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NONHARMONIC NONLINEAR FOURIER FRAMES AND CONVERGENCE OF CORRESPONDING FRAME SERIES

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Abstract

Having redundancy, frames, compared with basis, can provide more robust representation of a vector in application. By introducing nonharmonic nonlinear Fourier frames, a method is established to construct such frames by perturbation. Based on a special class of nonharmonic nonlinear Fourier frames, the convergence of its corresponding frame operator is investigated, and the convergence rate, associated with the coefficient sequence of the frame operator, is estimated. Finally, we also discuss the equiconvergence of two different (nonlinear or linear) Fourier (basis or frame) series of $f \in L^2(-\pi, \pi)$.

1. Introduction

Riesz bases and frames, being able to stably and flexibly represent a signal, have attracted much attention in the recent decade. Furthermore, nonlinear Fourier transform has been utilized in many interesting application such as tomography [16]. Recently Chen et al. [5] introduced a new family of nonlinear fourier bases and explored their time-frequency aspects. Moreover, the integral version of the nonlinear Fourier series, called Chirp transform, was studied in [14]. All these works used uniformly sampling for representation. But in several applications, such as magnetic resonance imaging (MRI), there are two main difficulties. On one hand, it is often difficult to sample uniformly; on the other hand, even though uniformly sampling can be realized, there is a common situation that sensing equipment may err in collecting uniform samples. In order to overcome these difficulties, data should be collected by non-uniformly sampling and nonharmonic (nonlinear) Fourier frames need to be introduced. Motivated by the stability of frames in application, we introduce nonharmonic

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nonlinear Fourier frames and discuss the convergence of corresponding series representations.

In [14], the authors introduced Chirp transform based on the nonlinear atoms and the nonlinear Fourier series originated from [5]. They showed that the set of functions given by

$$E_k^{\phi_N}(t) := e^{ik\phi_N(t)}, \quad \phi_N(t) = \phi(t)/N, \quad \forall k \in \mathbf{Z} \tag{1.1}$$

form an orthonormal basis of $L^2((-\pi,\pi), d\phi_N)$, where $N \in \mathbf{N}$ called the Blaschke index of ϕ. Unlike the classical linear orthonormal basis $\{e^{ikt}\}_{k\in\mathbf{Z}}$ of $L^2(-\pi,\pi)$, the more intricate phase function ϕ with instantaneous frequency make (1.1) seems to be more suitable for non-stationary time-frequency Fourier analysis. An important type of ϕ is that

$$\theta_a(t) := t + 2\arctan\frac{|a|\sin(t-t_a)}{1-|a|\cos(t-t_a)}, \tag{1.2}$$

where $a = |a|e^{it_a}$ and $|a| < 1$. For a given window function $g \in L^2(-\pi,\pi)$, Fu et al. [9] established the Balian-Low theorem for a new kind of Gabor Riesz bases, $\{e^{ik\phi_N(t)}g(t-n)\}_{k,n\in\mathbf{Z}}$. That is $g(t)$ can not be well localized in both time and frequency. In this article, the conditions for $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ to be a frame for $L^2((-\pi,\pi), d\phi_N)$ are discussed. Moreover, some new perturbation results of the nonlinear Fourier orthonormal bases $\{e^{ik\phi_N}\}_{k\in\mathbf{Z}}$ is established.

For convergence, in traditional Fourier series, if a 2π-periodic function $f \in \mathscr{C}$ has absolutely integrable n-th derivative, then its Fourier series $f(t) = \sum_{k\in\mathbf{Z}} c_k e^{ikt}$, with $c_k = \langle f, e^{ikt}\rangle$, converges uniformly and $|c_{\pm k}| \le \|f^{(n)}\|_1/k^n$ for $k > 0$. Bultheel and Carrette [2] also showed a similar result for Takenaka-Malmquist system, a rational (nonlinear) orthonormal system. Note that the convergence rate of Frourier coefficients plays an important role in applications. In this article we investigate frame operator $Sf = \sum_{k\in\mathbf{Z}} \langle f, e^{i\lambda_k\theta_a}\rangle_{\theta_a} e^{i\lambda_k\theta_a}$ under the same condition of f. On page 197 of [17], Theorem 15 discussed the equiconvergence of Fourier series and nonharmonic Fourier series of $f \in L^2(-\pi,\pi)$. Motivated by this, as nonlinear Fourier basis and frame are introduced, we investigate the equiconvergence of two different (nonlinear or linear) Fourier (basis or frame) series of f.

