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NOTE ON THE FILTRATIONS OF THE K-THEORY

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Abstract

Let X be a (colimit of) smooth algebraic variety over a subfield k of $\mathbf{C}$. Let $K_{alg}^0(X)$ (resp. $K_{top}^0(X(\mathbf{C}))$) be the algebraic (resp. topological) K-theory of k (resp. complex) vector bundles over X (resp. $X(\mathbf{C})$). When $K_{alg}^0(X) \cong K_{top}^0(X(\mathbf{C}))$, we study the differences of its three (gamma, geometrical and topological) filtrations. In particular, we consider in the cases $X = BG$ for algebraic group G over algebraically closed fields k, and $X = \mathbf{G}_k/T_k$ the twisted form of flag varieties G/T for non-algebraically closed field k.

1. Introduction

Let X be a (colimit of) smooth algebraic variety over a subfield k of $\mathbf{C}$. We consider the cases that

$$(1.1) \qquad K_{alg}^0(X) \cong K_{top}^0(X(\mathbf{C}))$$

where $K_{alg}^0(X)$ (resp. $K_{top}^0(X(\mathbf{C}))$) is the algebraic (resp. topological) K-theory generated by algebraic k-bundles (complex bundles) over X (resp. $X(\mathbf{C})$). In this assumption, we study the typical three filtrations

$$F_\gamma^i(X) \subset F_{geo}^i(X) \subset F_{top}^i(X(\mathbf{C}))$$

namely, the gamma and the geometric filtrations defined by Grothendieck [Gr], and the topological filtration defined by Atiyah [At]. Namely, we study induced maps of associated rings

$$gr_\gamma^*(X) \to gr_{geo}^*(X) \to gr_{top}^*(X(\mathbf{C})).$$

Atiyah showed that $gr_{top}^*(X(\mathbf{C}))$ is isomorphic to the infitite term $E_\infty^{*,0}$ of the AHss (Atiyah-Hirzebruch spectral sequence) converging to K-theory $K^*(X(\mathbf{C}))$. Moreover he showed that $gr_{top}^*(X(\mathbf{C})) \cong gr_\gamma^*(X)$ if and only if $E_\infty^{*,0}$ is generated by Chern classes in $H^*(X(\mathbf{C}))$. We will see that similar facts hold for $gr_{geo}^*(X)$. Namely, $gr_{geo}^{2*}(X) \cong AE_\infty^{2*,*,0}$ of the motivic AHss converging to motivic K-theory

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$AK^{*,*'}(X)$. Moreover we show that $gr^*_{geo}(X) \cong gr^*_\gamma(X)$ if and only if $AE_\infty^{2*,*,0}$ is generated by Chern classes in the Chow ring $CH^*(X) \cong H^{2*,*}(X)$.

Let G be a compact Lie group (e.g., a finite group) and G_k be the corresponding algebraic group over an algebraically closed field k. Then by Merkurjev and Totaro ([To1]), we have the isomorphisms

$$K^0_{alg}(BG_k) \cong R(G_k)^\wedge \cong R(G)^\wedge \cong K^0_{top}(BG),$$

where $R(G_k)^\wedge$ (resp. $R(G)^\wedge$) is the k-representation (resp. complex representation) ring completed by the augmentation ideal, and BG_k and BG are their classifying spaces.

Atiyah had conjectured in [At] that $F^i_\gamma(BG) = F^i_{top}(BG)$ for all finite groups. Weiss [Th] showed this does not hold for $G = A_4$. Counter examples of p-groups were given by Leary-Yagita [Le-Ya] when G is $rank_p(G) = 2$ of class 3 with $p \geq 5$. We will see for the same group G, $F^{2p+2}_\gamma(BG_k) \neq F^{2p+2}_{geo}(BG_k) = F^{2p+2}_{top}(BG_k)$.

We study these filtrations detailedly for connected groups $(O_n, SO_n, \ldots)$. In particular we show

Theorem 1.1. (*Let k be an algebraically closed field.*) *For $G = Spin_7$, there is an element x in $K^0_{alg}(BG_k)$ such that*

$$0 \neq x \in gr^4_\gamma(BG_k), \quad 0 \neq x \in gr^6_{geo}(BG_k), \quad 0 \neq x \in gr^8_{top}(BG).$$

These facts also hold for the extraspecial 2-group 2^{1+6}_+.

Remark. Quite recently B. Totaro published paper [To2]. In §15 in this paper, he gives examples such that

$$gr^*_{geo}(BG)_{(p)} \neq gr^*_{top}(BG)_{(p)}$$

for all primes p.

We consider the different type of examples, which satisfy (1.1). (See also [Ga-Za], [Za].) Here we do not assume that k is algebraically closed. Let us write by $M(X)$ the (pure) motive of X, and by $M_a = (M_n)$ the Rost motive for a nonzero pure symbol $a \in K^M_{n+1}(k)/p$ ([Ro1,2], [Su-Jo]). We consider the cases X such that

$$M(X) \cong M_n \otimes A(X) \tag{1.2}$$

where $A(X)$ is a sum of k-Tate motives. Then we can see that (1.1) is satisfied by the result from ([Vi-Ya], [Ya6]).

Some cases of flag manifolds G/P satisfy (1.2) ([Ca-Pe-Se-Za], [Ni-Se-Za], [Pe-Se-Za]). We consider the exceptional Lie group G_2. Let $G_{2,k}$ and T_k be the corresponding splitting reductive group and its splitting maximal torus. Let us write by $\mathbf{G}_{2,k}$ the nontrivial $G_{2,k}$-torsor (induced from a Rost cohomological invariant $0 \neq a \in K^M_3(k)/2$, [Ga-Me-Se]). (Namely, $\mathbf{G}_{2,k}/T_k$ is a twisted form of G_2/T.) Then for $p = 2$ $X = \mathbf{G}_{2,k}/T_k$ satisfies (1.2) ([Bo], [Pe-Se-Za]).

Note that $H^*(G_2/T)$ is torsion free, and we have

$$gr^*_{geo}(G_{2,k}/T_k) \cong gr^*_{top}(G_2/T) \cong H^*(G_2/T).$$

By using the fact that $CH^*(\mathbf{G}_{2,k}/T_k)$ is generated by Chern classes, we can show

THEOREM 1.2. *Let $\mathbf{G}_{2,k}$ be the nontrivial $G_{2,k}$-torsor for the Rost cohomological invariant in $K_3^M(k)/2$. Then we have*

$$gr^{2*}_{\gamma}(G_2/T) \cong gr^{2*}_{geo}(\mathbf{G}_{2,k}/T_k) \cong CH^*(\mathbf{G}_{2,k}/T_k).$$

From (1.1), the gamma filtration is defined purely topologically. Thus we see that this topological invariant is isomorphic to a purely algebraic geometric object such as the Chow ring of twisted form.

2. Filtrations

We first recall the topological filtration defined by Atiyah. Let Y be a topological space (e.g., a CW-complex). Let $K^*(Y)$ be the complex K-theory; the Grothendieck group generated by complex bundles over Y. Let Y^i be an i-dimensional skeleton of Y. Define the topological filtration of $K^*(Y)$ by

$$F^i_{top}(Y) = Ker(K^*(Y) \to K^*(Y^i))$$

and the associated graded algebra $gr^i_{top}(Y) = F^i_{top}(Y)/F^{i+1}_{top}(Y)$.

We consider the long exact sequence (exact couple)

$$\cdots \to K^*(Y^i/Y^{i-1}) \to K^*(Y^i) \to K^*(Y^{i-1}) \xrightarrow{\delta} K^{*+1}(Y^i/Y^{i-1}) \to \cdots$$

Here we have $K^*(Y^i/Y^{i-1}) \cong K^* \otimes H^*(Y^i/Y^{i-1})$, which induces the (well known) AHss

$$E_2^{*,*'}(Y) \cong H^*(Y) \otimes K^* \Rightarrow K^*(Y).$$

By the construction of the spectral sequence, we have

LEMMA 2.1 (Atiyah [At]). $gr^*_{top}(Y) \cong E_\infty^{*,0}(Y)$.

Next we consider the geometric filtration. Let X be a smooth algebraic variety over a subfield k of $\mathbf{C}$. Let $K^0_{alg}(X)$ be the algebraic K-theory which is the Grothendiek group generated by k-vector bundles over X. It is also isomorphic to the Grothendieck group genrated by coherent sheaves over X (we assumed X smooth). This K-theory can be written by the motivic K-theory $AK^{*,*'}(Y)$ ([Vo1,2], i.e.,

$$K^i_{alg}(X) = \bigoplus\nolimits_* AK^{2*-i,*}(X).$$

In particular $K^0_{alg}(X) = \bigoplus_* AK^{2*,*}(X)$.

The geometric filtration ([Gr]) is defined as

$$F_{geo}^{2i}(X) = \{[O_V] \mid codim_X\ V \geq i\}$$

(and $F_{geo}^{2i-1}(X) = F_{geo}^{2i}(X)$) where O_V is the structural sheaf of closed subvariety V of X.

We recall the algebraic cobordism $MGL^{*,*'}(-)$ [Vo1] and let us write $MGL^{2*,*}(X) = \Omega^*(X)$, in fact, this is isomorphic to the algebraic cobordism defined by Levine and Morel ([Le-Mo1,2], [Vo1,2]). Recall

$$\Omega^*(Spec(k)) = \Omega^*(pt.) \cong MU^{2*}(pt.) = MU^*$$

where $MU^* \cong \mathbf{Z}[x_1, x_2, \ldots]$, $|x_i| = -2i$ is the complex cobordism ring. Then we have the isomorphism

$$\Omega^*(X) \otimes_{MU^*} \mathbf{Z} \cong CH^*(X), \quad \Omega^*(X) \otimes_{MU^*} K^* \cong K_{alg}^0(X)$$

where the MU^* module structure of K^* is given by Todd genus (see §3 below). Each element $x \in \Omega^*(X)$ is represented by a projective map $x = [f : M \to X]$ with $codim_X\ M = i$ and M smooth ([Le-Mo1,2]), namely, $x = f_*(1_M)$ for $1_M \in \Omega^0(M)$ and f_* is the Gysin map. Then the geometric filtration is also defined as

$$F_{geo}^{2i}(X) = \{f_*(1_M) \mid f : M \to X \ \ and \ \ codim_X\ M \geq i\}$$

since $f_*(M) = [O_M]$ in $K_{alg}^0(X)$.

Here we recall the motivic AHss ([Ya3, 4])

$$AE_2^{*,*',*''}(X) \cong H^{*,*'}(X; K^{*''}) \Rightarrow AK^{*,*'}(X).$$

(Of course this spectral sequence is not defined using skeleton as the topological case. But we assume the existence of the AHss converging to the motivic K-theory $AK^{*,*'}(X)$.) Note that

$$AE_2^{2*,*,*''}(X) \cong H^{2*,*}(X; K^{*''}) \cong CH^*(X) \otimes K^{*''}.$$

Hence $AE_\infty^{2*,*,0}(X)$ is a quotient of $CH^*(X)$ by dimensional reason of degree of differential d_r (i.e., $d_r AE_r^{2*,*,*''}(X) = 0$). Thus we have

Lemma 2.2. $gr_{geo}^{2*}(X) \cong AE_\infty^{2*,*,0}(X)$.

