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THE CONFORMAL ROTATION NUMBER

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Abstract

The rotation number of a planar closed curve is the total curvature divided by 2π. This is a regular homotopy invariant of the curve. We shall generalize the rotation number to a curve on a closed surface using conformal geometry of ambient surface. This conformal rotational number is not integral in general. We shall show the fractional part is relevant to harmonic 1-forms of the surface.

1. Introduction

Let M be a connected oriented closed surface with a Riemannian metric g. The conformal Laplacian L_g is defined as $L_g u = -\Delta_g u + K_g$, where K_g is the Gauss curvature of g. If we denote by G_p a Green function of L_g with pole at $p \in M$, we have a flat surface $(M_p, g_p) = (M \backslash \{p\}, e^{2G_p} g)$. Then for a regular closed curve $\gamma : S^1 \to M_p$ we set $r(\gamma, p) = \dfrac{1}{2\pi} \int_\gamma \kappa \, ds$, which will be called *relative rotation number* or *conformal rotation number*, where κ is the curvature of γ with respect to g_p. This is a conformally invariant with repect to g, and a regular homotopy invariant of γ, but not in general integer valued. We think of $r(\gamma, p)$ as a function in $p \in M$. It turns out that $r(\gamma, p)$ is not continuous for $p \in \gamma(S^1)$ if $\chi(M) \neq 0$, but the differential extends smoothly on M, and we write α_γ for this 1-form. The main result of this paper is the following.

Theorem 1.1. *If $\chi(M) < 0$, then $\dfrac{1}{\chi(M)} \alpha_\gamma$ is the harmonic form whose de Rham cohomology class is the Poincaré dual of γ.*

We shall also explain the relation between our conformal rotation number and Reinhart's $\bmod \chi(M)$ invariant ([3]).

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2. Conformal Laplacian of a surface

Let (M, g) be a connected closed surface and K_g the Gauss curvature of g. The *conformal Laplacian* is defined as $L_g u = -\Delta_g u + K_g$.

Lemma 2.1. *If $\tilde{g} = e^{2u} g$, then we have*
(1) $L_g u = e^{2u} K_{\tilde{g}}$;
(2) $L_g v = e^{2u} L_{\tilde{g}}(v - u)$;
(3) $\Delta_g v = e^{2u} \Delta_{\tilde{g}} v$;
(4) $\int_M L_g u \, d\mu_g = 2\chi(M)$;
(5) $u - v = \mathrm{const}$ *if* $L_g u = L_g v$.

Proof. (4) is the Gauss-Bonnet theorem. The others are easily verified. □

Now we assume moreover that M is oriented. κ_g denotes the curvature of a regular curve γ.

Lemma 2.2. *Suppose $\tilde{g} = e^{2u} g$ and γ is a regular curve.*
(1) $\int_\gamma \kappa_{\tilde{g}} \, ds_{\tilde{g}} - \int_\gamma \kappa_g \, ds_g = -\int_\gamma (\partial_\nu u) \, ds_g$, *where ν is the unit normal vector of γ.*
(2) *If $\gamma = \partial U$ then* $\int_\gamma \kappa_{\tilde{g}} \, ds_{\tilde{g}} - \int_\gamma \kappa_g \, ds_g = \int_U (\Delta_g u) \, d\mu_g$.
(3) *If $\gamma = \partial U$ then* $\int_\gamma \kappa_{\tilde{g}} \, ds_{\tilde{g}} = 2\pi\chi(\overline{U}) - \int_U (L_g u) \, d\mu_g$.

Proof. (1) is a direct calculation. (2) follows from (1). (3) is Gauss-Bonnet. □

Definition 2.3. $G_p \in C^\infty(M_p)$ is called a *Green function* of L_g with pole at $p \in M$ if $L_g G_p = a\delta_p^g$ for some constant a, where δ_p^g is the Dirac δ-function at p with repect to the metric g.