This paper is organized as follows. We introduce some definitions and notations about frame and Chirp transform in Section 1. Moreover, we establish sufficient and necessary conditions for $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ being a frame for $L^2((-\pi,\pi), d\phi)$. In Section 2, some perturbation results about $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ are given. We discuss the convergence of frame operator with respect to a class of nonharmonic nonlinear Fourier frames, and estimate the convergence rate of its corresponding coefficient sequence in Section 3. Based on nonlinear (or linear) Fourier frame (or basis), the equiconvergence of different series of $f \in L^2(-\pi,\pi)$ is also investigated.

Frames were originally studied in the context of nonharmonic Fourier series [8]. A thorough discussion can be found in [3] or [7]. For nonharmonic

Fourier series one can also refer [17]. Now let us recall some basic notations and results.

1.1. Frame and Riesz base. A set of $\{f_k\}_{k\in\mathbf{Z}}$ is a **frame** for a separable Hilbert space $\mathscr{H}$, if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{k\in\mathbf{Z}} |\langle f, f_k\rangle|^2 \le B\|f\|^2 \tag{1.3}$$

for all $f \in \mathscr{H}$. The constants A and B are called frame bounds, and the frame is called **tight** if $A = B$.

Let $\{f_k\}_{k\in\mathbf{Z}}$ be a frame for $\mathscr{H}$. Then the frame operator

$$S : \mathscr{H} \to \mathscr{H}, \quad Sf = \sum_{k\in\mathbf{Z}} \langle f, f_k\rangle f_k \tag{1.4}$$

is bounded, positive, and invertible. Thus for any $f \in \mathscr{H}$, one has

$$f = SS^{-1}f = \sum_{k\in\mathbf{Z}} \langle f, S^{-1}f_k\rangle f_k. \tag{1.5}$$

Recall that $\{f_k\}_{k\in\mathbf{Z}}$ is a **Riesz basis** for $\mathscr{H}$ if $\{f_k\}_{k\in\mathbf{Z}}$ is complete and there exist constants $0 < A \le B < \infty$ such that

$$A\sum |c_k|^2 \le \|\sum c_k f_k\|^2 \le B\sum |c_k|^2 \tag{1.6}$$

for all finite sequence $\{c_k\}$ of complex scalars.

1.2. Chirp transform and series. Firstly, we consider the function spaces $L^2((-\pi,\pi), d\mu)$, where μ is a positive measure. The norm and inner product in $L^2((-\pi,\pi), d\mu)$ are defined as

$$\|f\|_\mu = \left(\int_{-\pi}^{\pi} |f(t)|^2\, d\mu(t)\right)^{1/2} \quad \text{and} \quad \langle f, g\rangle_\mu = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, d\mu(t),$$

respectively. If $\mu(t) = t$ for $t \in \mathbf{R}$, $L^2((-\pi,\pi), d\mu)$ is the common space $L^2(-\pi,\pi)$. Furthermore, we assume in this paper that μ satisfy the following assumption.

Assumption 1. Let μ be a positive measure, and satisfy
(1) $\mu : \mathbf{R} \to \mathbf{R}$ with $\mu \in C^1$ and $\mu' > 0$;
(2) $\mu(t + 2k\pi) = \mu(t) + 2Nk\pi$ for all $t \in \mathbf{R}$, $k \in \mathbf{Z}$ and some fixed $N > 0$.

Under the Assumption 1, it is direct to get the following properties of μ. Readers are referred to [14] for more details.