Proof. Let $q : \Omega^*(X) \otimes K^* \to K^*(X)$. Then

$$F_{geo}^{2i}(X) = q\{f_*(1_M) \in \Omega^*(X) \mid f : M \to X \ \ and \ \ codim_X\ M \geq i\}.$$

Let $q' : \Omega^*(X) \to CH^*(X)$ and $q'' : CH^*(X) \to E_\infty^{2*,*,0}$. Then $q \mid (\Omega^*(X) \otimes 1) = q''q'$. Thus we have

$$F_{geo}^{2i}(X)/F_{geo}^{2i+2}(X) = q'' CH^i(X)$$

since q' is an epimorphism. □

Lemma 2.3. *Let $t_{\mathbf{C}} : K_{alg}^0(X) \to K_{top}^0(X(\mathbf{C}))$ be the realization map. Then $F_{geo}^i(X) \subset (t_{\mathbf{C}}^*)^{-1} F_{top}^i(X(\mathbf{C}))$.*

Proof. Let us write $K^0_{top}(X(\mathbf{C}))$ simply by $K(X)$. The Gysin map $f_* : K(M) \to K(X)$ is defined by using Thom isomorphism

$$K(M) \cong K(Th_X(M)) \to K(X).$$

Let $codim_X\, M \geq i$. For an $2i$-skeleton X^{2i} of $X(\mathbf{C})$, we can show that the map

$$K(Th_X(M)) \to K(X) \to K(X^{2i})$$

is zero. Because the above composition map is rewritten

$$K(Th_X(M)) \to K(Th_X(M)^{2i}) \to K(X^{2i}).$$

Its first map is zero, because $H^*(Th_X(M)) = 0$ for $* < 2i$ and the exact sequence (exact couple) for K-theory for skeletons of X (see the definition of the AHss). □

At last, we consider the gamma filtration. Let $\lambda^i(x)$ be the exterior power of the vector bundle $x \in K^0_{alg}(X)$ and $\lambda_t(x) = \sum \lambda^i(x)t^i$. Let us denote

$$\lambda_{t/(1-t)}(x) = \gamma_i(x) = \sum \gamma^i(x)t^i.$$

The Gamma filtration is defined as

$$F^{2i}_\gamma(X) = \{\gamma^{i_1}(x_1) \cdot \ldots \cdot \gamma^{i_m}(x_m) \mid i_1 + \cdots + i_m \geq i,\, x_j \in K^0_{alg}(X)\}.$$

Then we can see $F^i_\gamma(X) \subset F^i_{geo}(X)$ (Proposition 12.5 in [At], Atiyah proved $F^i_\gamma(X) \subset F^i_{top}(X)$ in $K_{top}(X)$. However the arguments work also in $K^0_{alg}(X)$ and this fact is well known [Ga-Za]. [Ju].) Let $\varepsilon : K^0_{alg}(X) \to \mathbf{Z}$ be the augmentation map and $c_i(x) \in H^{2i,i}(X)$ the Chern class. Recall $q'' : CH^*(X) \to E^{2*,*,0}_\infty$ be the quotient map. Then (p. 63 in [At]) we have

$$q''(c_n(x)) = [\gamma^n(x - \varepsilon(x))].$$

Lemma 2.4 (Atiyah). *The condition $F^{2*}_\gamma(Y) = F^{2*}_{top}(Y)$ (resp. $F^{2*}_\gamma(X) = F^{2*}_{geo}(X)$) is equivalent to that $E^{2*,0}_\infty(Y)$ (resp. $AE^{2*,*,0}_\infty(X)$) is (multiplicatively) generated by Chern classes in $H^{2*}(Y)$ (resp. $CH^*(X)$).*

3. Morava K-theory (K-theory localized at p)

In this paper, we assume that p is a fixed prime number and consider only cohomology theories (Chow rings) localized at this prime p. Namely, for the notation $A^*(X)$ means $A^*(X)_{(p)}$ in this paper. In particular, $\mathbf{Z}$ *always means* $\mathbf{Z}_{(p)}$ and $MU^*(X)$ means $MU^*(X)_{(p)}$ throughout this paper.

Let $AMU^{*,*'}(X) = MGL^{*,*'}(X)$ and recall $MU^* = \mathbf{Z}[x_1, \ldots, x_n, \ldots]$, $deg(x_i) = (-2i, -i)$. Given a sequence $S = (x_{i_1}, x_{i_2}, \ldots)$ of generators, we can construct generalized cohomology theory (in the $\mathbf{A}^1$-homotopy category) such that

$$t_{\mathbf{C}} : AMU(S)^{*,*'}(X) \to MU(S)^*(X(\mathbf{C})) \quad \textit{with } MU(S)^* = MU^*/(S).$$

In particular letting $x_{p^n-1} = v_n$ and $S = (x_i \mid i \neq p^n - 1)$, we have the motivic BP-theory ([Ya3,5])

$$ABP^{*,*'}(X) \quad with \ \ MU^*/(S) \cong BP^* = \mathbf{Z}[v_1, v_2, \dots].$$

Then we have the isomorphisms ([Ya3])

$$ABP^{*,*'}(X) \cong MGL^{*,*'}(X) \otimes_{MU^*} BP^*,$$
$$MGL^{*,*'}(X) \cong ABP^{*,*'}(X) \otimes_{BP^*} MU^*.$$

Similarly, we can construct the motivic connective Morava K-theory such that

$$Ak(n)^{*,*'}(X) \quad with \ \ k(n)^* = \mathbf{Z}/p[v_n],$$

and the integral connected K-theory $A\tilde{k}(n)^{*,*'}(X)$ with $\tilde{k}(n) = \mathbf{Z}[v_n]$. Moreover let the (usual) motivic Morava K-theory

$$AK(n)^{*,*'}(X) = Ak(n)^{*,*'}(X)[v_n^{-1}], \quad A\tilde{K}(n)^{*,*'}(X) = A\tilde{k}(n)^{*,*'}(X)[v_n^{-1}].$$

By the Landweber exact functor theorem ([Ra], [Ha]), it is well known that

$$AK^{*,*'}(X) \cong (AMU^{*,*'}(X) \otimes_{MU^*} \mathbf{Z}) \otimes \mathbf{Z}[B, B^{-1}]$$

where the MU^*-module structure of $\mathbf{Z}$ is given by the Todd genus, and B is the Bott periodicity with $deg(B) = (-2, -1)$. Since the Todd genus of v_1 (resp. v_i, $i > 1$) is 1 (resp. 0), we can write

$$AK^{*,*'}(X) \cong ABP^{*,*'}(X) \otimes_{BP^*} \mathbf{Z}[B, B^{-1}] \quad identifying \ \ B^{p-1} = v_1.$$

Then we have

Lemma 3.1. *There is a natural isomorphism*

$$A\tilde{K}^{*,*'}(X) \cong A\tilde{K}(1)^{*,*'}(X) \otimes_{\tilde{K}(1)^*} \mathbf{Z}[B, B^{-1}] \quad identifying \ \ v_1 = B^{p-1}.$$

Proof. Recall that we have the natural map (by the construction of $AMU(S)$)

$$\rho : ABP^{*,*'}(X) \otimes_{BP^*} \mathbf{Z}[B, B^{-1}] \to A\tilde{K}(1)^{*,*'}(X) \otimes_{\tilde{K}(1)^*} \mathbf{Z}[B, B^{-1}].$$

Of course, the functor

$$A \mapsto A \otimes_{\tilde{K}(1)^*} \mathbf{Z}[B, B^{-1}] \cong A \otimes \mathbf{Z}\{1, B, \dots, B^{p-2}\}$$

is exact, and we have the spectral sequence

$$E_2^{*,*',*''}(A\tilde{K}(1)) \otimes_{\tilde{K}(1)^*} \mathbf{Z}[B, B^{-1}] \Rightarrow A\tilde{K}(1)^{*,*'}(X) \otimes_{\tilde{K}(1)^*} \mathbf{Z}[B, B^{-1}].$$

Since for a $BP^*(BP)$ module A, the functor

$$A \mapsto A \otimes_{BP^*} \mathbf{Z}[B, B^{-1}]$$

is exact from the Landweber exact functor theorem, we have the spectral sequence from the AHss for $ABP^{*,*'}(X)$

$$E_2^{*,*',*''}(ABP) \otimes_{BP^*} \mathbf{Z}[B, B^{-1}] \Rightarrow ABP^{*,*'}(X) \otimes_{BP^*} \mathbf{Z}[B, B^{-1}],$$

which is compatible with the map ρ. The E_2-term of the both spectral sequences are isomorphic to

$$H^{*,*'}(X; \mathbf{Z}) \otimes \mathbf{Z}[B, B^{-1}].$$

Therefore the two spectral sequences are isomorphic. □

We also note from the arguments in the above proof.

LEMMA 3.2. *Let $E(ABP)_r^{*,*',*''}$ (resp. $E(A\tilde{K}(1))_r^{*,*',*''}$) be the AHss coverging to $ABP^{*,*'}(X)$ (resp. $A\tilde{K}(1)^{*,*'}(X)$). Then we have*

$$E(ABP)_r^{*,*',*''} \otimes_{BP^*} \tilde{K}(1)^* \cong E(A\tilde{K}(1))_r^{*,*',*''}).$$

From above lemmas, it is sufficient to consider the Morava K-theory $A\tilde{K}(1)^{*,*'}(X)$ when we want to study $AK^{*,*'}(X)$. Hereafter of this paper, we only consider the theories $A\tilde{K}(1)^{*,*'}(X)$ and $A\tilde{k}(1)^{*,*'}(X)$ instead of $AK^{*,*'}(X)$ or $K_{alg}^*(X)$. (We only consider the cohomology theories and Chow rings localied at p.)

We assume the following assumption

$$(*) \quad K_{alg}^0(X) \cong K_{top}^0(X(\mathbf{C})) \quad (and\ K_{top}^1(X(\mathbf{C})) = 0).$$

That is equivalent to

$$(*) \quad A\tilde{K}(1)^{2*,*}(X) \cong \tilde{K}(1)^{2*}(X(\mathbf{C})) \quad (and\ \tilde{K}(1)^{2*+1}(X(\mathbf{C})) = 0).$$

From Lemma 2.3, we have

$$F_\gamma(X) \subset F_{geo}^i(X) \subset F_{top}^i(X(\mathbf{C})).$$

Here we note that the gamma filtrations of topogical and algebraic geometrical are same, i.e., $F_\gamma^*(X) \cong F_\gamma^*(X(\mathbf{C}))$. So we have the maps of associated graded rings

$$gr_\gamma^*(X) \to gr_{geo}^*(X) \to gr_{top}^*(X(\mathbf{C})).$$

LEMMA 3.3. $gr_\gamma^2(X) = gr_{geo}^2(X)$.

