamma] \subset \left\{ \begin{pmatrix} kx_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} : k \in \mathbf{Z} \right\} \subset N.$$

We will make extensive use of this fact when we are proving that certain Lie groups do not have lattices.

2.3. Five-dimensional contact Lie groups

A *contact Lie group* is a $(2n+1)$-dimensional Lie group G with a left-invariant differential form η such that $\eta \wedge d\eta^n \neq 0$. Set $\mathscr{H} = ker\, \eta$. Then $\mathscr{H}$ is a left-invariant $2n$-dimensional subbundle of TG so that $\mathscr{H}$ induces a subspace of the Lie algebra $\mathfrak{g}$ of G, which we will also denote as $\mathscr{H}$. An element $X \in \mathfrak{g}$ is called *horizontal*, if $X \in \mathscr{H}$. A submanifold of G is called *totally isotropic*, if its tangent space in G is horizontal everywhere. A totally isotropic submanifold of (maximal) dimension n is called a *Legendrian* submanifold of G.

Let ξ be the unique left-invariant vector field in $\mathfrak{g}$ defined by

$$\eta(\xi) = 1,$$
$$d\eta(\xi, *) = 0.$$

Then $\mathfrak{g} = \langle \xi \rangle_{\mathbf{R}} \oplus \mathscr{H}$, and ξ is called the *Reeb vector field* of η.

LEMMA 2.7. *Let (G, η) be a solvable contact Lie group with nilradical N. Let $\mathfrak{n}$ be the Lie algebra of N. Then $\mathfrak{n}$ is not contained in $\mathscr{H}$.*

Proof. $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. Thus, if $\mathfrak{n} \subset \mathscr{H}$, $[X, Y] \in \mathfrak{n} \subset \mathscr{H}$ for any X, Y. If this were the case, then $d\eta(X, Y) = -\frac{1}{2}\eta([X, Y]) = 0$ for any X, Y. Thus, $d\eta = 0$ on G, a contradiction. □

A Lie algebra $\mathfrak{g}$ is said to be *decomposable* if it is the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Such a Lie algebra has a contact form if and only if $\mathfrak{g}_1$ has a contact form and $\mathfrak{g}_2$ an exact symplectic form, or vice versa.

LEMMA 2.8. *If a contact Lie algebra (resp. group) is unimodular, then it is necessarily nondecomposable.*

Proof. A decomposable Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is unimodular if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular. But as noted above, if $\mathfrak{g}$ had a contact form, then $\mathfrak{g}_1$ (or $\mathfrak{g}_2$) would have an exact symplectic form. And due to the existence of a left invariant radiant vector field for the associated left invariant affine connection, a Lie group with a left invariant exact symplectic form cannot be unimodular (see [11]). This, applied to any Lie group with Lie algebra $\mathfrak{g}_1$, would lead to a contradiction. □

Corollary 2.9. *Let $G = N \rtimes_b T$ be a $(2n+1)$-dimensional, simply-connected splittable solvable Lie group with nilradical N and homomorphism $b : T \to Aut(N)$. Let $\eta \in \mathfrak{g}^*$ be a left-invariant contact structure on G. Then*

1. *The subspace $\mathfrak{n} \cap \mathscr{H}$ has codimension 1 in $\mathfrak{n}$.*
2. *$dim\, T \leq n$ and $dim\, \mathfrak{n} \geq n+1$.*
3. *For every $X \in T$, there is an $X' \in \mathfrak{n} \cap \mathscr{H}$ such that $d\eta(X, X') = 1$.*

2.3.1. *Five-dimensional solvable contact Lie algebras*

In [9], the first author classified the five-dimensional simply connected contact Lie groups (via their Lie algebras) with the following theorem.

Theorem 2.10 (Diatta [9]). *Let G be a five-dimensional Lie group with Lie algebra $\mathfrak{g}$.*

1. *Suppose G is non-solvable. Then G is a contact Lie group if and only if $\mathfrak{g}$ is one of the following Lie algebras:*
   (a) *$aff(\mathbf{R}) \oplus \mathfrak{sl}(2, \mathbf{R})$, $aff(\mathbf{R}) \oplus \mathfrak{so}(3, \mathbf{R})$ (decomposable cases) or*
   (b) *$\mathfrak{sl}(2, \mathbf{R}) \ltimes \mathbf{R}^2$ (non-decomposable case).*
2. *Suppose that G is solvable such that $\mathfrak{g}$ is non-decomposable with trivial center $Z(\mathfrak{g})$. Then*
   (a) *If the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ has dimension three and is non-Abelian, then $\mathfrak{g}$ is a contact Lie algebra.*
   (b) *If $[\mathfrak{g}, \mathfrak{g}]$ has dimension four, then $\mathfrak{g}$ is contact if and only if*
      i. *$dim(Z([\mathfrak{g}, \mathfrak{g}])) = 1$ or*
      ii. *$dim(Z([\mathfrak{g}, \mathfrak{g}])) = 2$ and there is a $v \in \mathfrak{g}$ such that $Z([\mathfrak{g}, \mathfrak{g}])$ is not an eigenspace of ad_v.*

The first statement of this result taken with Lemma 2.8 implies that the only unimodular non-solvable five-dimensional contact Lie group is $\mathfrak{sl}(2, \mathbf{R}) \ltimes \mathbf{R}^2$. Furthermore, the second statement in conjunction with the list of five-dimensional solvable Lie algebras in [4] yields the list of all five-dimensional solvable contact Lie algebras, a total of 24 distinct nondecomposable Lie algebras and families of Lie algebras. Among these, exactly 12 are unimodular. They are listed below along with an example of a contact form η. The label for each Lie algebra refers to that algebra's position in the original list in [9] and will serve as the name of that Lie algebra (or corresponding simply connectd Lie group) throughout this paper.