Proposition 1.1. *Let* $\mu : \mathbf{R} \to \mathbf{R}$ *satisfy Assumption* 1. *Then*

(i) *The nonlinear phase function* $\mu_N|_{(-\pi,\pi)} : (-\pi,\pi) \to (-\pi,\pi)$ *is a strictly increasing bijective mapping, where* $\mu_N(t) = \mu(t)/N$;

(ii) *There exists a constant* K *such that*

$$|\mu_N(t) - t| \le K, \quad \forall t \in \mathbf{R},$$

and one may take $K = \max_{t \in (-\pi,\pi)} |\mu_N(t) - t|$;

(iii) $\mu'(t)$ *is a* 2π*-periodic function*;

(iv) $\mu'(t) \simeq 1$, *i.e.*, $0 < \min_{t \in (-\pi,\pi)} \mu'(t) \le \mu'(t) \le \max_{t \in (-\pi,\pi)} \mu'(t) < \infty$.

Remark 1.2. Obviously, $\theta_a(t)$ defined in (1.2) is a function satisfying Assumption 1. Moreover, there are many other functions satisfying Assumption 1, for example the functions $Nx + \sin(x)$ for any $N \in \mathbf{N}$ with $N \ge 2$.

It is easy to know that μ_N has same properties as μ, by an abuse of notation, μ_N is used sometimes instead of μ. The following function ϕ satisfies the Assumption 1, but is restricted on the interval $(-\pi,\pi)$.

Definition 1.3. For $f \in L^2(-\pi,\pi)$, we define the discrete Chirp transform as

$$\mathscr{T}_{\phi_N} f(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{ik\phi_N(t)}\, d\phi_N, \quad k \in \mathbf{Z}. \tag{1.7}$$

Next, let $\mathscr{T}$ be discrete Fourier transform on $L^2(-\pi,\pi)$,

$$\mathscr{T} f(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt, \quad k \in \mathbf{Z}.$$

We also need the ϕ-pushforward operator $\Phi_\phi f := f \circ \phi^{-1}$. Note that Φ_ϕ is an isometry and its inverse satisfies $\Phi_\phi^{-1} = \Phi_{\phi^{-1}}$.

Now we have the relationship between discrete Chirp transform $\mathscr{T}_{\phi_N}$ and discrete Fourier transform $\mathscr{T}$,

$$\mathscr{T}_{\phi_N} = \mathscr{T} \circ \Phi_{\phi_N} \quad \text{and inverse} \quad \mathscr{T}_{\phi_N}^{-1} = \Phi_{\phi_N^{-1}} \circ \mathscr{T}^{-1}. \tag{1.8}$$

Proposition 1.4 ([14, Theorem 3.2]). *For any* $f \in L^2((-\pi,\pi), d\phi_N)$, *we have*

$$f(t) = \sum_{k \in \mathbf{Z}} c_k e^{ik\phi_N(t)}$$

with

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ik\phi_N}\, d\phi_N.$$

Form now on, we concentrate on non-stationary time Fourier frame $E_{\Lambda}^{\phi_N} := \{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ of $L^2((-\pi,\pi),d\phi_N)$, where $\{\lambda_k\}_{k\in\mathbf{Z}} \subseteq \mathbf{R}$ and $L^2((-\pi,\pi),d\phi_N)$ is equipped with the inner product

$$\langle f,g\rangle_{\phi_N} = \int_{\pi}^{\pi} f(t)\overline{g(t)}\,d\phi_N(t), \quad f,g \in L^2((-\pi,\pi),d\phi_N)$$

and the induced norm $\|\cdot\|_{\phi_N}$. Specially, if $\phi_N(t) = t$ for $t \in \mathbf{R}$, many valuable works were obtained, see [4, 8, 10, 11, 12, 15, 17] and references therein.