Proof. If $0 \neq x \in gr_\gamma^2(X)$, then $x = c_1(\xi) \in A\tilde{K}(1)^{2*,*}(X)$ for some bundle ξ. In $CH^*(X)$, we know $c_1(\xi) = c_1(det(\xi))$ which is determined by the line bundle $det(\xi)$. Line bundles are determined by $Pic(X) = CH^1(X)$. So $0 \neq x \in CH^1(X)$. □

LEMMA 3.4. *If an element* $y \in A\tilde{K}(1)^{2*,*}(X)$ *is represented by* $0 \neq y$ (*resp.* y'', y''') *in* $gr_\gamma^i(X)$ (*resp.* $gr_{geo}^j(X)$, $gr_{top}^k(X(\mathbf{C}))$), *then*

$$i \leq j \leq k, \quad \text{and} \quad i = k = j \ mod(2(p-1)).$$

Proof. The element y is represented

$$y = v_1^s y' \in A\tilde{K}(1)^{2*,*}(X)/F_\gamma^{2i+1} \quad y = v_1^t y'' \in A\tilde{K}(1)^{2*,*}(X)/F_{geo}^{2j+1}$$

for some $s, t \in \mathbf{Z}$. □

Remark. The above fact does not hold for $y \in K_{top}^0(X)$ (which is a sum of $\tilde{K}(1)^{2*,*}(X)$, $0 \leq * \leq p-2$). Let us write

$$y = b^k y_k + b^{k+1} y_{k+1} + \cdots + b^{k+p-2} y_{k+p-2},$$

with $b^i \in \tilde{K}(1)^{-2i}$ and $y_i \in F_{top}^{2i}(Y)$. Suppose $j < k$. Then this means that there is s such that $0 \neq y_s \in gr_{geo}^j(X)$ with $s - j = 0 \ mod(2p-2)$. Of course if $s \neq k$, then $k - j \neq 0 \ mod(2p-2)$.

To study the difference of $F_{geo}^*(X)$ and $F_{top}^*(X(\mathbf{C}))$, we consider AHss $E_r^{*,*'}(BP)$ converging to $BP^*(X)$. Suppose that

$$[v_1 \otimes x] \in BP^{*'} \otimes H^*(X(\mathbf{C})) \cong E(BP)_2^{*,*'}$$

is an permanent cycle, but $[x] \in H^*(X(\mathbf{C}))$ itself is not (i.e., $d_r(x) \neq 0$ for some r). Let $x' \in BP^*(X(\mathbf{C}))$ be a corresponding element for $[v_1 \otimes x]$ in $E_\infty^{*,*'}$

LEMMA 3.5. *Let* $x \in H^{2*}(X(\mathbf{C}))$ *and* $x' \in BP^{*'}(X(\mathbf{C}))$ *be elements with the assumption above. Suppose that*

$$0 \neq x' \in BP^{*'}(X(\mathbf{C})) \otimes_{BP^*} \mathbf{Z}[v_1, v_1^{-1}] \cong \tilde{K}(1)^*(X(\mathbf{C}))$$

and that $x' \in BP^{*'}(X(\mathbf{C})) \otimes_{BP^*} \mathbf{Z}$ *is in the image of the Totaro cycle map*

$$CH^{*'}(X) \to BP^{2*'}(X(\mathbf{C})) \otimes_{BP^*} \mathbf{Z}.$$

Then $0 \neq x' \in gr_{top}^{2*}(X(\mathbf{C}))$, *but* $0 \neq x' \in gr_{geo}^{2(*-p+1)}(X))$.

Proof. In this case $*' = * - (p-1)$ in the above arguments. Let $x \in H^{2i}(X(\mathbf{C}))$. In fact $x' \in Im(CH^{i-p+1}(X))$ and $0 \neq x' \in gr_{geo}^{2(i-p+1)}(X(\mathbf{C}))$, but $0 \neq x' = [v_1 \otimes x] \in gr_{top}^{2i}(X(\mathbf{C}))$. □

Next we consider the cases $gr_\gamma^*(X) \cong gr_{top}(X(\mathbf{C}))$. From the Atiyah theorem (Lemma 2.4), the following lemma is immediate.

LEMMA 3.6. *Suppose* (∗) *and suppose that the infinity term* $E_\infty^{2*,0}(\tilde{K}(1))$ (*of the AHss for* $\tilde{K}(1)^*(X(\mathbf{C}))$) *is generated by Chern classes in* $H^*(X)$ *for all* $* \geq N$. *Then for all* $* \geq N$, *we have*

$$gr_\gamma^{2*}(X) \cong E_\infty^{2*,0}(\tilde{K}(1)^*(X(\mathbf{C}))) \quad \text{for all } * \geq N.$$

LEMMA 3.7 (Lemma 2.8 in [Ya4]). *Suppose* $(*)$ *and that* $H^*(X(\mathbf{C}))$ *is generated by Chern classes. Then we have*

$$CH^*(X) \cong H^*(X(\mathbf{C})) \quad for \ * \leq p-1.$$

Moreover if $X(\mathbf{C})$ *is simply connected* (*resp.* 3-*connected*), *then we have an isomorphisms for* $* \leq p$ (*resp.* $* \leq p+1$)

$$CH^*(X) \otimes \mathbf{Z}_p \cong H^{2*}(X(\mathbf{C}); \mathbf{Z}_p).$$

Proof. By the assumption, we see

$$gr_\gamma^{2*}(X) \cong gr_{geo}^{2*}(X) \cong gr_{top}^{2*}(X(\mathbf{C})).$$

To compute the last graded ring, we consider AHss

$$E_2^{*,*'}(\tilde{K}(1)) \cong H^*(X; \tilde{K}(1)^{*'}) \Rightarrow \tilde{K}(1)^*(X(\mathbf{C})).$$

Here $\tilde{K}(1)^* \cong \mathbf{Z}[v_1, v_1^{-1}]$ with $|v_1| = -2p+2$. It is well known that the first non zero differential is

$$d_{2p-1}(x) = v_1 \otimes Q_1(x) \ mod(p).$$

So each element in $H^{2*}(X(\mathbf{C}))$ is not targent of any differential d_r when $* \leq p-1$. (Of course $d_r(x) = 0$ for Chern classes x.) Hence we have $gr_{top}^{2*}(X(\mathbf{C})) \cong H^{2*}(X(\mathbf{C}))$ for $* \leq p-1$.

Similarly, considering AHss converging to $A\tilde{K}(1)^{*,*'}(X)$, we have the isomorphism $gr_{geo}^{2*}(X) \cong CH^*(X)$ for $* \leq p-1$. Here we use the fact $E_2^{2*,*,0}(A\tilde{K}(1)) \cong CH^*(X)$. Thus the isomorphism of the geometric and toplogical filtrations, gives the first statements.

From the isomorphism

$$H^{1,1}(X; \mathbf{Z}/p) \cong H^1(X(\mathbf{C}); \mathbf{Z}/p) = 0.$$

we see that $H^{1,1}(X; \mathbf{Z})$ is p-divisible. Since the image of the differential of p-divisible elements are also p-divisible,

$$\begin{aligned} H^{2p}(X(\mathbf{C})) &\cong gr_{top}^{2p}(X) \\ &\cong gr_{geo}^{2p}(X) \cong CH^{2p}(X)/(p - divisible). \end{aligned}$$

Hence we have the second isomorphism. (In 3-connected cases, the isomorphism is seen similarly for $* \leq p+1$.) □

Remark. The first statement in the above lemma is also proved by the Riemann-Roch formula without denominators, namely, the composition map

$$CH^i(X) \to gr_{geo}^i(X) \xrightarrow{c_i} CH^i(X)$$

is multiplication by $(-1)^{i-1}(i-1)!$. Hence we get $CH^i(X) \cong gr_{geo}^i(X)$ for $i \leq p$. Moreover we know that $CH^i(X)$ is represented by the i-th Chern class $c_i(\xi)$ for some bundle ξ.

Remark. Lemma 2.8 in [Ya4] was not correct (assumed $gr_{geo}^*(X) = gr_{top}(X(\mathbf{C}))$ there). Hence the assumption of Lemma 2.8 in [Ya4] is not sufficient, and it should be changed as above Lemma 3.7.

4. Classifying spaces BG for finite groups

Let G be a compact Lie group (e.g., a finite group) and G_k be the corresponding algebraic group over an algebraically closed field k in $\mathbf{C}$. Then by Merkurjev and Totaro ([To1]), we have the isomorphisms

$$K_{alg}^0(BG_k) \cong R(G_k)^\wedge \cong R(G)^\wedge \cong K_{top}^0(BG), \tag{1.1}$$

where $R(G_k)^\wedge$ (resp. $R(G)^\wedge$) is the k-representation (resp. complex representation) ring completed by the augmentation ideal and $K_{alg}^0(BG_k)$ (resp. $K_{top}^0(BG)$) is the K-theory generated by k-bundles (resp. complex bundles) of the classifying space BG_k (resp. BG).

When k is algebraically closed, we write BG_k by BG simply. For Section 4–6, we assume k is algebraically closed.

In this section, we consider cases that G are finite groups. At first, we consider the case $G = \mathbf{Z}/p^r$. Then $H^*(BG) \cong \mathbf{Z}[y]/(p^r y)$, $|y| = 2$ and $y_1 = c_1(e)$ for a nonzero linear representation e. So all three filtrations are the same. The similar fact holds for its product.

THEOREM 4.1 ($p = 2$, $r = 1$ case by Atiyah [At]). *Let $q = p^r$ and $G = \bigoplus^n \mathbf{Z}/q$. Then*

$$gr_{top}^*(BG) \cong \mathbf{Z}[y_1, \dots, y_n]/(qy_i, y_i^q y_j - y_i y_j^q).$$

Hence the three filtrations are the same.

Proof. Let $Q_0' = \beta_q$ be the higher Bockstein. The integral cohomology is isomorphic to a subring of the mod q cohomology

$$H^*(BG) \subset H^*(BG; \mathbf{Z}/q), \quad \textit{when } * > 0.$$

Here $H^*(BG; \mathbf{Z}/q) \cong \mathbf{Z}/q[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n)$ with $Q_0'(x_i) = y_i$, and we know

$$H^*(BG) \cong \mathbf{Z}/q[y_1, \dots, y_n]\{Q_0'(x_{i_1} \cdots x_{i_s}) \mid 1 \le i_1 < \cdots, i_s \le n\}$$

with $Q_0'(x_{i_1} \cdots x_{i_s}) = \sum_k (-1)^{k-1} y_{i_k} x_{i_1} \cdots \hat{x}_{i_k} \cdots x_{i_s}$.

We consider the AHss converging to $\tilde{K}(1)^*(BG)$. We define the weight degree for elements in this AHss by

$$w(v_1) = 0, \quad w(y_i) = 0, \quad w(x_i) = 1$$

so that $w(Q_0'(x_{i_1}\cdots x_{i_s}))=s-1$. We will prove

$$(1)\qquad (weight=0)\cap E_{2q}^{*,*'}\cong \mathbf{Z}/q[y_1,\dots,y_n]/(y_i^q y_j-y_iy_j^q)\quad for\ *>0,$$

$$(2)\qquad (weight=1)\cap E_{2q}^{*,*'}=0.$$

Then we can prove this theorem by the following arguments.

We consider the AHss converging to the motivic $A\tilde{K}(1)^*(BG)$. The weight $w(x)$ of an element $x\in H^{*,*'}(X:\mathbf{Z}/q)$ is defined as $2*'-*$. Since $x_i\in H^{1,1}(BG;\mathbf{Z}/q)$ and $y_i\in H^{2,1}(BG;\mathbf{Z}/q)$, their weights are in fact $w(x_i)=1$ and $w(y_i)=0$. The degree of the motivic AHss is

$$deg(d_{2r-1})=(2r-1,r-1,-2(r-1))\quad with\ (r-1)=0\ mod(p-1),$$

namely, $w(d_{2r-1})=-1$ which means

$$d_{2r-1}(weight=s)=(weight=s-1).$$

From (2), $(weight=0)$-parts are not a target of any diffrential d_{2r-1} for $r>q$. By the naturality of realization map from the motivic AHss to the usual AHss, we get the same fact for the AHss for $\tilde{K}(1)^*(BG)$. Since $\tilde{K}(1)^*(BG)$ is generated by only $weght=0$ elements, we have the theorem.