2.3.2. *Five-dimensional unimodular solvable contact Lie algebras*

Below is the list of unimodular solvable contact Lie algebras of dimension 5.

Central extensions

D1 $[e_2, e_4] = e_1$, $[e_3, e_5] = e_1$, $\eta := e_1^*$. This is the Heisenberg Lie algebra $\mathfrak{h}_5$. See Section 2.2.

D2 $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $\eta := e_1^*$. This is the central extension $\mathfrak{b} \times_\omega \mathbf{R}e_1$, where $\omega = e_3^* \wedge e_4^* + e_2^* \wedge e_5^*$ and $\mathfrak{b} = \mathfrak{h}_3 \oplus \mathbf{R}e_4$, as in 2.2.

D3 $[e_3, e_4] = e_1$, $[e_2, e_5] = e_1$, $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$, $\eta = e_1^*$. This is the central extension $\mathfrak{b} \times_\omega \mathbf{R}e_1$, where $\omega = e_3^* \wedge e_4^* + e_2^* \wedge e_5^*$ and $\mathfrak{b} = span(e_2, e_3, e_4, e_5)$ with Lie bracket $[e_3, e_5] = e_2$, $[e_4, e_5] = e_3$.

D5 $[e_2, e_3] = e_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = -e_3$, $[e_4, e_5] = e_1$, $\eta = e_1^*$. This is $\mathfrak{b} \times_\omega \mathbf{R}e_1$, where $\omega = e_2^* \wedge e_3^* + e_4^* \wedge e_5^*$ and $\mathfrak{b} = span(e_2, e_3, e_4, e_5)$ with Lie bracket $[e_2, e_5] = e_2$, $[e_3, e_5] = -e_3$.

D11 $[e_2, e_3] = e_1$; $[e_2, e_5] = e_3$; $[e_3, e_5] = -e_2$; $[e_4, e_5] = \varepsilon e_1$; $\varepsilon = \pm 1$; $\eta = e_1^*$. Here $\mathfrak{g} = \mathfrak{b} \times_\omega \mathbf{R}e_1$, where $\omega = e_2^* \wedge e_3^* + e_4^* \wedge e_5^*$ and $\mathfrak{b} = span(e_2, e_3, e_4, e_5)$ with Lie bracket $[e_2, e_5] = e_3$; $[e_3, e_5] = -e_2$.

Semi-direct products

D4 $[e_2, e_3] = e_1$, $[e_1, e_5] = (1 + p)e_1$, $[e_2, e_5] = e_2$, $[e_3, e_5] = pe_3$, $[e_4, e_5] = -2(p + 1)e_4$, $p \neq -1$, $\eta = e_1^* + e_4^*$. Here $\mathfrak{g}$ is the semi-direct product $(\mathfrak{h}_3 \oplus \mathbf{R}e_4) \rtimes \mathbf{R}e_5$ where $\mathfrak{h}_3 \oplus \mathbf{R}e_4$ is as in Section 2.2.

D8 $[e_2, e_3] = e_1$; $[e_1, e_5] = 2e_1$; $[e_2, e_5] = e_2 + e_3$; $[e_3, e_5] = e_3$; $[e_4, e_5] = -4e_4$; $\eta = e_1^* + e_4^*$. This is the semi-direct product $(\mathfrak{h}_3 \oplus \mathbf{R}e_4) \rtimes \mathbf{R}e_5$.

D10 $[e_2, e_3] = e_1$; $[e_1, e_5] = 2pe_1$; $[e_2, e_5] = pe_2 + e_3$; $[e_3, e_5] = -e_2 + pe_3$; $[e_4, e_5] = -4pe_4$, $p \neq 0$; $\eta = e_1^* + e_4^*$. This is the semi-direct product $(\mathfrak{h}_3 \oplus \mathbf{R}e_4) \rtimes \mathbf{R}e_5$.

D13 $[e_2, e_3] = e_1$; $[e_1, e_5] = -\frac{1}{2}e_1$; $[e_2, e_5] = -\frac{3}{2}e_2$; $[e_3, e_5] = e_3 + e_4$; $[e_4, e_5] = e_4$; $\eta = e_1^* + e_4^*$; $p \neq 0$. This is the semi-direct product $(\mathfrak{h}_3 \oplus \mathbf{R}e_4) \rtimes \mathbf{R}e_5$.

D15 $[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$, $[e_1, e_5] = \frac{2}{3}e_1$, $[e_2, e_5] = -\frac{1}{3}e_2$, $[e_3, e_5] = -\frac{4}{3}e_3$, $[e_4, e_5] = e_4$, $\eta = e_1^* + e_3^*$. This is the semi-direct product $\mathfrak{b} \rtimes \mathbf{R}e_5$ where $\mathfrak{b}$ is the nilpotent Lie algebra $\mathfrak{b} = span(e_1, e_2, e_3, e_4)$ (Note that this is the 15th entry of the list in [9] with $p = -\frac{4}{3}$.)

D18 $[e_1, e_4] = e_1$, $[e_3, e_4] = -e_3$, $[e_2, e_5] = e_2$, $[e_3, e_5] = -e_3$; $\eta = e_1^* + e_2^* + e_3^*$.