Let I be a countable index. A set $\Lambda := \{\lambda_k\}_{k\in I}$ is **separated** if there is some $\delta > 0$ such that $|\lambda_k - \lambda_j| \geq \delta$ for all $j \neq k$ and the constant δ is called a separated constant. If Λ is a finite union of separated sets, we say that Λ is **relatively separated**. Given a relatively separated set Λ and $r > 0$, let $n^-(r)$ denote the minimal number of elements form Λ to be found in an interval of length r. The lower Beurling density of Λ is defined by

$$D^-(\Lambda) = \lim_{r\to\infty} \frac{n^-(r)}{r}.$$

THEOREM 1.5. *Let $\Lambda = \{\lambda_k\}_{k\in\mathbf{Z}}$ be a relatively separated sequence of real numbers. Then the following holds*:

(a) *If $D^-(\Lambda) > 1$, then $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ forms a frame for $L^2((-\pi,\pi),d\phi_N)$;*

(b) *If $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ forms a frame for $L^2((-\pi,\pi),d\phi_N)$, then $D^-(\Lambda) \geq 1$.*

Proof. The ϕ_N-pushforward operator Φ_{ϕ_N} and [7, Theorem 7.6.4] will be used for proving. Here we just show the sufficient case (a), and case (b) can be proved similarly. If $D^-(\Lambda) > 1$, following [7, Theorem 7.6.4], we know that $\{e^{i\lambda_k t}\}$ is a frame of $L^2(-\pi,\pi)$, this means that for any $f \in L^2(-\pi,\pi)$, there exists $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbf{Z}} |\langle f, e^{i\lambda_k t}\rangle|^2 \leq B\|f\|^2. \tag{1.9}$$

Note that Φ_{ϕ_N} is an isometry, one obtains that

$$\langle f, e^{i\lambda_k t}\rangle = \langle \Phi_{\phi_N}^{-1} f, \Phi_{\phi_N}^{-1} e^{i\lambda_k t}\rangle_{\phi_N} = \langle \Phi_{\phi_N^{-1}} f, e^{i\lambda_k\phi_N(t)}\rangle_{\phi_N}$$

and

$$\|f\|^2 = \|\Phi_{\phi_N^{-1}} f\|_{\phi_N}^2.$$

Thus (1.9) can be rewritten as,

$$A\|\Phi_{\phi_N^{-1}} f\|_{\phi_N}^2 \leq \sum_{k\in\mathbf{Z}} |\langle \Phi_{\phi_N^{-1}} f, e^{i\lambda_k t}\rangle_{\phi_N}|^2 \leq B\|\Phi_{\phi_N^{-1}} f\|_{\phi_N}^2.$$

Now replacing $\Phi_{\phi_N} f$ by f in the above inequality, we can obtain the required result. The proof is completed. $\square$

2. Perturbation results of nonlinear Fourier basis

In this section we focus on the stability of frames under perturbation. Perturbation theory is one of main research issues in frames and its origins lie in the celebrated works of Paley and Wiener [13] and Levinson [12]. Furthermore, Kadec [11] gave the best constant followed from a theorem of [12, Page 48].

For non-harmonic frames $\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}$ of exponentials in $L^2(-\pi,\pi)$, Kadec's celebrated 1/4 theorem in [11] tells us that $\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$, when λ_k is close to k for all $k\in\mathbf{Z}$, that is to say,

$$\sup_{k\in\mathbf{Z}}|\lambda_k-k|\le L, \quad \text{with } L<1/4.$$

When chirp transform and series are introduced, $\{e^{ik\phi_N(t)}\}_{k\in\mathbf{Z}}$ is exactly an orthonormal basis of $L^2((-\pi,\pi),d\phi_N)$ and we have the following theorem.

Theorem 2.1. *Let $\{\lambda_k\}_{k\in\mathbf{Z}}$ be a real sequence. If there exists a constant $L<1/4$ such that*

$$\sup_{k\in\mathbf{Z}}|\lambda_k-k|\le L, \tag{2.1}$$

then $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ is a Riesz basis for $L^2((-\pi,\pi),d\phi_N)$ with bounds $(\cos(\pi L)-\sin(\pi L))^2$ *and* $(2-\cos(\pi L)+\sin(\pi L))^2$.