The first nonzero differential is known $d_{2q-1}(x_i)=v_1^{1+p+\cdots+p^{r-1}}y_i^q$ [Ya3]. Hereafter let $v_1=1$ for ease of notations. We see (1) from

$$d_{2q-1}(Q_0'(x_1x_2))=d_{2q-1}(y_1x_2-y_2x_1)=y_1y_2^q-y_1^qy_2.$$

Now we prove (2). Let $x\in Ker(d_{2s-1})$ and $x=\sum a_{ij}Q_0'(x_ix_j)$. Then (since d_r is a derivation)

$$d_{2q-1}(x)=\sum a_{ij}(y_iy_j^q-y_i^qy_j)=0\quad in\ \mathbf{Z}/q[y_1,\dots,y_n].$$

Here we consider them in $mod(x_i,y_i\,|\,i\ge 4)$. Then we see $a_{12}=a_{12}'y_3$ and we see (by dividing $y_1y_2y_3$)

$$a_{12}'(y_1^{q-1}-y_2^{q-1})+a_{23}'(y_2^{q-1}-y_3^{q-1})+a_{31}'(y_3^{q-1}-y_1^{q-1})=0.$$

This implies that $a_{12}'\in ideal(y_1^{q-1},y_2^{q-1},y_3^{q-1})$. Moreover we see that a_{12} contains y_3^q. Similarly a_{23}, a_{13} contains y_1^q and y_2^q respectively.

On the other hand, we see

$$\begin{aligned}d_{2q-1}(Q_0'(x_1x_2x_3))&=d_{2q-1}\Big(\sum y_1x_2x_3\Big)\\&=\sum y_1y_2^qx_3-\sum y_1x_2y_3^q=\sum y_1y_2^qx_3-\sum y_3x_1y_2^q\\&=\sum y_2^q(y_1x_3-y_3x_1)=-\sum y_1^qQ_0'(x_2x_3)\end{aligned}$$

Taking off $a''d_{2r-1}Q_0'(x_1x_2x_3)$ for some adequate $a''\in\mathbf{Z}/q[y_1,\dots,y_n]$, we can prove (2). □

Recall that a group G is called an extraspecial p-group if its center $Z(G) \cong \mathbf{Z}/p$ and there is a central extension

$$0 \to \mathbf{Z}/p \to G \to \bigoplus^{2n} \mathbf{Z}/p \to 0.$$

For each prime p, such groups have only two types, namely, p_+^{1+2n}, p_-^{1+2n}. (e.g., $2_+^{1+2} \cong D_8$ the dihedral group (of order 8), $2_-^{1+2} \cong Q_8$ the quaternion group). We here only write down the case p_+^{1+2} for $p \geq 3$. The cohomology is known ([Ya1,4])

$$H^{even}(BG) \cong (Y \oplus B) \otimes \mathbf{Z}[c_p]/(p^2 c_p)$$

where $Y = \mathbf{Z}[y_1, y_2]/(py_i, y_1 y_2^p - y_1^p y_2)$, $B = \mathbf{Z}/p\{c_2, \dots, c_{p-1}\}$ and $y_i = c_1(e_i)$ and $c_i = c_i(\xi)$ for some linear representations e_i and p-dimensional representation ξ. Hence the even dimensional part of this cohomology is generated by Chern classes and all three filtrations are the same. The odd degree part is

$$H^{odd}(BG) \cong Y \otimes \mathbf{Z}/p[c_p]\{a_1, a_2\}/(y_2 a_1 - y_1 a_2, y_2^p a_1 - y_1^p a_2) \quad |a_i| = 3.$$

THEOREM 4.2. *Let $G = p_+^{1+2}$ and $p \geq 3$. Then*

$$gr_{top}^*(BG) \cong Y \oplus (\mathbf{Z}\{c_p\} \oplus B) \otimes \mathbf{Z}[c_p]/(p^2 c_p).$$

Proof. We know the Milnor cohomology operation

$$v_1^{-1} d_{2p-1} = Q_1 : H^{odd}(BG) \to H^{even}(BG)$$

is injective and $Q_1(a_i) = y_i c_p$. Hence we see

$$\begin{aligned} gr\, \tilde{K}(1)^*(BG) &\cong E_\infty^{*,*'} \cong \tilde{K}(1)^* \otimes H^{even}(BG)/(Q_1 H^{odd}(BG)) \\ &\cong \tilde{K}(1)^* \otimes H^{even}(BG)/(y_i c_p). \end{aligned}$$ □

When $p \geq 5$, the groups of $rank_p\, G = 2$ are classified by Blackburn. When groups are of class 2 (i.e., $[G,[G,G]] = 1$), cohomology rings are generated by Chern classes ([Le-Ya], [Ya1]), and hence all three filtrations are the same. Define the class 3 p-group (i.e., $[G,[G,G]] \neq 1$) by

$$G(4,1) = \langle a, b, c \mid a^p = b^p = c^{p^2} = [b,c] = 1, [a, b^{-1}] = c^p, [a,c] = b \rangle.$$

Let $G = G(4,1)$. Then there is an element $x_{p+1} \in H^{2p+2}(BG)$ [Le-Ya], [Ya] such that it is a permanent cycle in AHss for $\tilde{K}(1)^*(BG)$ and x_{p+1} is not represented by Chern class. But all elemnts in $H^{even}(BG)$ is represented by transfers of Chern classes [Ya1]. Of course Chow rings have the transfer map. Hence we have

THEOREM 4.3. *Let $p \geq 5$ and $G = G(4,1)$. Then $gr_{top}^*(BG) \cong gr_{geo}^*(BG)$ but $gr_\gamma^i(BG) \not\cong gr_{geo}^i(BG)$ for $i = 4, 2p+2$.*

Proof. The first isomorphism follows from that all elements in $H^{even}(BG)$ is represented by transfer of Chern classes. The second statement follows from that

x_{p+1} is not represented by Chern classes and the element $x_{p+1} \in E_{\infty}^{2p+2,0}$ represents a nonzero element in $gr_{\gamma}^{4}(BG)$ from Lemma 3.4. $\square$

5. Connected groups with $p = 2$

Throughout this section, let $p = 2$. At first we consider the case $G = O_n$. The mod 2 cohomology of the classifying space BO_n of the n-th orthogonal group is

$$H^*(BO_n; \mathbf{Z}/2) \cong H^*((B\mathbf{Z}/2)^n; \mathbf{Z}/2)^{S_n} \cong \mathbf{Z}/2[w_1, \dots, w_n]$$

where S_n is the n-th symmetry group, w_i is the Stiefel-Whiteney class which restricts the elementary symmetric polynomial in $\mathbf{Z}/2[x_1, \dots, x_n]$. Each element w_i^2 is represented by Chern class c_i of the induced representation $O_n \subset U_n$. Let us write w_i^2 by c_i.

Recall the Milnor operation Q_i which is defined $Q_0 = \beta$ and $Q_i = [Q_{i-1}, P^{p^{i-1}}]$. Let us write by $Q(i)$ the exteria algebra $\Lambda(Q_0, \dots, Q_i)$. W. S. Wilson ([Wi], [Ko-Ya]) found a good $Q(i)$-module decomposition for BO_n, namely,

$$H^*(BO_n; \mathbf{Z}/2) = \bigoplus_{i=-1} Q(i)G_i \quad \textit{with} \ \ Q_0 \cdots Q_i G_i \in \mathbf{Z}/2[c_1, \dots, c_n].$$

Let us write by $P(n)^* = BP^*/(p, \dots, v_{i-1})$. The BP^*-theory is then computed

$$gr\ BP^*(BG)/p \cong \bigoplus P(i+1)^* Q_0 \cdots Q_i G_i.$$

Hence we have $K(1)^*(BG) \cong K(1)^*(G_{-1} \oplus Q_0 G_0)$.

Moreover, by Wilson, it is known that

$$BP^*(BO_n) \cong BP^*[[c_1, \dots, c_n]]/(c_1 - c_1^*, \dots, c_n - c_n^*)$$

where c_i^* is the conjugation of c_i. Hence $\tilde{K}(1)^*(BG)$ is generated by Chern classes from $H^*(BG)$. Thus from Lemma 2.4, all filtrations are same.

Here G_{k-1} is quite complicated (see for details [Wi]), namely, it is generated by symmetric functions

$$\sum x_1^{2i_1+1} \cdots x_k^{2i_k+1} x_{k+1}^{2j_1} \cdots x_{k+q}^{2j_q}, \quad k + q \le n,$$

with $0 \le i_1 \le \cdots \le i_k$ and $0 \le j_1 \le \cdots \le j_q$; and if the number of j equal to j_u is odd, then there is some $s \le k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

Thus when $k \le 1$, there is not above j_u, that means numbers of $j = j_u$ are always even.

THEOREM 5.1. *Let $G = O_n$. Then all three fitrations are the same, and $gr_{top}^*(BG) \cong A \oplus B/2$ with ($y_i = x_i^2$ so that $\sum y_1 = c_1$)*

$$A = \mathbf{Z}\Big\{\sum (y_1 y_2)^{j_1} \cdots (y_{2s-1} y_{2s})^{j_s}\Big\} \quad B = \mathbf{Z}\Big\{\sum y_1^i (y_2 y_3)^{j_1} \cdots (y_{2s} y_{2s+1})^{j_s}\Big\}.$$

(*Note $A/2 = G_{-1}$ and $B/2 = Q_0 G_0$.*)

Example. When $G = O_2$, we have the isomorphism

$$gr^*_{top}(BG) \cong \mathbf{Z}[c_2] \oplus \mathbf{Z}/2[c_1].$$

When $G = SO_{odd}$, (since $SO_{odd} \times \mathbf{Z}/2 \cong O_{odd}$), the situations are same. Let $G = SO_{2n}$. Then from Field, we have ([Fi], [Ma-Vi], [In-Ya])

$$CH^*(BG) \cong \mathbf{Z}[c_2, \dots c_{2n}]\{y_{2n}\} \oplus CH^*(BO_{2n})/(c_1),$$
$$BP^*(BG) \cong BP^*[c_2, \dots, c_{2n}]\{y_{2n}\} \oplus BP^*(BO_{2n})/(F_1)$$

where $F_1 = Ker(Bdet^*)$ and $y_{2n}^2 = (-1)^n 2^{2n-2} c_{2n}$. Hence

$$y_{2n} = (-1)^* 2^{n-1} w_{2n} \in H^*(BG)_{(2)}.$$

THEOREM 5.2. *Let $G = SO_{2n}$ and $n \geq 3$. Then*

$$gr^*_{top}(BG) = gr^*_{geo}(BG) \cong \mathbf{Z}[c_2, c_4, \dots, c_{2n}]\{y_{2n}\} \oplus gr^*_{top}(BO_{2n})/(c_1).$$

However we have $gr^{2n}_{\gamma}(BG) \not\cong gr^{2n}_{geo}(BG)$.