D20 $[e_1, e_4] = -2e_1$; $[e_2, e_4] = e_2$; $[e_3, e_4] = e_3$; $[e_2, e_5] = -e_3$; $[e_3, e_5] = e_2$. The last two Lie algebras above are the 2-step solvable Lie algebra $\mathscr{R}^3 \rtimes \mathbf{R}^2$ where the Abelian subalgebra $\mathscr{R}^3 = span(e_1, e_2, e_3)$ is the derived ideal and $\mathscr{R}^2 = span(e_4, e_5)$ is also Abelian.

Inspection of the list above yields the following corollary.

Corollary 2.11. *Let G be a five-dimensional simply-connected solvable contact Lie group. Then G is splittable.*

See Appendix I for a list of descriptions of the nilradicals for each of these Lie groups.

3. Five-dimensional contact Lie groups with uniform lattices

The following theorem indicates which of the simply-connected contact Lie groups in Theorem 2.10 have uniform lattices.

Theorem 3.1. *Let G be a five-dimensional connected and simply connected contact Lie group with a uniform lattice. Then one of the following statements is true.*

1. *G is the central extension of a solvable symplectic Lie group with a lattice that extends to G. In particular, G is one of the following groups:*
   (a) $\mathscr{H}eis^5 = \mathbf{R}^4 \times_{\omega_1} \mathbf{R}$, *where ω_1 is the standard symplectic form on $\mathbf{R}^4$,*
   (b) $(\mathscr{H}eis^3 \times \mathbf{R}) \times_{\omega_2} \mathbf{R}$, *where ω_2 is the symplectic form on $\mathscr{H}eis^3 \times \mathbf{R}$, or*
   (c) $B_j \times_{\omega_j} \mathbf{R}$ $(j = 3, 4, 5)$, *where ω_j is the symplectic form on $B_j = \mathbf{R}^3 \rtimes_{F_j} \mathbf{R}$ with $F_j : \mathbf{R} \to Gl(3, \mathbf{R})$ defined by the matrices*
      i. $F_3(t) = \begin{pmatrix} 1 & -t & \frac{1}{2}t^2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}$,
      ii. $F_4(t) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
      iii. $F_5(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
2. *G is a solvable semi-direct product and is one of the following groups:*
   (a) $\mathbf{R}^3 \rtimes_{b_1} \mathbf{R}^2$, *where $b_1 : \mathbf{R}^2 \to Gl(3, \mathbf{R})$ is given by* $b_1(s, t) = \begin{pmatrix} e^{-s} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{s+t} \end{pmatrix}$.
   (b) $\mathbf{R}^3 \rtimes_{b_2} \mathbf{R}^2$, *where $b_2 : \mathbf{R}^2 \to Gl(3, \mathbf{R})$ is given by* $b_2(s, t) = \begin{pmatrix} e^{2s} & 0 & 0 \\ 0 & e^{-s}\cos(t) & -e^{-s}\sin(t) \\ 0 & e^{-s}\sin(t) & e^{-s}\cos(t) \end{pmatrix}$.

Corollary 3.2. *Let X be a compact five-dimensional contact manifold uniformized by a five-dimensional contact Lie group G. Then G is solvable.*

4. Proof of Theorem 3.1

We will prove Theorem 3.1 by showing (1) that the Lie groups stated in the Theorem have lattices and (2) that the rest of the five-dimensional contact Lie groups given by Theorem 2.10 and the list in Subsection 2.3.2 do not. As stated before, each solvable Lie group (or Lie algebra) will be referred to by its label in the list, e.g. **D2**, **D13**. For ease of reading, we have relegated several technical results to appendices at the end of the paper. In Appendix I (Section 5), the reader will find a description of the nilradical of each solvable Lie algebra in the list in Subsection 2.3.2 as well as matrix representations of db and β for the splitting $\mathfrak{n} \rtimes_\beta T$.

4.1. Positive cases

The groups listed in Theorem 3.1 are the simply connected Lie groups with Lie algebras **D1**, **D2**, **D3**, **D5**, **D11**, **D18**, and **D20**, respectively. Due to the variety of specific procedures used, we prove the existence of lattices in several individual propositions. The overall methodology for the non-nilpotent cases is that utilized by Sawai in [25] and by Sawai and Yamada in [26], in which the existence of specific lattices is proven.

PROPOSITION 4.1. *The Lie groups with Lie algebras* **D1**, **D2** *and* **D3** *have lattices.*

Proof. The Lie algebras **D1**, **D2** and **D3** are all nilpotent. Recall that a nilpotent Lie group has a lattice if and only if its Lie algebra also has a **Q**-algebra, i.e., there is a basis on which all the coefficients of all of the bracket relations are in **Q** (See pp. 46–47 of [22]). The bases of the Lie algebras for these Lie groups as given in Appendix I (Section 2.3.2) all satisfy this property. Thus, groups with Lie algebras **D1**, **D2** and **D3** have lattices. □

PROPOSITION 4.2. *The Lie group with Lie algebra* **D5** *has a lattice.*

Proof. Let G be the simply-connected, connected Lie group with Lie algebra **D5**. Then $G = (\mathscr{H}eis^3 \times \mathbf{R}) \rtimes_b \mathbf{R}$. For this proposition, we use a representation of the multiplication on $\mathscr{H}eis^3$ different from that described in the previous section. Namely, for $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathscr{H}eis^3$, let

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' + \frac{1}{2}(y'z - yz') + x \\ y' + y \\ z' + z \end{pmatrix}.$$