Proof. First we claim that $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ is a Riesz basis for $L^2((-\pi,\pi),d\phi_N)$ if and only if $\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$. Here we just show the sufficient part, because of the necessary part is similar. By the definition of Riesz basis, $\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}$ is complete in $L^2(-\pi,\pi)$ and there exists constants $A,B>0$ such that

$$A\sum_k|c_k|^2\le\left\|\sum_k c_k e^{i\lambda_k t}\right\|^2\le B\sum_k|c_k|^2 \tag{2.2}$$

for every finite scalar sequence $\{c_k\}$. Noting that ϕ_N-pushforward operator

$$\Phi_{\phi_N}:L^2((-\pi,\pi),d\phi_N)\to L^2(-\pi,\pi)$$

is an isometry and its inverse satisfies $\Phi_{\phi_N}^{-1}=\Phi_{\phi_N^{-1}}$, we have

$$\left\|\sum_k c_k e^{i\lambda_k t}\right\|^2=\left\|\Phi_{\phi_N^{-1}}\left(\sum_k c_k e^{i\lambda_k t}\right)\right\|_{\phi_N}^2=\left\|\sum_k c_k e^{i\lambda_k\phi_a(t)}\right\|_{\phi_N}^2.$$

Then (2.2) is equivalent to

$$A\sum_k|c_k|^2\le\left\|\sum_k c_k e^{i\lambda_k\phi_N(t)}\right\|_{\phi_N}^2\le B\sum_k|c_k|^2. \tag{2.3}$$

Suppose that $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ is not complete in $L^2((-\pi,\pi),d\phi_N)$. Then there exists $0\neq f_0\in L^2((-\pi,\pi),d\phi_N)$ such that $f_0\in\mathscr{H}_0^{\perp}$ with $\mathscr{H}_0=\overline{\operatorname{span}\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}}$. Therefore,

$$0=\langle f_0,e^{i\lambda_k\phi_N(t)}\rangle_{\phi_N}=\langle\Phi_{\phi_N}f_0,\Phi_{\phi_N}e^{i\lambda_k\phi_N(t)}\rangle=\langle\Phi_{\phi_N}f_0,e^{i\lambda_k t}\rangle$$

for any $k\in\mathbf{Z}$. This means $\Phi_{\phi_N}f_0\in\mathscr{H}_1^{\perp}$, where $\mathscr{H}_1=\Phi_{\phi_N}(\mathscr{H}_0)=\overline{\operatorname{span}\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}}$. On the other hand

$$0\neq\Phi_{\phi_N}f_0=f_0(\phi^{-1}(t))\in L^2(-\pi,\pi).$$

It is a contradiction to the fact that $\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}$ is complete in $L^2(-\pi,\pi)$.

Now, under the condition of (2.1) with $L<1/4$, by Kadec's 1/4 theorem, one obtains that $\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}$ is a Riesz basis for $L^2(-\pi,\pi)$ and the first conclusion follows.

Next we shall obtain the bounds. For any finite nonzero sequence c_k, $k\in\mathbf{Z}$, we have

$$\left\|\sum_k c_k e^{i\lambda_k\phi_N(t)}\right\|_{\phi_N}=\left\|\sum_k c_k e^{i\lambda_k t}\right\|\leq\left\|\sum_k c_k e^{ikt}\right\|+\left\|\sum_k c_k e^{i(\lambda_k-k)t}\right\|.$$

Since $\{e^{ikt},k\in\mathbf{Z}\}$ is an orthonormal basis of $L^2(-\pi,\pi)$, we have

$$\left\|\sum_k c_k e^{ikt}\right\|=\|\{c_k\}_{k\in\mathbf{Z}}\|_2.$$

By [1, Theorem 1], one obtains

$$\left\|\sum_k c_k e^{i(\lambda_k-k)t}\right\|\leq(1-\cos(\pi L)+\sin(\pi L))^2\|\{c_k\}_{k\in\mathbf{Z}}\|_2.$$

Thus

$$\begin{aligned}(\cos(\pi L)-\sin(\pi L))^2\sum_k|c_k|^2&\leq\left\|\sum_k c_k e^{i\lambda_k\phi_N(t)}\right\|_{\phi_N}^2\\&\leq(2-\cos(\pi L)+\sin(\pi L))^2\sum_k|c_k|^2.\end{aligned}$$

This completes the proof. □

Similar to Theorem 2.1, we can also expand [4, Theorem 16.9], which was proved independently by Balan [1] and Christensen [6], to nonlinear Fourier frames case as follows.