We note when $G = SO_4$, all the three filtrations are same, since y_4 is represented by Chern classes. By Field, it is shown that just $(n-1)!y_{2n}$ (for $n > 2$) is represented by Chern classes (Theorem 8, Corollary 2 in [Fi]). Thus we have

PROPOSITION 5.3. *Let $G = SO_{2(p+1)}$ and $p \neq 2$. Then*

$$gr^*_{\gamma}(BG) \cong \mathbf{Z}_{(p)}[c_2, \dots, c_{2p+2}] \otimes (\mathbf{Z}_{(p)}\{1, y'\} \oplus \mathbf{Z}/p\{y\})$$

with $|y'| = 2(p+1)$ and $|y| = 4$.

Next, we consider the exceptional Lie group G_2. Let $G = G_2$. Its $mod(2)$ cohomology is well known

$$H^*(BG; \mathbf{Z}/2) \cong \mathbf{Z}/2[w_4, w_6, w_7]$$

and integral cohomology is

$$H^*(BG) \cong \mathbf{Z}[w_4, c_6] \otimes (\mathbf{Z}\{1\} \oplus \mathbf{Z}/2[w_7]\{w_7\}).$$

We can compute the AHss for $BP^*(BG)$ ([Ko-Ya], [Sc-Ya])

$$gr\ BP^*(BG) \cong \mathbf{Z}[c_4, c_6] \otimes (BP^*\{1, 2w_4\} \oplus P(3)^*[c_7]\{c_7\}).$$

Here we can show the element $\{2w_4\}$ is represented by a Chern class c_2'. We see $\tilde{K}(1)^*(BG) \cong \tilde{K}(1)^*[c_4, c_6] \otimes \{1, 2w_4\})$, and ([Ya3], [Gu])

$$CH^*(BG) \cong BP^*(BG) \otimes_{BP^*} \mathbf{Z} \cong \mathbf{Z}[c_2', c_4, c_6, c_7]/((c_2')^2 - 4c_4, 2c_7).$$

Theorem 5.4. *Let $G = G_2$. Then all three filtrations are the same*

$$gr^*_{top}(BG) \cong CH^*(BG)/(c_7) \cong \mathbf{Z}[c_2', c_4, c_6]/((c_2')^2 - 4c_4).$$

Next we study the case $G = Spin_7$. Its $mod(2)$ cohomology is

$$H^*(BG; \mathbf{Z}/2) \cong \mathbf{Z}/2[w_4, w_6, w_7, w_8].$$

The infinity term of the AHss for $BP^*(BG)$ is still computed

$$\begin{aligned} gr\, BP^*(BG) \cong \mathbf{Z}[c_4, c_6] \otimes (&BP^*[c_8]\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\} \\ &\oplus P(3)^*[c_7]\{c_7\} \oplus P(4)^*[c_7, c_8]\{c_7c_8\}). \end{aligned}$$

Hence we see

$$gr\, \tilde{K}(1)^*(BG) \cong \tilde{K}(1)^*[c_4, c_6, c_8]\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\}.$$

Here it is known that $2w_4$, $2w_8$, $2w_4w_8$ are represented by Chern classes. Write them by c_2', c_4', c_6'. But it is proved (Theorem 6.2 in [Sc-Ya]) that v_1w_8 is not represented by (transfer) of Chern classes while it is in the image of cycle map. Let $cl(\xi) = [v_1w_8]$ ([Gu], Lemma 9.6 in [Ya], §9 in [Ka-Te-Ya]). Totraro's conjecture also holds this case

$$\begin{aligned} CH^*(BG) &\cong BP^*(BG) \otimes_{BP^*} \mathbf{Z} \\ &\cong \mathbf{Z}[c_4, c_6, c_8] \otimes (\mathbf{Z}\{1, c_2', c_4', c_6'\} \oplus \mathbf{Z}/2\{\xi\} \oplus \mathbf{Z}/2[c_7]\{c_7\}) \end{aligned}$$

with $|\xi| = 6$. Moreover, we can prove

Lemma 5.5. *Let $G = Spin_7$. Any element $x \in BP^*(BG)$ such that*

$$0 \neq x = [v_1w_8]a \in BP^*(BP) \quad \text{with } a \in \mathbf{Z}[c_4, c_6, c_8],$$

can not be generated by Chern classes of BP^*-theory.

Proof. Let $N = Z(G) \cong \mathbf{Z}/2$ be the center of G and $N \oplus A$ is a maximal elementary abelian 2-subgroup of G, so $A \cong (\mathbf{Z}/2)^3$. A representation ξ of G is said to be a spin representation, if $\xi \mid N \neq 0$. For a nonspin representation η, we know the total Chern class

$$c(\eta)|_{N \oplus A} = c(\eta)|_A \in BP^*[c_4, c_6, c_7].$$

For a spin representation χ, we have

$$(\chi)|_N = (1 + u)^s \in BP^*(BN) \cong BP^*[u]/([2](u)) \quad |u| = 2$$

where $[2](u) = 2u + v_1u^2 + \cdots$ is the 2-th product of the BP^*-formal group laws. Here we note $s = 8s'$ since $c_8|_N = u^8$. It is known that $v_1w_8|_N = v_1u^4$ [Sc-Ya]. Then

$$c(\chi)|_N = (1 + 8u + 28u^2 + \cdots + u^8)^{s'}.$$

Here we can compute in $BP^*(BN)$ by using $[2](u) = 0$

$$8u = 4v_1u^2 = 2v_1^2u^3 = v_1^3u^4, \quad 28u^2 = 14v_1u^3 = 7v_1^2u^4, \ldots.$$

Thus we see that v_1u^4 is not represented by the restriction of Chern classes. (However $v_1^2u^4$ has its possibility, in fact $|v_1w_8| = 4$ and it is represented by the Chern class c_2.)

Of course $c(\chi \oplus \eta) = c(\chi)c(\eta)$, we get the lemma. □

THEOREM 5.6. *Let $G = Spin_7$. Then*

$$gr^*_{top}(BG) \cong \mathbf{Z}[c_4, c_6, w_8]\{1, c_2'\},$$
$$gr^*_{\alpha}(BG) \cong \mathbf{Z}[c_4, c_6, c_8](\mathbf{Z}\{1, c_2', c_4', c_6'\} \oplus \mathbf{Z}/2\{\xi\})$$

where $deg(\xi) = 6$ (resp. $= 4$) if $\alpha = geo$ (if $\alpha = \gamma$).

Remark. $\tilde{K}(1)^*(BG)$ is generated as a $\tilde{K}(1)^*[c_4, c_6, c_8]$-module by

$$\{1,\ 2w_4,\ 2w_8,\ 2w_4w_8,\ v_1w_8\}.$$

Since $v_1^{-1} \in \tilde{K}(1)^*$, we have $w_8 \in \tilde{K}(1)^*(BG)$. Hence $\tilde{K}(1)^*(BG)$ is generatd as a $K(1)^*[c_4, c_6, c_8]$-algebra by $\{1, 2w_4, w_8\}$.

Remark. The graded ring $gr^*_{top}(BG)$ is also written as $gr^*_{\alpha}(BG)$ in the right hand side ring of the second isomorphism in the above theorem, with identifying $\xi = w_8,\ c_4' = 2w_8,\ c_6' = c_2'w_8$.

Recall that 2_+^{1+2n} is the extraspecial 2-group, which is isomorphic to the central product of n-copies of the dihedral group D_8 of order 8. Let $G = 2_+^{1+6}$. There is an inclusion $i : G \subset Spin_7$ and its induced map $i^* : H^*(B\,Spin_7; \mathbf{Z}/2) \to H^*(BG; \mathbf{Z}/2)$ is also injective by Quillen [Qu]. Let $j : \mathbf{Z}/2 \cong Z(G) \subset G$. Then it is know [Qu], [Sc-Ya] $j^*i^*(w_8) = u^4 \in \mathbf{Z}[u]/(2u) \subset H^*(BZ(G))$. Hence we have in $\tilde{K}(1)^*$-theory

$$j^*i^*(v_1w_8) = v_1u^4 \neq 0 \in \tilde{K}(1)^*(BZ(G)) \cong \tilde{K}(1)^*[u]/(2u - v_1u^2).$$

This element $v_1 \otimes w_8$ is not generated by Chern classes also in $H^*(BG)$. Hence we have

COROLLARY 5.7. *Let $G = 2_+^{1+6}$. Then there is an element $x \in A\tilde{K}(1)^*(BG)$ such that*

$$0 \neq x \in gr^4_{\gamma}(BG), \quad x = \xi \in gr^6_{geo}(BG), \quad \textit{and} \quad x = w_8 \in gr^8_{top}(BG).$$

6. Connected groups for p odd

In this section, we assume $p \geq 3$. At first we consider the case $G = PGL_p$. Its mod p cohomolgy is given by Vistoli and Kameko-Yagita ([Vi], [Ka-Ya]), namely, there is a short exact sequence

$$0 \to M/p \to H^*(BG; \mathbf{Z}/p) \to N \to 0$$

where $M \cong \mathbf{Z}[x_4, x_6, \dots, x_{2p}]$ additively (but not as rings), and $N \cong N' \otimes \Lambda(Q_0, Q_1)\{u_2\}$, $|u_2| = 2$ for some $\mathbf{Z}/p$-module N'. ($H^{even}(BG)_{(p)}$ is not generated by Chern classes (in facts $Q_0Q_1(u_2)$ is not represented by a Chern class).

The BP-theory $BP^*(BG)$ is also studied. There is a short exact sequence

$$0 \to BP^* \otimes M \to gr\, BP^*(BG) \to N'' \to 0$$

where $gr\, N'' \cong P(3)^* \otimes N'\{Q_0Q_1(u)\}$. In particular, $Q_0Q_1(u_2)$ is v_1-torsion, and hence its becomes zero in $\tilde{K}(1)^*(BG)$. Therefore we see additively $gr^*\, \tilde{K}(1)^0(BG) \cong M$. Totaro's conjecture also holds this case. Thus we have

Theorem 6.1. *Let $G = PGL_p$. Then*

$$gr^*_{top}(BG) \cong gr^*_{geo}(BG) \quad (\cong M \text{ additively}).$$

When $p = 3$, the ring structure of M is known

$$(*) \quad M/3 \cong \mathbf{Z}/3[c_2, c_3', c_6]/(c_2^3 = (c_3')^2)$$

where c_2, c_3', c_6 are Chern classes for some representations. Hence

$$M_{(3)} \cong gr^*_{\gamma}(BPL_3)_{(3)} \cong gr^*_{geo}(BPL_3)_{(3)}.$$

The fact $(*)$ is explicitly written

$$c_2 = c_2(sl_3), \quad c_3' = c_3(Sym^3(E)), \quad c_6 = c_6(sl_3)$$

in the notation in Theorem 1.1 and Proposition 1.2 in [Ve] by Vezzosi and Theorem 3.7 (a) in Vistoli [Vit]. Vistoli gives corrected generators and relations (for example, $\chi = 0$ for χ in [Ve]).

However, for $p \geq 5$, it seems unknown that M above is generated by Chern classes or not.