That is, the x-coordinate of the product is the sum of the two x-coordinates plus one half of the signed area of the parallelogram in the plane defined by the origin, $\begin{pmatrix} y' \\ z' \end{pmatrix}$ and $\begin{pmatrix} y \\ z \end{pmatrix}$. Set $A\left(\begin{pmatrix} y' \\ z' \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}\right) = \frac{1}{2}(y'z - yz')$. With this version of the Heisenberg group, multiplication on G is given by

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \\ t' \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \\ t \end{pmatrix} = \begin{pmatrix} x' - \frac{1}{2}(e^{t'}z'y - e^{-t'}y'z) - t'w + x \\ y' + e^{-t'}y \\ z' + e^{t'}z \\ w' + w \\ t' + t \end{pmatrix}$$

Let n be a positive integer greater than 2. Let t_0 be a positive real number such that $(e^{t_0})^2 - ne^{t_0} + 1 = 0$. Set $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} e^{-t_0} \\ e^{t_0} \end{pmatrix} \in \mathbf{R}^2$ and $x_0 = A(v_1, v_2)$. Note that the lattice of $\mathbf{R}^2$ generated by the vectors v_1 and v_2 is preserved by the linear transformation $\begin{pmatrix} e^{-t_0} & 0 \\ 0 & e^{t_0} \end{pmatrix}$.

Let $\Gamma \in G$ be the discrete subset given by

$$\Gamma = \mathbf{Z}\begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{Z}\begin{pmatrix} 0 \\ v_1 \\ 0 \\ 0 \end{pmatrix} + \mathbf{Z}\begin{pmatrix} 0 \\ v_2 \\ 0 \\ 0 \end{pmatrix} + \mathbf{Z}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{x_0}{t_0} \\ 0 \end{pmatrix} + \mathbf{Z}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ t_0 \end{pmatrix}.$$

By definition of the quantities and vectors involved, we can see that Γ is actually a *subgroup* of G (c.f. Sawai [25] and Sawai and Yamada [26]). Thus, G has a lattice. $\square$

PROPOSITION 4.3. *The Lie group with Lie algebra* **D11** *has a lattice.*

Proof. Let G be the simply-connected, connected Lie group with Lie algebra **D11**. The group structure of G is given by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \\ \theta_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \pm \theta_1 w_2 + y_1((\sin\theta_1)y_2 + (\cos\theta_1)z_2) \\ y_1 + (\cos\theta_1)y_2 - (\sin\theta_1)z_2 \\ z_1 + (\sin\theta_1)y_2 + (\cos\theta_1)z_2 \\ w_1 + w_2 \\ \theta_1 + \theta_2 \end{pmatrix}.$$

Then the subgroup of G generated by the elements

$$\left\{ \begin{pmatrix} \pi \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{\pi} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{\pi} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pi \end{pmatrix} \right\}$$

is discrete in G and hence a lattice. $\square$

PROPOSITION 4.4. *The Lie group with Lie algebra* **D18** *has a lattice.*

Proof. Let $G = \mathbf{R}^3 \rtimes_{b_1} \mathbf{R}^2$ be the Lie group corresponding to Lie algebra **D18**. In order to show that G has a lattice, we need to produce a basis of $\mathbf{R}^3$, $\{v_1, v_2, v_3\}$, and a basis of $\mathbf{R}^2$, $\{p_1, p_2\}$ such that

$$\Gamma = \langle v_1, v_2, v_3 \rangle_{\mathbf{Z}} \rtimes_{b_1} \langle p_1, p_2 \rangle_{\mathbf{Z}}$$

is a subgroup of G. We do this by setting

$$T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -5 \\ 0 & 1 & 6 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -4 & -4 & -3 \\ 21 & 16 & 11 \\ -4 & -3 & -2 \end{pmatrix}.$$

It is easily verified that $T_1 \circ T_2 = T_2 \circ T_1$.

Furthermore, the characteristic polynomials of T_1 and T_2 are given respectively as

$$f_1(X) = X^3 - 6X^2 + 5X - 1,$$
$$f_2(X) = X^3 - 10X^2 + 17X - 1,$$

each of which have three distinct roots. The roots of f_1 are

$$\alpha_1 = 0.30797853\ldots$$
$$\beta_1 = 0.64310413\ldots$$
$$\gamma_1 = 5.0489173\ldots,$$

and the roots of f_2 are

$$\alpha_2 = 2.088146\ldots$$
$$\beta_2 = 7.8508551\ldots$$
$$\gamma_2 = 0.06099892\ldots,$$

Thus, T_1 and T_2 are simultaneously diagonalizable. In fact, there is a $\Phi \in Gl(3, \mathbf{R})$ such that, for $j = 1, 2$,

$$\Phi T_j \Phi^{-1} = \begin{pmatrix} \alpha_j & 0 & 0 \\ 0 & \beta_j & 0 \\ 0 & 0 & \gamma_j \end{pmatrix}.$$

Set

$$p_1 = \begin{pmatrix} \ln \alpha_1 \\ \ln \beta_1 \end{pmatrix} = \begin{pmatrix} -1.7772\ldots \\ -0.441449\ldots \end{pmatrix}, \quad p_2 = \begin{pmatrix} \ln \alpha_2 \\ \ln \beta_2 \end{pmatrix} = \begin{pmatrix} 2.06062\ldots \\ 0.736277\ldots \end{pmatrix},$$