Lemma 2.4.
(1) $a = 2\pi\chi(M)$.
(2) *G_p is unique up to an additive constant.*
(3) *$G_p - u$ is a Green function of $e^{2u} g$.*

Proof. From Lemma 2.1. □

We remark that G_p has no pole at p if $\chi(M) = 0$.

Corollary 2.5. *(M_p, g_p) is flat and its homothety class depends only on the conformal class of (M, g).*

Proposition 2.6. *There is a Green function G_p, and $G_p(x) + \chi(M) \log d(x, p)$ is continuous at p.*

Proof. We have a metric $\tilde{g} = e^{2\lambda} g$ which is flat near p. Let $u \in C^\infty(M_p)$ be a function such that $u(x) = -\chi(M) \log \tilde{d}(x, p)$ near p and we have $-\Delta_{\tilde{g}} u = 2\pi\chi(M)\delta_p^{\tilde{g}}$ near p. Put

$$v(x) = \begin{cases} 0 & x = p \\ \Delta_{\tilde{g}} u & \text{otherwise.} \end{cases}$$

Then $v \in C^\infty(M)$ and $\Delta_{\tilde{g}} u = v - 2\pi\chi(M)\delta_p^{\tilde{g}}$. It follows from the Gauss-Bonnet theorem that $\int_M (K_{\tilde{g}} - v)\, d\mu_{\tilde{g}} = 0$. Hence we have $w \in C^\infty(M)$ such that $\Delta_{\tilde{g}} w = K_{\tilde{g}} - v$, and $G_p = u + w - \lambda \in C^\infty(M \backslash \{p\})$ is the desired Green function. □

We will give a proof of the following classical theorem.

Proposition 2.7. *Any metric g is conformal to a metric of constant curvature.*

Proof. Case $\chi(M) = 0$: The Poisson equation $-\Delta_g u + K_g = 0$ is solvable. Case $\chi(M) < 0$: Let $u \in C^\infty(M)$ be a solution of $L_g u = 2\pi\chi(M)/\int_M d\mu_g$, and put $u_+ = u - \min_x u(x)$ and $u_- = u - \max_x u(x)$. Because $\chi(M) < 0$ the method of sub- and super-solutions (pp. 35–36 of [2]) is applicable, and we get $v \in C^\infty(M)$ such that $L_g v = e^{2v} \cdot 2\pi\chi(M)/\int_M d\mu_g$. Case $\chi(M) > 0$: Take $p \in M$ and consider (M_p, g_p). We have $g_p = \lambda d(p, x)^{-2\chi(M)} g$, where λ is a fucntion continuous at p. Hence (M_p, g_p) is a complete flat surface with one end because $\chi(M) > 0$. Therefore (M_p, g_p) is isometric to either $(\mathbf{R}^2, g_0)$ or $(S^1 \times \mathbf{R}/\pm 1, g_0)$. That is, (M, g) is conformal to (S^2, g_0) or $(\mathbf{R}\mathrm{P}^2, g_0)$. □

We set $G(x, y) = G_x(y)$, $x \neq y \in M$, and call it a *Green kernel* of L_g.

Proposition 2.8. *We can choose a Green kernel so that $G(x, y) = G(y, x)$.*

Proof. Suppose $\tilde{g} = e^{2u} g$ has constant Gauss curvature $\tilde{K}$. Take a Green function $\tilde{G}_p$ of $L_{\tilde{g}}$. Note that $\tilde{G}_p$ is integrable and set $G'(x, y) = \tilde{G}_x(y) - \int_M G_x(y)\, dy / \int_M dy$. It is not hard to see that $G'(x, y) = G'(y, x)$. $G(x, y) = G'(x, y) + u(x) + u(y)$ is the desired Green kernel. □

Remark 2.9. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of $-\Delta_{\tilde{g}}$, and ϕ_i be eigenfunction with eigenvalue λ_i such that $\int_M \phi_i \phi_j = \delta_{ij}$. Then $G'(x, y) = 2\pi\chi(M) \sum_{k>0} \dfrac{1}{\lambda_k} \phi_k(x)\phi_k(y)$. (cf. [1].)