For exceptional Lie groups, we can compute $BP^*(BG)$ except for $(G, p) = (E_8, p = 3)$. So we know $gr^*_{top}(BG)$, but it seems not so easy to compute $CH^*(BG)$ now, and $gr^*_{geo}(BG)$ seems unknown. For example, when $G = F_4$ we can compute $BP^*(BG)$. The $mod(3)$ cohomology is generated by $x_4, x_8, x_9, x_{20}, x_{21}, \dots$ (by Toda). The BP-theory is computed

$$gr\, BP^*(BG) \cong BP^*[c_{18}, c_{24}]\{1, 3x_4\} \oplus BP^* \otimes E \oplus P(3)^*[x_{26}]\{x_{26}\}$$

where $E = \mathbf{Z}[x_4, x_8]\{ab \mid a, b \in \{x_4, x_8, x_{20}\}\}$. Hence we have

$$gr\, \tilde{K}(1)^*(BG) \cong \tilde{K}(1)^* \otimes (\mathbf{Z}[c_{18}, c_{24}]\{1, 3x_4\} \oplus E).$$

It is now unknown whether the element $x_8^2 \in E$ (or $x_8x_4^2 \in E$) is in the image of the cycle map (see (2.4) and the proof of Lemma 3.1 in [Ya2]). If it is so, then $gr^*_{geo}(BG) \cong gr^*_{top}(BG)$, otherwise $gr^i_{geo}(BG) \not\cong gr^i_{top}(BG)$ for $i = 12, 16$.

7. Rost motives

In this section, we do not assume that k is algebraically closed. At first, we recall the (generalized) Rost motive ([Ro1,2]). Let $M(X)$ be the motive of (smooth) variety X. For a non zero symbol $a = \{a_0, \dots, a_n\}$ in the mod 2 Milnor K-theory $K_{n+1}^M(k)/2$, let $\phi_a = \langle\langle a_0, \dots, a_n \rangle\rangle$ be the $(n+1)$-fold Pfister form. Let X_{ϕ_a} be the projective quadric of dimension $2^{n+1}-2$ defined by ϕ_a. The Rost motive $M_a(= M_{\phi_a})$ is a direct summand of the motive $M(X_{\phi_a})$ representing X_{ϕ_a} so that $M(X_{\phi_a}) \cong M_a \otimes M(\mathbf{P}^{2^n-1})$.

Moreover for an odd prime p and nonzero symbol $0 \neq a \in K_{n+1}^M/p$, we can define ([Ro2], [Vo4,5], [Su-Jo]) the generalized Rost motive M_a, which is irreducible and is split over K/k if and only if $a|_K = 0$ (as the case $p = 2$).

The Chow group of the Rost motive is well known. Let $\bar{k}$ be an algebraic closure of k, $X|_{\bar{k}} = X \otimes_k \bar{k}$, and $i_{\bar{k}} : CH^*(X) \to CH^*(X|_{\bar{k}})$ the restriction map.

Lemma 7.1 (Rost [Ro1,2], [Vo4], [Vi-Ya], [Ya5,6]). *The Chow group $CH^*(M_a)$ is only dependent on n. There are isomorphisms*

$$CH^*(M_a) \cong \mathbf{Z}\{1\} \oplus (\mathbf{Z}\{c_0\} \oplus \mathbf{Z}/p\{c_1, \dots, c_{n-1}\})[y]/(c_i y^{p-1})$$
$$\text{and} \quad CH^*(M_a|_{\bar{k}}) \cong \mathbf{Z}[y]/(y^p)$$

where $2\,deg(y) = |y| = 2(p^{n-1} + \cdots + p + 1)$ and $|c_i| = |y| + 2 - 2p^i$. Moreover the restriction map is given by $i_{\bar{k}}(c_0) = py$ and $i_{\bar{k}}(c_i) = 0$ for $i > 0$.

Remark. The element y does not exist in $CH^*(M_a)$ while $c_i y$ exists. Usually $CH^*(M_a)$ is defined only additively, however when $CH^*(M_a)$ has the natural ring structure (e.g., $p = 2$), the multiplications are given by $c_i \cdot c_j = 0$ for all $0 \leq i, j \leq n-1$.

For the simplicity of notation, hereafter we always *write by* $\Omega^*(X)$ the BP^*-version of the algebraic cobordism

$$\Omega^*(X) \otimes_{MU^*} BP^* \cong ABP^{2*,*}(X).$$

Hence we *mean* $\Omega^* = BP^*$ hereafter.

Let I_n be the ideal in Ω^* generated by $v_0, \dots, v_{n-1}$, i.e.,

$$I_n = (p = v_0, v_1, \dots, v_{n-1}) \subset \Omega^*.$$

Then it is well known that I_n and I_∞ are the only prime ideals stable under the Landweber-Novikov cohomology operations ([Ra]) in Ω^*.

The category of cobordism motives is defined and studied in [Vi-Ya]. In particular, we can define the algebraic cobordism of motives. The following is the main result in [Vi-Ya] (in [Ya5] for odd primes).

Lemma 7.2 ([Vi-Ya], [Ya5]). *The restriction map*

$$i_{\bar{k}} : \Omega^*(M_a) \to \Omega^*(M_a|_{\bar{k}}) \cong \Omega^*[y]/(y^p)$$

is injective and there is an Ω^*-module isomorphism

$$\Omega^*(M_a) \cong \Omega^*\{1\} \oplus I_n\{y, \dots, y^{p-1}\} \subset \Omega^*[y]/(y^p)$$

such that $v_i y = c_i$ in $\Omega^*(M_a) \otimes_{\Omega^*} \mathbf{Z} \cong CH^*(M_a)$.

We consider the following assumption for X.

Assumption (*). There is an isomorphism of motives

$$M(X) \cong M_n \otimes A(X) \quad \text{with } A(X) \cong \bigoplus_s \mathbf{T}^{i_s}$$

where $\mathbf{T}$ is the k-Tate module.

Lemma 7.3. *Suppose Assumption (*). Then*

$$K^0_{alg}(X) \cong K^0_{alg}(X|_{\bar{k}}) \cong K^0_{top}(X(\mathbf{C})).$$

Proof. Since $M(X|_{\bar{k}})$ is a sum of $\bar{k}$-Tate modules, we have the isomorphism $K^0_{alg}(X|_{\bar{k}}) \cong K^0_{top}(X(\mathbf{C}))$ from

$$K^0_{alg}(\mathbf{T}) \cong K^0_{alg}(S^{2,1}|_{\bar{k}}) \cong K^0_{top}(S^2).$$

For the first isomorphism, we only need to show $K^0_{alg}(M_n) \cong K^0_{alg}(M_n|_{\bar{k}})$. Recall

$$\Omega^*(M_n) \cong BP^* \oplus Ideal(p, v_1, \dots, v_{n-1})[y]/(y^p)$$

by $c_i \mapsto v_i y$. Hence $v_i c_1 = v_1 c_i$. Therefore for $i > 1$, we see $c_i = 0$ in $A\tilde{K}(1)^{2*,*}(M_n)$ where $v_i = 0$. So we have

$$\begin{aligned} A\tilde{K}(1)^{2*,*}(M_n) &\cong \tilde{K}(1)^*\{1\} \oplus \tilde{K}(1)^*\{c_0, c_1\}[y]/(v_1 c_0 = pc_1, y^{p-1}) \\ &\cong \tilde{K}(1)^*\{1\} \oplus \tilde{K}(1)^*\{c_1\}[y]/(y^{p-1}) \\ &\cong \tilde{K}(1)^*\{1\} \oplus \tilde{K}(1)^*\{v_1 y\}[y]/(y^{p-1}) \\ &\cong \tilde{K}(1)^*[y]/(y^p) \cong A\tilde{K}(1)^{2*,*}(M_n|_{\bar{k}}). \end{aligned}$$

□

8. Flag manifolds G/T

Now we consider the flag variety G/T. Let G be a simply connected Lie group and T the maximal torus. Moreover we assume that its cohomology is

$$H^*(G; \mathbf{Z}/p) \cong \mathbf{Z}/p[y]/(y^p) \otimes \Lambda(x_1, \dots, x_\ell)$$

with $|y| = 2(p+1)$ and $|x_i| = odd$. Then it is well known that the cohomology of G/T is torsion free ([Tod]) and

$$H^*(G/T) \cong \mathbf{Z}[y, t_1, \dots, t_\ell]/(f_y, b_1, \dots, b_\ell)$$

where $f_y = y^p \ mod \ Ideal(t_i)$ and $(b_1, \dots, b_\ell)$ is a regular sequence in $\mathbf{Z}[t_1, \dots, t_\ell]$.

Let k be a subfield of $\mathbf{C}$ which contains primitive p-th root of the unity. Let us denote by G_k the split reductive group over k which corresponds G. By definition, a G_k-torsor $\mathbf{G}_k$ over k is a variety over k with a free G_k-action such that the quotient variety is $Spec(k)$. A G_k-torsor over k is called trivial, if it is isomorphic to G_k or equivalently it has a k-rational point. In this paper by $\mathbf{G}_k$, we mean a nontrivial torsor at any finite extension K/k coprime to p.

Let H be a subgroup of G. Given a torsor $\mathbf{G}_k$ over k, we can form the twisted form of G/H by

$$(\mathbf{G}_k \times G_k/H_k)/G_k \cong \mathbf{G}_k/H_k.$$

Letting $X = G/T$, we consider cases such that Assumption $(*)$ in §7 hold. By [Pe-Se-Za], exceptional Lie groups $(G_2, p = 2)$ and $(F_4, p = 3)$ are such cases. The filtrations of K-theory of such spaces are also studied by Garibardi and Zainouline ([Ga-Za], [Za], [Ju]) as the twisted gamma filtrations.

At first, we consider the case $(G, p) = (G_2, 2)$. We recall the cohomology from Toda-Watanabe [To-Wa],

$$H^*(G/T; \mathbf{Z}) \cong \mathbf{Z}[t_1, t_2, y]/(t_1^2 + t_1t_2 + t_2^2, t_2^3 - 2y, y^2)$$

with $|t_i| = 2$ and $|y| = 6$. Let P be the maximal parabolic subgroup such that G/P is isomorphic to a quadric. Then we have $H^*(P/T) \cong \mathbf{Z}\{1, t_1\}$ (see [To-Wa], [Ya6])

$$H^*(G/P; \mathbf{Z}) \cong \mathbf{Z}[t_2, y]/(t_2^3 - 2y, y^2) \cong \mathbf{Z}\{1, y\} \otimes \{1, t_2, t_2^2\}$$

Of course this is isomorphic to $gr_{top}^*(G/P)$.

Since G/P is a quadric, we have the decomposition ([Bo], §7 in [Pe-Se-Za])

$$M(\mathbf{G}_k/P_k) \cong M_2 \oplus M_2(1) \oplus M_2(2).$$

Theorem 8.1 (Theorem 5.2 in [Ya6]). *There is a ring isomorphism*

$$\begin{aligned} gr_\gamma^*(G/P) &\cong gr_{geo}^*(\mathbf{G}_k/P_k) \cong CH^*(\mathbf{G}_k/P_k) \\ &\cong \mathbf{Z}_{(2)}[t_2, u]/(t_2^6, 2u, t_2^3u, u^2) \cong \mathbf{Z}_{(2)}[t_2]/(t_2^6) \oplus \mathbf{Z}/2[t_2]/(t_2^3)\{u\} \end{aligned}$$

with $|t_2| = 2$, $|u| = 4$.