Note that the slope of p_1 as a vector is approximately 2.66786 and that of p_2 is approximately 2.79871. Thus, p_1 and p_2 are linearly independent vectors in $\mathbf{R}^2$. By definition of p_1 and p_2, we have

$$b(p_j) = \begin{pmatrix} \alpha_j & 0 & 0 \\ 0 & \beta_j & 0 \\ 0 & 0 & \gamma_j \end{pmatrix}$$

for each $j = 1, 2$ so that $b(p_j) \circ \Phi = \Phi \circ T_j$ for each j. In particular, if we set $v_k = \Phi(\varepsilon_k)$ for $k = 1, 2, 3$, where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the standard basis of $\mathbf{R}^3$, then $b(p_j)(v_k) \in \langle v_1, v_2, v_3 \rangle_Z$ for each $j = 1, 2$ and $k = 1, 2, 3$. Thus, $\Gamma = \langle v_1, v_2, v_3 \rangle_{\mathbf{Z}} \rtimes_{b_1} \langle p_1, p_2 \rangle_{\mathbf{Z}}$ is a discrete, uniform subgroup of G. This proves the claim.

PROPOSITION 4.5. *The Lie group with Lie algebra* **D20** *has a lattice.*

Proof. Let $G = \mathbf{R}^3 \rtimes_{b_1} \mathbf{R}^2$ be the Lie group corresponding to Lie algebra **D20**. As with the previous claim, we need to produce a basis of $\mathbf{R}^3$, $\{v_1, v_2, v_3\}$, and a basis of $\mathbf{R}^2$, $\{p_1, p_2\}$ such that

$$\Gamma = \langle v_1, v_2, v_3 \rangle_{\mathbf{Z}} \rtimes_{b_2} \langle p_1, p_2 \rangle_{\mathbf{Z}}$$

is a subgroup of G.

Here, we do this by setting

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 1 & 1 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then $U_1 U_2 = U_2 U_1$. Also, the characteristic polynomials of U_1 and U_2 are given respectively as

$$f_1(X) = X^3 - 3X^2 + 2X - 1,$$
$$f_2(X) = X^3 + X^2 - 1,$$

each of which have exactly one real root—$f_1(2.3247\ldots) = 0 = f_2(0.7549\ldots)$—and two complex roots.

Thus, there is a $\Psi \in Gl(3, \mathbf{R})$ such that, for $j = 1, 2$,

$$\Psi U_j \Psi^{-1} = \begin{pmatrix} \alpha_j^2 & 0 & 0 \\ 0 & \alpha_j^{-1} \cos(\beta_j) & -\alpha_j^{-1} \sin(\beta_j) \\ 0 & \alpha_j^{-1} \sin(\beta_j) & \alpha_j^{-1} \cos(\beta_j) \end{pmatrix},$$

where $\alpha_1 \approx \sqrt{2.3247}$, $\beta_1 \approx 1.0300$, $\alpha_2 \approx \sqrt{0.7549}$ and $\beta_2 \approx 2.4378$.

Set

$$p_1 = \begin{pmatrix} \ln \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0.4217\ldots \\ 1.0300\ldots \end{pmatrix}, \quad p_2 = \begin{pmatrix} \ln \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -0.1405\ldots \\ 2.4378\ldots \end{pmatrix},$$

Thus, p_1 and p_2 are linearly independent vectors in $\mathbf{R}^2$. By definition of p_1 and p_2, we have

$$b(p_j) = \begin{pmatrix} \alpha_j^2 & 0 & 0 \\ 0 & \alpha_j^{-1} \cos(\beta_j) & -\alpha_j^{-1} \sin(\beta_j) \\ 0 & \alpha_j^{-1} \sin(\beta_j) & \alpha_j^{-1} \cos(\beta_j) \end{pmatrix}$$

for each $j = 1, 2$ so that $b(p_j) \circ \Psi = \Psi \circ U_j$ for each j. In particular, if we set $v_k = \Psi(\varepsilon_k)$ for $k = 1, 2, 3$, where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the standard basis of $\mathbf{R}^3$, then $b(p_j)(v_k) \in \langle v_1, v_2, v_3 \rangle_Z$ for each $j = 1, 2$ and $k = 1, 2, 3$. Thus, $\Gamma = \langle v_1, v_2, v_3 \rangle_{\mathbf{Z}} \rtimes_{b_1} \langle p_1, p_2 \rangle_{\mathbf{Z}}$ is a discrete, uniform subgroup of G. This proves the claim.

4.2. Negative cases

4.2.1. *Solvable*

We now show that the rest of the solvable contact Lie groups of five dimensions do not have lattices. Appendix I lists the corresponding homomorphism $\beta : T \to der(\mathfrak{n})$ and $db : T \to Aut(\mathfrak{n})$ of each group. There are only two classes of such Lie groups remaining from the list in Subsection 2.3.2, namely, those whose nilradical is $\mathscr{H}eis^3 \times \mathbf{R}$ (**D4, D5, D8, D10, D11, D13**) and one whose nilradical is a semidirect product $\mathbf{R}^3 \rtimes_f \mathbf{R}$ (**D15**).

Proposition 4.6. *None of the Lie groups corresponding to the Lie algebras* **D4, D8, D10, D13** *have lattices.*

Proof. Let $\mathfrak{g}$ be one of the Lie algebras **D4, D8, D10, D13**. Let G be the simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then we have the short exact sequence

$$0 \to N \to G \to \mathbf{R} \to 0,$$

where $N = \mathscr{H}eis^3 \times \mathbf{R}$ and the sequence splits ($G = N \rtimes_b \mathbf{R}$).