In the case of nonpositive Euler characteristic (M_p, g_p) is no longer complete, that is, p may be regarded as a singular point rather than a point at infinity. The following describes a local picture around p.

Proposition 2.10. *Suppose $\chi(M) \leq 0$. Then there are neighborhoods U of p and V of* 0 *in $\mathbf{R}^2$, and a mapping $f : U \to V$ such that f is a ramified covering of degree $1 - \chi(M)$ branched at p with $f(p) = 0$ and $g_p = f^* g_0$, where g_0 is the*

Euclidean metric. In particular, there is a local coordinates x^1 *and* x^2 *around* p *such that* $g_p = |x|^{-2\chi(M)}\delta_{ij}\, dx^i dx^j$ *near* p.

Proof. Routine and omitted. □

3. Rotation number relative to a reference point

Let (M, g) be as before and $\gamma : S^1 \to M$ be a regular closed curve. For $p \in M \backslash \gamma(S^1)$ we set

$$r(\gamma, p) = \frac{1}{2\pi}\int_\gamma \kappa\, ds,$$

where the curvature κ of γ and the line element ds are with respect to the flat metric g_p.

Lemma 3.1.
(1) $r(\gamma, p)$ *depends only on the conformal class of* g.
(2) *If* $\tilde{\gamma}$ *is regularly homotopic to* γ *in* M_p, *then* $r(\tilde{\gamma}, p) = r(\gamma, p)$.
(3) *Suppose that* $\tilde{\gamma}$ *is regularly homotopic to* γ *in* M, *and that in the course of homotopy the point* p *is passed once in such a way that* p *is in the left of* γ *and in the right of* $\tilde{\gamma}$. *Then* $r(\tilde{\gamma}, p) = r(\gamma, p) + \chi(M)$.

Proof. (1) Since $\kappa\, ds$ is invariant under homothety of ambient metric, the result follows from Corollary 2.5. (2) We have only to consider a regular homotopy whose support is very small. Then the result is evident because g_p is flat. (3) Let D be a sufficiently small disk around p with smooth boundary $c = \partial D$. Then it is easy to see that $r(\tilde{\gamma}, p) - r(\gamma, p) = 1 - r(c, p)$. On the other hand we have, from Lemma 2.2 (3), $r(c, p) = 1 - \chi(M)$. □

Corollary 3.2. $r_\gamma : M \to \mathbf{R}/\chi(M)\mathbf{Z}$; $r_\gamma(p) = r(\gamma, p) \bmod \chi(M)$ *is well-defined and smooth.*

Proof. From Lemma 2.2 (1) and Proposition 2.8 it follows that $r(\gamma, p)$ is smooth in $p \notin \gamma(S^1)$. The result then follows from Lemma 3.1. □

Proposition 3.3. *If* γ *is null homologous in* $H_1(M, \mathbf{Z})$, *then*
(1) $r(\gamma, p) \in \mathbf{Z}$;
(2) $r(\gamma, p)$, *as a function of* p, *is locally constant for* $p \notin \gamma(S^1)$;
(3) $r(\gamma) := r(\gamma, p) \bmod \chi(M)$ *is well-defined.*

Proof. Let q be a point on γ. Since g_p is flat, we have holonomy $\varphi : \pi_1(M_p, q) \to SO(2) = U(1)$. This is explicitly given as $\varphi([c]) = \exp(-2\pi r(c, p))$. Since $U(1)$ is Abelian, and $H_1(M_p, \mathbf{Z}) = H_1(M, \mathbf{Z})$, φ induces a homomorphism $\varphi : H_1(M, \mathbf{Z}) \to U(1)$. Hence $\varphi([\gamma]) = 1$, which implies (1). From Corollary 3.2 we get (2) because of (1). Then (3) follows from Lemma 3.1 (3). □

For a regular closed curve γ on $M = S^2$ we have $r(\gamma) = 0$ or $1 \bmod 2$. It is easy to see that this is a complete invariant of regular homotopy on S^2 (see also [4]).