Proof. Recall that from Lemma 7.2,

$$\Omega^*(M_2) \cong \Omega^*\{1, 2y, vy\} \subset \Omega^*\{1, y\}.$$

From the decomposition of the motive, we have the Ω^*-module isomorphism

$$\Omega^*(\mathbf{G}_k/P_k) \cong \Omega^*\{1, v_1y, 2y\} \otimes \{1, t_2, t_2^2\} \subset \Omega^*(G_k/P_k).$$

Since $CH^*(X) \cong \Omega^*(X) \otimes_{\Omega^*} \mathbf{Z}$, we have the isomorphism

$$CH^*(\mathbf{G}_k/P_k) \cong \mathbf{Z}\{1, 2y\}\{1, t_2, t_2^2\} \oplus \mathbf{Z}/2\{v_1y\}\{1, t_2, t_2^2\}.$$

(Note $2v_1y = v_1(2y) \in \Omega^{<0}\Omega^*(\mathbf{G}_k/P_k)$.)

Here the multiplications are given as follows. Since $2y = t_2^3 \; mod(\Omega^{<0})$ in $\Omega^*(G_k/T_k)$, we can take $2y = t_2^3 \in CH^*(\mathbf{G}/P_k)$ so that

$$\mathbf{Z}\{1, 2y\}\{1, t_2, t_2^2\} = \mathbf{Z}[t_2]/(t_2^6) \subset CH^*(\mathbf{G}/P_k).$$

Let us write $u = v_1 y$ in $CH^*(\mathbf{G}_k/T_k)$. Then $t_2^3 u = 2yv_1 y = 0$ and $u^2 = v_1^2 y^2 = 0$ in $\Omega^*(\mathbf{G}_k/T_k) \otimes_{\Omega^*} \mathbf{Z}$. Hence we have the second isomorphism in the theorem.

Since $|u| = 4$, the element u is represented by Chern classes, we see the first isomorphism. $\square$

Remark. The space $\mathbf{G}_k/T_k$ is isomorphic to the quadric defined by the maximal neighbor of the 3-Pfister form. Hence its Chow ring is computed in [Ya6].

It is well known that the representations (over $\mathbf{C}$)) are written as

$$R(G/T) \cong R(T)/R(G).$$

Therefore each element which is represented by Chern classes is written as an element in $\Omega^*(\mathbf{G}_k/T_k)$

$$c(\xi) = \prod(1 + \lambda_1 t_1 + \lambda_2 t_2) \in \Omega^*[t_1, t_2] \quad \lambda_i \in \mathbf{Z}/2$$

modulo $((t_1, t_2)\Omega^{<0}\Omega^*(G_k/T_k))$. By the similar arguments, we have (see Theorem 5.3 in [Ya6])

Theorem 8.2. *There are ring isomorphisms*

$$gr_\gamma^*(G/T) \cong CH^*(\mathbf{G}_k/T_k) \cong \mathbf{Z}[t_1, t_2]/(t_2^6, 2u, t_2^3 u, u^2)$$

where $u = t_1^2 + t_1 t_2 + t_2^2$.

Proof. The Chow ring is isomorphic to

$$(*) \quad CH^*(\mathbf{G}_k/T_k) \cong CH^*(\mathbf{G}_k/P_k)\{1, t_1\}$$
$$\cong (\mathbf{Z}\{1, 2y\} \oplus \mathbf{Z}/2\{v_1 y\})\{1, t_2, t_2^2\}\{1, t_1\}.$$

Here $2y = t_2^3$. Since $v_1 y \in (t_1, t_2)$ and $v_1 y = 0 \in CH^*(G_k/T_k)$, we see

$$v_1 y = \lambda(t_1^2 + t_1 t_2 + t_2^2) \quad mod((t_1, t_2)\Omega^{<0}\Omega^*(G_k/T_k))$$

for $\lambda \in \mathbf{Z}$. We can take $\lambda = 1 \; mod(2)$. Otherwise $v_1 y = 0 \in \Omega^*(G_k/T_k)/2$, which is an $\Omega^*/2$-free, and this is a contradiction. Hence we can take $t_1^2 + t_1 t_2 + t_2^2$ as $v_1 y$. Hence in $CH^*(\mathbf{G}_k/T_k)$ we have the relation

$$(t_2^3)^2 = 0, \quad (t_2^3)u = 0, \quad u^2 = 0, \quad 2u = 0. \qquad \square$$

Next we consider the case $(G, p) = (F_4, 3)$. Let $\mathbf{G}_k$ be a nontrivial G_k-torsor at 3 as previous sections. Let P_k be a maximal parabolic subgroup of G_k given by the first three vertexes

$$\overset{1}{\circ} - - \overset{2}{\circ} \Rightarrow = \overset{3}{\circ} - - \overset{4}{\circ}$$

of the Dynkin diagram. Then Nikolenko-Semenov-Zainoulline ([Ni-Se-Za]) showed that there is an isomorphism

$$M(\mathbf{G}_k/P_k) \cong \bigoplus_{i=0}^{7} M_2(i).$$

We first recall the ordinary cohomology of G/P ([Is-To], [Du-Za]).

$$H^*(G/P)_{(3)} \cong \mathbf{Z}[t,y]/(r_8, r_{12}), \quad |t| = 2, \quad |y| = 8$$

where $r_8 = 3y^2 - t^8$ and $r_{12} = 26y^3 - 5t^{12}$. Hence we can rewrite

$$H^*(G/P) \cong \mathbf{Z}\{1, t, \dots, t^7\} \otimes \{1, y, y^2\}.$$

Recall the Rost motive $CH^*(M_2|_{\bar{k}}) \cong \mathbf{Z}[y]/(y^3)$,

$$CH^*(M_2) \cong \mathbf{Z}\{1\} \oplus \mathbf{Z}\{3y, 3y^2\} \oplus \mathbf{Z}/3\{v_1 y, v_1 y^2\}.$$

Of course, the above $y \in CH^*(M_a)$ can be identified with the same named element in $H^*(G_k/P_k)$ by the restriction map $CH^*(M_a) \to CH^*(M_a|_{\bar{k}}) \subset CH^*(G_k/P_k)$. From the above isomorphism, we have the decomposition

$$(*) \quad CH^*(\mathbf{G}_k/P_k) \cong \mathbf{Z}\{1, t, \dots, t^7\} \otimes (\mathbf{Z}\{1, 3y, 3y^2\} \oplus \mathbf{Z}/3\{v_1 y, v_1 y^2\}).$$

The ring structure is given as follows.

Proposition 8.3 (Theorem 6.2 in [Ya6]).

$$\begin{aligned} gr^*_{geo}(\mathbf{G}_k/P_k) &\cong CH^*(\mathbf{G}_k/P_k) \\ &\cong \mathbf{Z}[t, b, a_1, a_2]/(t^{16}, t^8 b, b^2 = 3t^8, ba_i, 3a_i, t^8 a_i, a_1 a_2) \\ &\cong \mathbf{Z}\{1, t, \dots, t^7\} \otimes (\mathbf{Z}\{1, \sqrt{3}t^4, t^8\} \oplus \mathbf{Z}/3\{a_1, a_2\}) \end{aligned}$$

where $|b| = 8$ *and* $|a_1| = 4$, $|a_2| = 12$.

Proof. From the relation r_8 in $CH^*(G/P)$, we have

$$3y^2 = t^8 + vx \in \Omega^*(G/P) \quad \textit{for } v \in \Omega^{<0}.$$

Hence we can take t^8 instead of $3y^2$ in $(*)$. Of course

$$(3y)^2 = 3t^8 + 3vx \in \Omega^*(G_k/P_k).$$

Hence we write by $b = \sqrt{3}t^4$ the element $3y$. Write by a_1, a_2 the elements $v_1 y$, $v_1 y^2$ respectively. Elements in $I_\infty \Omega^{<0} \subset \Omega(G_k/P_k)$ reduces to zero in $CH^*(\mathbf{G}_k/T_k)$. Therefore we have the desired multiplicative results. □

The element $b = 3y$ is represented by a Chern class $c_4(\xi)$ for some ξ by the Riemann-Roch theorem without denominators. Unfortunately, we do not know if $a_2 = v_1 y^2$ are Chern classes in $CH^*(\mathbf{G}_k/P_k)$ or not.

Proposition 8.4. *If* $a_2 = v_1 y^2 \in CH^*(\mathbf{G}_k/P_k)$ *is represented by a Chern class, then* $gr_\gamma(G/P) \cong CH^*(\mathbf{G}_k/P_k)$. *Otherwise*

$$gr_\gamma(G/P) \cong \mathbf{Z}[t, b, a_1]/(t^{16}, t^8 b, b^2 = 3t^8, ba_1, 3a_1, t^8 a_1, a_1^3)$$

where $|b| = 8$ *and* $|a_1| = 4$.

Proof. If $v_1 y^2$ is not represented by Chern class of $CH^*(\mathbf{G}_k/Pk)$ (or $\Omega^*(\mathbf{G}_k/P_k)$), then the corresponding nonzero element in $gr_\gamma(G/T)$ is $v_1^2 y^2$, which is written as $(v_1 y)^2 = (a_1)^2$. □

9. Filtrations of the Morava K-theory

For most groups G in the preceding sections, it is known that $K(n)^{odd}(BG) = 0$ (while Kriz gave some examples with $K(n)^{odd}(BG) \neq 0$). Hereafter, we only consider spaces X such that

$$K(n)^{odd}(X(\mathbf{C})) = \tilde{K}(n)^{odd}(X(\mathbf{C})) = 0, \tag{9.1}$$

$$K(n)^*(X(\mathbf{C})) \cong AK(n)^{2*,*}(X). \tag{9.2}$$

Then we can define the three filtrations for the Morava $K(n)$-theory

$$F(n)_{top}^{2i} = Ker(K(n)^*(X(\mathbf{C}) \to K(n)^*(X(\mathbf{C})^{2i}),$$
$$F(n)_{geo}^{2i} = \{f_*(1_M) \mid f : M \to X \text{ and } codim_X M \geq i\}$$
$$F(n)_\gamma^{2i} = \{c_{i_1}^{K(n)}(x_1) \cdot \ldots \cdot c_{i_m}^{K(n)}(x_m) \mid i_1 + \cdots i_m \geq i\},$$

and let us write the associated graded algebras

$$gr(n)_\gamma^*(X), \quad gr(n)_{geo}^*(X), \quad gr(n)_{top}^*(X(\mathbf{C})).$$

Here $c_{i_s}^{K(n)}(x_s)$ is the Chern class for $AK(n)^{*,*'}$-theory for some k-representation $x_s : X \to BGL_N$. This Chern class is induced from the isomorphism

$$AK(n)^{2*,*}(BGL_N) \cong K(n)^* \otimes_{BP^*} \Omega^*(BGL_N),$$

in fact, it is well known that in $\Omega^*(X)$, we can define Chern classes canonically (see [Mo-Le] for example). However each element in $K(n)^*(X(\mathbf{C}))$ (for $n \geq 2$) need not to be represented by $K(n)^*$-theory Chern classes. Hence we need the assumption

$$F_\gamma^0 = K(n)^*(X). \tag{9.3}$$

(However, we also consider the cases where (9.3) is not assumed.) Of course the assumptions are satisfied for $K(1)^*$-theory, if they are so for $\tilde{K}(1)^*$-theory.

Recall $P(n)^*(X)$ be the cohomology theory with the coefficient

$$P(n)^* = BP^*/(p, v_1, \ldots, v_{n-1}).$$

It is well known, for all X,

$$P(n)^*(X) \otimes_{BP^*} K(n)^* \cong K(n)^*(X).$$

Let us write by $E(P(n))_r^{*,*'}$ (resp. $E(K(n))_r^{*,*'}$) the AHss converging to $P(n)^*(X)$ (resp. $K(n)^*(X)$). Then we have

$$E(P(n))_r^{*,*'} \otimes_{BP^*} K(n)^* \cong E(K(n))_r^{*,*'}.$$

If (9.1)–(9.3) are satisfied, then $K(n)$-version (exchanging $BP^*(X)$ to $P(n)^*(X)$) of all lemmas in §2 also hold.