Suppose G has a lattice Γ. By Theorem 2.5, $N \cap \Gamma$ is a lattice of N; and, by Theorem 2.6, Γ is isomorphic to a group of the form $(N \cap \Gamma) \rtimes_b \langle t_0 \rangle_{\mathbf{Z}}$ for some $t_0 \in \mathbf{R}^+$. Recall also that a lattice of N necessarily contains a subgroup given as the span over $\mathbf{Z}$ of $g_0 = \begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ for some positive $x_0 \in \mathbf{R}$. Thus, $g_1 = b(t_0)(g_0)$ and $g_1^{-1} = b(-t_0)(g_0)$ are both elements of $N \cap \Gamma$.

We now show the non-existence of lattices in G by looking at each of the possible cases.

A. If $\mathfrak{g}$ is **D4**, then

$$g_1 = \begin{pmatrix} e^{-(p+1)t_0} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} e^{(p+1)t_0} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since g_1 and g_2 are integer multiples of g_0, we have either $p = -1$ or $t_0 = 0$. Both of these possible cases are contradictions.

B. If $\mathfrak{g}$ is **D8**, then

$$g_1 = \begin{pmatrix} e^{-2t_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} e^{2t_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since g_1 and g_2 are integer multiples of g_0, we have $t_0 = 0$. This is a contradiction.

C. If $\mathfrak{g}$ is **D10**, then

$$g_1 = \begin{pmatrix} e^{-2pt_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} e^{2pt_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since g_1 and g_2 are integer multiples of g_0, we have either $p = 0$ or $t_0 = 0$. Both of these possible cases are contradictions.

D. Finally, if $\mathfrak{g}$ is **D13**, then

$$g_1 = \begin{pmatrix} e^{(1/2)t_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} e^{-(1/2)t_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since g_1 and g_2 are integer multiples of g_0, we have $t_0 = 0$. This is a contradiction.

This proves the proposition. □

PROPOSITION 4.7. *A Lie group with Lie algebra* **D15** *does not have a lattice.*

Proof. The simply connected Lie group with Lie algebra **D15** is given by $G = (\mathbf{R}^3 \rtimes_f \mathbf{R}) \rtimes_b \mathbf{R}$, where $f : \mathbf{R} \to \mathbf{R}^3$ is given by

$$f(w) = \begin{pmatrix} 1 & -w & \frac{1}{2}w^2 \\ & 1 & -w \\ & & 1 \end{pmatrix}$$

and $b : \mathbf{R} \to \mathbf{R}^3 \rtimes_f \mathbf{R}$ is given by

$$b(t) = \begin{pmatrix} e^{-(2/3)t} & & & \\ & e^{(1/3)t} & & \\ & & e^{(4/3)t} & \\ & & & e^{-t} \end{pmatrix}.$$

The nilradical of group **D15** is of the subgroup $N = (\mathbf{R}^3 \rtimes_f \mathbf{R}) \rtimes_b (0)$ with multiplication given by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} x + x' - wy' + \frac{1}{2}w^2 z' \\ y + y' + wz' \\ z + z' \\ w + w' \end{pmatrix}.$$

so that

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}^{-1} = \begin{pmatrix} -x - yw - \frac{1}{2}zw^2 \\ -y - zw \\ -z \\ -w \end{pmatrix}.$$

And, for $g_j = \begin{pmatrix} x_j \\ y_j \\ z_j \\ w_j \end{pmatrix} \in N, \ j = 1, 2,$

$$g_1 g_2 g_1^{-1} g_2^{-1} = \begin{pmatrix} y_1 w_2 - y_2 w_1 - \frac{1}{2} z_1 w_2^2 + \frac{1}{2} z_2 w_1^2 \\ z_1 w_2 - z_2 w_1 \\ 0 \\ 0 \end{pmatrix}.$$

Let $N_1 = [N, N]$ and $N_2 = [[N, N], N]$. Then $N_2 = \mathbf{R} \times (0, 0, 0)$, and $[N_2, N] = (0)$, that is, N is 3-step nilpotent.

Suppose G has lattice Γ. By Theorem 2.5, $\Gamma_0 = \Gamma \cap N$ is a lattice of N. By Theorem 2.2, $\Gamma_1 = \Gamma \cap N_1$ is a lattice of N_1, and $\Gamma_2 = \Gamma \cap N_2$ is a lattice of $N_2 \cong \mathbf{R}$. Let x_0 be the unique positive real number such that

$$\Gamma_2 = \left\{ \begin{pmatrix} nx_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} : n \in \mathbf{Z} \right\}.$$

Furthermore, by Theorem 2.6, Γ is isomorphic to a group $\tilde{\Gamma}$ satisfying the short, exact sequence

$$0 \to \Gamma_0 \to \tilde{\Gamma} \to \Gamma_0 \backslash \tilde{\Gamma} \to 0$$

induced from the short exact sequence

$$0 \to N \to G \to N \backslash G \to 0.$$

Since $N\backslash G \cong \mathbf{R}$, there is a $t_0 \in \mathbf{R}^+$ such that $b(\pm t_0)(\Gamma_0) \subset \Gamma_0$. Since $b(t_0)$ and $b(-t_0)$ are non-trivial group homomorphisms from Γ_0 to itself, they must preserve the central series of Γ_0, that is,

$$b(\pm t_0)(\Gamma_1) \subset \Gamma_1 \quad \text{and} \quad b(\pm t_0)(\Gamma_2) \subset \Gamma_2.$$

Thus, both

$$b(t_0)\begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-(2/3)t_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b(-t_0)\begin{pmatrix} x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{(2/3)t_0}x_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \left\{ \begin{pmatrix} nx_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} : n \in \mathbf{Z} \right\},$$

implying that $e^{\pm(2/3)t_0} \in \mathbf{Z}$. So, $t_0 = 0$, which is a contradiction, and thus no lattice exists on G. $\square$

4.2.2. *Non-solvable: the general case of* $\mathbf{R}^n \rtimes Sl(n,\mathbf{R})$.