Theorem 2.2. *Let* $\{\mu_k\}_{k\in\mathbf{Z}}$, $\{\lambda_k\}_{k\in\mathbf{Z}}$ *be two real sequences and* $\{e^{i\mu_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ *be a frame for* $L^2((-\pi,\pi),d\phi_N)$ *with bounds* A *and* B. *If there exists a constant constant* $L<1/4$ *such that,*

$$\sup_{k\in\mathbf{Z}}|\lambda_k-\mu_k|\le L, \quad \text{and} \quad 1-\cos\pi L+\sin\pi L<A/B, \tag{2.4}$$

then $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ *is also a frame for* $L^2((-\pi,\pi),d\phi_N)$ *with bounds* $A(1-B/A(1-\cos\pi L+\sin\pi L))^2$ *and* $B(2-\cos\pi L+\sin\pi L)^2$.

The following proposition analyzes the limit case of Theorem 2.1 and one can refer [4, Theorem 16.9] for the classical nonharmonic Fourier frame case.

Proposition 2.3. *Let* $\{\lambda_k\}_{k\in\mathbf{Z}}$ *be a real sequence satisfying* $\sup_{k\in\mathbf{Z}}|\lambda_k-k|=1/4$. *Then* $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ *is either an Riesz basis or not a frame of* $L^2((-\pi,\pi),d\phi_N)$.

Proof. For any $x\in[0,1]$, define $\lambda_k(t)=k+x(\lambda_k-k)$, then

$$\sup|\lambda_k(t)-k|=x\sup|\lambda_k-k|=\frac{x}{4}.$$

and

$$\sup|\lambda_k(x)-\lambda_k|=\sup|k+x(\lambda_k-k)-\lambda_k|=\sup|(1-x)(\lambda_k-k)|=\frac{1-x}{4}.$$

Suppose that $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ is a frame for $L^2((-\pi,\pi),d\phi_N)$ with bounds $0<A\le B<\infty$. By Theorem 2.2, if $1-\cos\left(\frac{1-x}{4}\pi\right)+\sin\left(\frac{1-x}{4}\pi\right)<A/B$ and $x\in(0,1]$, which means $1\ge x>x_0:=\frac{4}{\pi}\arcsin\left(\frac{\sqrt{2}}{2}\left(1-\frac{A}{B}\right)\right)$, then $\{e^{i\lambda_k(x)\phi_N(t)}\}_{k\in\mathbf{Z}}$ is a frame with bounds

$$A(x):=A\left(1-\frac{B}{A}\left(1-\cos\left(\frac{1-x}{4}\pi\right)+\sin\left(\frac{1-x}{4}\pi\right)\right)\right)^2$$

and

$$B(x):=B\left(2-\cos\left(\frac{1-x}{4}\pi\right)+\sin\left(\frac{1-x}{4}\pi\right)\right)^2.$$

While if $x<1$, by Theorem 2.1 we have $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ is an Riesz basis, with the same bounds as frame. Then

$$A(x)\sum_{k\in\mathbf{Z}}|c_k|^2\le\sum_{k\in\mathbf{Z}}c_ke^{i\lambda_k(x)\phi_N(t)}\le B(x)\sum_{k\in\mathbf{Z}}|c_k|^2$$

for any finite nonzero sequence $\{c_k\}_{k\in\mathbf{Z}}$. Let $x \to 1$. By sign-preserving theorem of limit, we have

$$A\sum_{k\in\mathbf{Z}}|c_k|^2 \le \sum_{k\in\mathbf{Z}} c_k e^{i\lambda_k\phi_N(t)} \le B\sum_{k\in\mathbf{Z}}|c_k|^2.$$

Then $\{e^{i\lambda_k\phi_N(t)}\}_{k\in\mathbf{Z}}$ is also an Riesz basis with bounds the same as frame. The proof is completed. □

3. Convergence

In this section, we give some generalizations about the convergence and estimation of the coefficients, of the frame operator with respect to a special class of nonharmonic nonlinear Fourier frame. Finally, Based on nonlinear (or linear) Fourier frame (or basis), the equiconvergence of different series of $f \in L^2(-\pi,\pi)$ is investigated.