Lemma 9.1. *Suppose* (9.1) *for all* $n \geq 1$, *and that* $\Omega^*(X)/p \cong BP^*(X(\mathbf{C}))/p$ *and it is generated by* $(BP^*$-$)$*Chern classes. Then* (9.2) *and* (9.3) *are satisfied and* $gr(n)_\gamma^*(X) \cong gr(n)_{geo}^*(X)$.

Proof. We consider the maps

$$\Omega^*(X) \otimes_{BP^*} K(n)^* \xrightarrow{\rho_1} AK(n)^{2*,*}(X) \xrightarrow{\rho_2} K(n)^*(X(\mathbf{C})).$$

Here the map ρ_1 is an epimorphism because $\Omega^*(X)$ (resp. $AK(n)^{2*,*}(X)$) is generated as a BP^*-module (resp. $K(n)^*$-module) by elements in $CH^*(X)$.

On the other hand by Ravenel-Wilson-Yagita [Ra-Wi-Ya], we know that (9.1) implies

$$K(n)^*(X(\mathbf{C})) \cong K(n)^* \otimes_{BP^*} BP^*(X(\mathbf{C})).$$

From the supposion in the theorem, we see that $\rho_2\rho_1$ is an isomorphism. This means that ρ_1, ρ_2 are also isomorphisms. $\square$

The assumptions in the above lemma are satisfied for $X = BG$, $G =$ *finite abelian*, $p_\pm^{1+2}$, O_n, G_2 and PGL_3 $(p = 3)$.

Of course $gr_{top}^*(X)$ and $gr(n)_{top}^*(X)$ are quite different. Let $G = \mathbf{Z}/p$. Then

$$K(n)^*(BG) \cong K(n)^*(y]/(y^{p^n}).$$

and this is generated by Chern classes in $H^*(BG; \mathbf{Z}/p)$.

Theorem 9.2. *Let* $G = \bigoplus^s \mathbf{Z}/p$. *Then all three filtrations of* $K(n)^*(BG)$ *are same and*

$$gr(n)_{top}^*(BG) \cong \mathbf{Z}/p[y_1, \dots, y_s]/(y_1^{p^n}, \dots, y_s^{p^n}).$$

Similarly, we have

Theorem 9.3. *Let* $G = O_m$ *and* $p = 2$. *Then all three filtrations of* $K(n)^*(BG)$ *are same and*

$$gr(n)_{top}^*(BG) \cong \left\{\sum y_1^{i_1} \cdots y_s^{i_s} (y_{s+1} y_{s+2})^{j_{s+1}} \cdots (y_{2k+1} y_{2k+2})^{j_{2k+1}}\right\}$$

where $0 \leq i_1 \leq \cdots \leq i_s < 2^n \leq i_s \leq \cdots \leq i_k$.

For example, $gr(n)_{top}^* \cong \mathbf{Z}/2[c_2] \oplus \mathbf{Z}/2\{c_1^i c_2^j \mid i + 2j < 2^n\}$.

Next we consider the case $G = SO_{2m}$ Recall for $m \geq 3$, y_{2m} is not represented by Chern classes

Theorem 9.4. *Let* $G = SO_{2m}$, $p = 2$ *and* $m > 2$. *Then*

$$gr(n)^*_{geo}(BG) \cong \mathbf{Z}[c_2, c_4, \dots, c_{2m}]\{y_{2m}\} \oplus gr(n)^*_{geo}(BO_{2m})/(c_1).$$

However $gr(n)^*_{\gamma}(BG) \not\cong gr(n)^*_{geo}(BG) \not\cong gr(n)^*_{top}(BG)$.

Proof. We only need the second non-isomorphism of the second statement. Since $y_{2m} = (-1)^* 2^{m-1} w_{2m} \in H^*(BG)$ is zero in $H^*(BG; \mathbf{Z}/2)$. Hence $0 \neq y_{2m} \in P(n)^*(BG)$ is represented in the AHss converging to $P(n)^*(BG)$ as element in $E_\infty^{*,*'}$ with $*' < 0$ and $* > 2m$. □

Next consider the case $G = G_2$ (and $p = 2$). By the computation of the AHss for $P(1)^*(BG)$ $(= BP^*(BG; \mathbf{Z}/2))$, we have

$$K(1)^*(BG) \cong K(1)^*[c_4, c_6]\{1, v_1 w_6\}.$$

By the direct computation of the AHss for $K(2)^*(BG)$, we see

$$K(2)^*(BG) \cong K(2)^*[c_4, c_6]\{1, w_4 w_6\}.$$

Thus we have

Theorem 9.5. *Let* $G = G_2$ *and* $p = 2$. *Then*

$$gr(i)^*_\alpha(BG) \cong \mathbf{Z}/2[c_4, c_6]\{1, a\}$$

where $a^2 = \begin{cases} c_4 c_6 \ \ |a| = 10 & \textit{if } \ i = 2. \ \alpha = top \\ c_6 \ \ |a| = 6 & \textit{if } \ i = 1. \ \alpha = top \\ 0 \ \ |a| = 4 & \textit{if } \ i = 1, 2. \ \alpha \neq top. \end{cases}$.

Proof. The above a is represented as $a = w_4 w_6$ (resp. w_6, $v_1 w_6$, $v_2 w_4 w_6$) when $i = 2$, $\alpha = top$ (resp. $i = 1$, $\alpha = top$, $i = 1$ $\alpha \neq top$), and $i = 2$ $\alpha \neq top$)). □

When $n \geq 1$, the geometric and topological filtrations are quite different.

Theorem 9.6. *Let* G *be a simply connected simple Lie group such that* $H^*(G)$ *has* p*-torsion. Then for* $n \geq 1$

$$gr(n)^4_{geo}(BG) \neq 0 \quad \textit{but} \quad gr(n)^4_{top}(BG) = 0.$$

Proof. The space BG is 3-connected and $H^4(BG) \cong \mathbf{Z}$ (so $H^4(BG; \mathbf{Z}/p) \cong \mathbf{Z}/p$). Let us write by x its 4-dimensional generator. To see $gr(n)^4_{top}(BG) = 0$, we only need to show

$$(*) \quad d_{2p^n - 1}(x) = v_n \otimes Q_n(x) \neq 0$$

in the AHss converging to $K(n)^*(BG)$.

For these groups, it is well known that there are embedding $G_2 \subset G$ for $p = 2$, ($F_4 \subset G$ for $p = 3$ and $G = E_8$ for $p = 5$). We will prove $(*)$ for $G = F_4$ and $p = 3$, then we can see $(*)$ for the other groups when $p = 3$. (The other primes cases are similar).

Let $G = F_4$ and $p = 3$. Then G has a maximal elementary p-group $A \cong (\mathbf{Z}/3)^3$. We consider the restriction map for $i : A \subset G$,

$$i^* : H^*(BG; \mathbf{Z}/p) \to H^*(BA; \mathbf{Z}/p) \cong \mathbf{Z}/p[y_1, y_2, y_3] \otimes \Lambda(x_1, x_2, x_3).$$

The restriction image is $i^*(x) = Q_0(x_1x_2x_3)$ (see [Ka-Te-Ya]). Hence we show

$$i^*(Q_n(x)) = Q_nQ_0(x_1x_2x_3) = \sum y_1^{p^n} y_2x_3 \neq 0.$$

By [Ka-Ya2], it is known that $px \in H^4(BG)$ is represented as the Chern class c_2 for some representation. Hence $gr(n)^4_{geo}(BG) \neq 0$. Thus we have the theorem. $\square$

Now we recall arguments for quadrics. Let $m = 2m' + 1$, and let us write the quadratic form $q(x)$ defined by

$$q(x_1, \dots, x_m) = x_1x_2 + \cdots + x_{m-2}x_{m-1} + x_m^2$$

and the projective quadric X_q defined by the quadratic form q. Then it is well known that (in fact $SO(m)$ acts on the affine quadric in $\mathbf{A}^m - 0$)

$$X_q \cong SO(m)/(SO(m-2) \times SO(2)).$$

Let $G = SO(m)$ and $P = SO(m-2) \times SO(2)$. Then the quadric q is always split over k and we know $CH^*(G_k/P_k) \cong CH^*(X_q)$.

In particular we consider the case $m = 2^{n+1} - 1$. Let $\rho = \{-1\} \in K_1^M(k)/2 = k^*/(k^*)^2$. We consider fields k such that

$$0 \neq \rho^{n+1} \in K_{n+1}^M(k)/2.$$

Define the quadratic form q' by $q'(x_1, \dots, x_m) = x_1^2 + \cdots + x_m^2$. Then this q' is a subform of $\langle\langle -1, \dots, -1 \rangle\rangle = \phi_{\rho^{n+1}}$ the $(n+1)$-th Pfister form associated to ρ^{n+1}. (That is, q' is the maximal neighbor of the $(n+1)$-th Pfister form.) Of course $q|_{\bar{k}} = q'|_{\bar{k}}$ and we can identify $\mathbf{G}_k/P_k \cong X_{q'}$. From Lemma 7.2 (or Rost's result), we know

$$CH^*(X_{q'}|_{\bar{k}}) \cong \mathbf{Z}[t, y]/(t^{2^n-1} - 2y, y^2).$$

As stated in §7, there is a decomposition of motives

$$M(X_{q'}) \cong M_n \otimes \mathbf{Z}/2[t]/(t^{2^n-1}).$$

Hence we have the additive isomorphism

$$CH^*(X_{\phi'_a}) \cong \mathbf{Z}[t]/(t^{2^n-1}) \otimes (\mathbf{Z}\{1, c_{n,0}\} \oplus \mathbf{Z}/2\{c_{n,1}, \dots, c_{n,n-1}\}).$$

With identification $t^{2^n-1} = 2y = c_{n,0}$, and $u_i = c_{n,i}$ for $i > 0$, we also get the ring isomorphism

Theorem 9.7 ([Ya6]). *Let $0 \neq \rho^{n+1} \in K_{n+1}^M(k)/2$ and let $\mathbf{G}_k/P_k$ be the above quadric $X_{q'}$. Then there is a ring isomorphism*

$$CH^*(\mathbf{G}_k/P_k) \cong \mathbf{Z}[t]/(t^{2^{n+1}-2}) \oplus \mathbf{Z}/2[t]/(t^{2^n-1})\{u_1, \dots, u_{n-1}\}$$

where $u_i = v_i y \in \Omega^*(\mathbf{G}_k/p) \otimes_{\Omega^*} Z_{(2)}$ so $u_i u_j = 0$. Hence for $1 \leq i \leq n-1$, we have

$$gr(i)_{geo}(\mathbf{G}_k/P_k) \cong \mathbf{Z}[t]/(t^{2^{n+1}-2}) \oplus \mathbf{Z}/2[t]/(t^{2^n-1})\{u_i\}.$$

Proof. In $K(i)^*(X)$, we see $v_j = 0$ for $i \neq j$. Since $v_j u_i = v_i u_j$, we see $u_j = 0$ for $i \neq j$. □

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