We note that the above definitions and arguments make sense for multiple curve $\gamma : S^1 \cup \cdots \cup S^1 \to M$. Thus we have

Corollary 3.4. *If γ is homologous to $\tilde{\gamma}$ in $H_1(M, \mathbf{Z})$, $r(\gamma, p) - r(\tilde{\gamma}, p) \in \mathbf{Z}$ for $p \in M \backslash (\gamma \cup \tilde{\gamma})$, and its residue class modulo $\chi(M)$, which will be denoted by $r(\gamma, \tilde{\gamma})$, is independent of p.*

Let $\mu_1, \ldots, \mu_{2g}$ be regular curves which generate $\pi_1(M)$, where $g = 1 - \chi(M)/2$. Then they constitute also a basis for $H_1(M, \mathbf{Z})$. Hence for γ, we have $n_i \in \mathbf{Z}$ such that γ is homologous to $\tilde{\gamma} = \sum n_i \mu_i$. The rotation number defined by Reinhart [3] is $r(\gamma, \tilde{\gamma})$ in our terminology.

Suppose N is a compact surface with boundary and $f : N \to M$ is an immersion. Obviously $c = f \,|\, \partial N$ is null homologous. In this setting we have a simple formula.

Lemma 3.5. *$r(c, p) + m_p \chi(M) = \chi(N)$, where $m_p = \# f^{-1}(p)$.*

Proof is easy and omitted.

Corollary 3.6. *If $\chi(M) \geq 0$ then $\chi(N) \geq r(c, p)$. If $\chi(M) \leq 0$ then $\chi(N) \leq r(c, p)$.*

4. Proof of Theorem 1.1

From Corollary 3.2 we have $r(\gamma, \cdot) \in C^\infty(M \backslash \gamma)$. Thus $\alpha_\gamma = dr(\gamma, \cdot) = dr_\gamma$ extends smoothly on M as a closed 1-form. Moreover Lemma 3.1 (3) yields the following.

$$\int_c \alpha_\gamma = \chi(M) \gamma \cdot c,$$

where c is a smooth 1-cycle and "$\cdot$" in the right hand side is the homology intersection. Therefore if $\chi(M) < 0$, $\dfrac{1}{\chi(M)} [\alpha_\gamma] \in H^1_{DR}(M)$ is the Poincaré dual of the cycle γ.

The key of the proof is Proposition 2.8. We write K for K_g.

$$-\Delta_x G_p(x) + K(x) = 0 \quad \text{if} \quad p \neq x.$$

We see from Proposition 2.8 that

$$-\Delta_p G_p(x) + K(p) = 0 \quad \text{if} \quad p \neq x.$$

Therefore ν being the unit normal vector of γ, we have

$$-\Delta_p \partial_\nu G_p(x) = -\partial_\nu \Delta_p G_p(x) = \partial_\nu(-\Delta_p G_p(x) + K(p)) = 0 \quad \text{if} \quad p \neq x.$$

This together with Lemma 2.2 (1) shows that $r(\gamma, \cdot)$ is harmonic in $M \backslash \gamma$, and hence α_γ is harmonic.

5. Supplementary remarks

Regular homotopy of closed curves is completely described by Smale [4] in terms of algebraic topology. We are interested in differential geometric interpretation of regular homotopy. Our conformal rotation number is not a complete invariant of regular homotopy. There is another non-trivial regular homotopy invariant $t(\gamma)$ (see [5]). It is of interest to understand $t(\gamma)$ from differential geometric point of view.

We distinguish the term "rotation number" from "winding number." The winding number is also generalized to a curve γ on a surface M, which is given as

$$w(\gamma, p_0, p_\infty) = -\frac{1}{\chi(M)}(r(\gamma, p_0) - r(\gamma, p_\infty)), \quad p_0, p_\infty \in M \backslash \gamma,$$

if $\chi(M) \neq 0$.

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