According to Theorem 2.10, the only unimodular non-solvable contact Lie group of dimension five is the group $\mathbf{R}^2 \rtimes Sl(2,\mathbf{R})$ of special affine transformations of the plane. We obtain the following more general result stating the nonexistence of uniform lattices in $\mathbf{R}^n \rtimes Sl(n,\mathbf{R})$, for every $n \geq 2$. See Theorem 4.8. Let us recall that $\mathbf{R}^n \rtimes Sl(n,\mathbf{R})$ is a contact Lie group [9]. We can exhibit a contact form η on the Lie algebra $\mathbf{R}^n \rtimes sl(n,\mathbf{R})$ of $\mathbf{R}^n \rtimes Sl(n,\mathbf{R})$ by looking at it as the subalgebra $\mathbf{R}^n \rtimes sl(n,\mathbf{R}) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix}, \text{ where } A \in sl(n,\mathbf{R}) \text{ and } v \in \mathbf{R}^n \right\}$ of the Lie algebra $\mathscr{G}l(n+1,\mathbf{R})$ of $(n+1)\times(n+1)$ real matrices. The $(n+1)\times(n+1)$ matrices $e_{i,j}$ all of whose entries are zero except the ij-th one which is equal to 1, form a basis of $\mathscr{G}l(n+1,\mathbf{R})$. Let us denote by $(e^*_{i,j})$ the corresponding dual basis. Then, $\eta := \sum_{i=1}^n e^*_{i,i+1}$ is a contact form on $\mathbf{R}^n \rtimes sl(n,\mathbf{R})$, with Reeb vector $\xi := \frac{1}{n}\sum_{i=1}^n e_{i,i+1}$. Now, we have the following.

THEOREM 4.8. *The group* $\mathbf{R}^n \rtimes Sl(n,\mathbf{R})$, *of special affine transformations of* $\mathbf{R}^n$, *has no uniform lattice, for every* $n \geq 2$.

Proof. Let $G := \mathbf{R}^n \rtimes Sl(n,\mathbf{R})$ and suppose Γ is a lattice in G. The radical of G is the subgroup $\mathbf{R}^n \times \{I\}$. Then $\Gamma' = \Gamma \cap \mathbf{R}^n \times \{I\}$ is a lattice of $\mathbf{R}^n \times \{I\}$ (Corollary 1.8 on p. 107 of [23]). Let $v_1,\dots,v_n \in \mathbf{R}^n$ be such that $(v_1,I)\cdots(v_n,I)$ generate Γ'. Let $A \in Sl(n,\mathbf{R})$ and $w \in \mathbf{R}^n$ such that $(w,A) \in \Gamma$. Then, for $j = 1,\dots,n$,

$$(w,A)(v_j,I)(w,A)^{-1} = (Av_j + w, A)(-A^{-1}w, A^{-1}) = (Av_j, I) \in \Gamma.$$

Hence the set M_Γ given by

$$M_\Gamma = \{A \in Sl(n, \mathbf{R}) : (w, A) \in \Gamma \text{ for some } w \in \mathbf{R}^n\}$$

preserves the lattice Γ' on $\mathbf{R}^n$. In particular, by the change of basis $v_j \mapsto e_j$ for $j = 1, \dots, n$, we can assume that $M_\Gamma \subset Sl(n, \mathbf{Z})$. Now, it is known that $Sl(n, \mathbf{Z})$ is a lattice of $Sl(n, \mathbf{R})$ but not a uniform lattice (e.g., see pp. 229–231 of [5] for the case where $n = 2$). In other words, there is a sequence $\{\gamma_j\} \subset Sl(n, \mathbf{R})$ such that its projection $Sl(n, \mathbf{Z})\backslash Sl(n, \mathbf{R})$ has no convergent subsequences. Thus, its projection in $M_\Gamma \backslash Sl(n, \mathbf{R})$ also has no convergent subsequences, which means that the sequence $\{[0, \gamma_j]\} \subset \Gamma \backslash \mathbf{R}^n \rtimes Sl(n, \mathbf{R})$ has no convergent subsequences. Therefore, $\Gamma \backslash \mathbf{R}^n \rtimes Sl(n, \mathbf{R})$ is not compact. Since Γ was assumed to be an arbitrary lattice of $\mathbf{R}^n \rtimes Sl(n, \mathbf{R})$, $\mathbf{R}^n \rtimes Sl(n, \mathbf{R})$ has no uniform lattices. □

5. Appendix I: List of nilradicals of the unimodular contact Lie algebras of dimension 5

The following is a list of all of the unimodular Lie algebras among those in the first author's list of solvable contact Lie groups in five dimensions from [9]. Their Lie brackets, in a basis $(e_1, e_2, e_3, e_4, e_5)$, are given in Section 2.3.2. Each of the corresponding Lie groups will be of the form $N \rtimes_b T$, where N is the nilradical, T is an Abelian group and $b : T \to Aut(N)$ a homomorphism. For each of these, the Lie algebra $\mathfrak{n}$ of the nilradical N of the simply-connected Lie group corresponding to each Lie algebra is provided as well as the Abelian group T. The transformations β and db are matrix representations (with respect to the given basis of $\mathfrak{n}$) of the corresponding homomorphisms $\beta : T \to der(\mathfrak{n})$ and $db(x) = exp(\beta(x)) : T \to Aut(\mathfrak{n})$ (for $x \in T$) induced from the semidirect product $N \rtimes_b T$.

D1 $\mathfrak{n} = \langle e_1, \dots, e_5 \rangle = \mathscr{H}_5$, $T = (0)$,

D2 $\mathfrak{n} = (\langle e_1, e_3, e_4 \rangle \oplus \langle e_2 \rangle) +_{df} \langle e_5 \rangle = (\mathscr{H}_3 \oplus \mathbf{R}) +_{df} \mathbf{R}$, $T = (0)$ where

$$ad(e_5) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad df(te_5) = \begin{pmatrix} 1 & \frac{1}{2}t^2 & 0 & -t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -t & 0 & 1 \end{pmatrix}.$$

D3 $\mathfrak{n} = (\langle e_1, e_3, e_4 \rangle \oplus \langle e_2 \rangle) +_{df} \langle e_5 \rangle = (\mathscr{H}_3 \oplus \mathbf{R}) +_{df} \mathbf{R}$, $T = (0)$ where

$$ad(e_5) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad df(te_5) = \begin{pmatrix} 1 & \frac{1}{2}t^2 & -\frac{1}{6}t^3 & -t \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -t & 0 & 1 \end{pmatrix}.$$

D4 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4 \rangle = \mathscr{H}_3 \oplus \mathbf{R}$, $T = \mathbf{R}e_5$,

$$\beta(e_5) = \begin{pmatrix} -(p+1) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 2(p+1) \end{pmatrix},$$

$$db(te_5) = \begin{pmatrix} e^{-(p+1)t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-pt} & 0 \\ 0 & 0 & 0 & e^{2(p+1)t} \end{pmatrix}.$$

D5 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4 \rangle = \mathscr{H}_3 \oplus \mathbf{R}$, $T = \mathbf{R}e_5$,

$$\beta(e_5) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad db(te_5) = \begin{pmatrix} 1 & 0 & 0 & -t \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

D8 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4 \rangle = \mathscr{H}_3 \oplus \mathbf{R}$, $T = \mathbf{R}e_5$,

$$\beta(e_5) = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad db(te_5) = \begin{pmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & -te^{-t} & e^{-t} & 0 \\ 0 & 0 & 0 & e^{4t} \end{pmatrix}.$$

D10 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4 \rangle = \mathscr{H}_3 \oplus \mathbf{R}$, $T = \mathbf{R}e_5$,

$$\beta(e_5) = \begin{pmatrix} -2p & 0 & 0 & 0 \\ 0 & -p & 1 & 0 \\ 0 & -1 & -p & 0 \\ 0 & 0 & 0 & 4p \end{pmatrix},$$

$$db(te_5) = \begin{pmatrix} e^{-2pt} & 0 & 0 & 0 \\ 0 & e^{-pt}\cos(-t) & -e^{-pt}\sin(-t) & 0 \\ 0 & e^{-pt}\sin(-t) & e^{-pt}\cos(-t) & 0 \\ 0 & 0 & 0 & e^{4pt} \end{pmatrix}.$$

D11 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4 \rangle = \mathscr{H}_3 \oplus \mathbf{R}$, $T = \mathbf{R}e_5$,

$$\beta(e_5) = \begin{pmatrix} 0 & 0 & 0 & \pm 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad db(te_5) = \begin{pmatrix} 1 & 0 & 0 & \pm t \\ 0 & \cos(t) & -\sin(t) & 0 \\ 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

D13 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4 \rangle = \mathscr{H}_3 \oplus \mathbf{R}$, $T = \mathbf{R}e_5$,

$$\beta(e_5) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad db(te_5) = \begin{pmatrix} e^{(1/2)t} & 0 & 0 & 0 \\ 0 & e^{(3/2)t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & -te^{-t} & e^{-t} \end{pmatrix}.$$

D15 $\mathfrak{n} = \langle e_1, \ldots, e_4 \rangle = \langle e_1, e_2, e_3 \rangle +_{f_*} \langle e_4 \rangle$, with $f_*(e_4) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $T = \mathbf{R}e_5$,

$$\beta(e_5) = \begin{pmatrix} -\frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad db(te_5) = \begin{pmatrix} e^{-(2/3)t} & 0 & 0 & 0 \\ 0 & e^{(1/3)t} & 0 & 0 \\ 0 & 0 & e^{(4/3)t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}.$$

D18 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle$, with

$$\beta(se_4 + te_5) = \begin{pmatrix} -s & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & s+t \end{pmatrix}, \quad db(se_4 + te_5) = \begin{pmatrix} e^{-s} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{s+t} \end{pmatrix}.$$

D20 $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle$, with

$$\beta(se_4 + te_5) = \begin{pmatrix} 2s & 0 & 0 \\ 0 & -s & -t \\ 0 & t & -s \end{pmatrix},$$

$$db(se_4 + te_5) = \begin{pmatrix} e^{2s} & 0 & 0 \\ 0 & e^{-s}\cos(t) & -e^{-s}\sin(t) \\ 0 & e^{-s}\sin(t) & e^{-s}\cos(t) \end{pmatrix}$$

Acknowledgement. The authors thank the referees of this paper for their attention to detail and excellent suggestions. The second author thanks Dr. Patrick Chen at John Carroll University for his valuable insight on lattices and semi-direct products.

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