Recall some results about Fourier series of 2π-periodic functions. For any function $f(t) \in \mathscr{C}^q$, i.e. with continuous q-th derivative and $q \in \mathbf{N}$, we have that

$$f(t) = \sum_{k\in\mathbf{Z}} c_k e^{ikt} \quad \text{with} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt.$$

The Fourier series is uniformly convergent and the Fourier coefficients are bounded:

$$|c_0| \le \|f\|_1 \quad \text{and} \quad |c_{\pm k}| \le \|f^{(q)}\|_1/k^q, \quad \text{for } k>0, \tag{3.1}$$

where $f^{(q)}(t)$ stands for the q-th derivative of the function $f(t)$. Similar results have been shown by Bultheel and Carrette [2, Theorem 1 and Corollary 2] for Takenaka-Malmquist system. In the first part of this section, we aim at characterizing the similar convergence and coefficient properties of the nonharmonic nonlinear frames $\{E_{\lambda_k}^{\theta_a}\}_{k\in\mathbf{Z}}$. Before that we need two lemmas:

Lemma 3.1. *Let* $h_k(z) = \left(\dfrac{1+\bar{a}z}{z+a}\right)^k$, *with* $k \in \mathbf{N}$ *and* $c = |a| < 1$. *If* $z = \rho e^{it}$, $\rho > 0$, *then*

$$\begin{cases} \left(\dfrac{1-\rho c}{\rho - c}\right)^k \le |h_k(z)| \le \left(\dfrac{1+\rho c}{\rho + c}\right)^k, & \rho > 1, \\ |h_k(z)| = 1, & \rho = 1, \\ \left(\dfrac{1+\rho c}{\rho + c}\right)^k \le |h_k(z)| \le \left(\dfrac{1-\rho c}{\rho - c}\right)^k, & \rho < 1. \end{cases} \tag{3.2}$$

Proof. Suppose $a = ce^{i\xi_0}$. When $k = 2$, we have

$$
\begin{aligned}
|h_2(z)| &= \left|\left(\frac{1+\bar{a}z}{z+a}\right)^2\right| = \left|\left(\frac{1+\bar{a}z}{z+a}\right)\right|^2 \\
&= \left|\frac{1+c\rho e^{i(t-\xi_0)}}{\rho+ce^{-i(t-\xi_0)}}\right|^2 = \frac{1+2\rho c\cos(t-\xi_0)+\rho^2c^2}{\rho^2+2\rho c\cos(t-\xi_0)+c^2}
\end{aligned}
$$

and define

$$
r(t) := \frac{1+2\rho c\cos(t)+\rho^2c^2}{\rho^2+2\rho c\cos(t)+c^2}, \quad t \in [-\pi,\pi],
$$

Note that

$$
r'(t) = \frac{2\rho c(1-\rho^2)(1-c^2)\sin(t)}{(\rho^2+2\rho c\cos(t)+c^2)^2}, \quad t \in [-\pi,\pi].
$$

one can easily show that,

$$
\begin{cases}
\left(\frac{1-\rho c}{\rho-c}\right)^2 \le |h_2(z)| \le \left(\frac{1+\rho c}{\rho+c}\right)^2, & \rho > 1, \\
|h_2(z)| = 1, & \rho = 1, \\
\left(\frac{1+\rho c}{\rho+c}\right)^2 \le |h_2(z)| \le \left(\frac{1-\rho c}{\rho-c}\right)^2, & \rho < 1.
\end{cases} \tag{3.3}
$$

Since $|h_k(t)| = |(h_2(t))|^{k/2}$, the argument for $k \in \mathbf{N}$ follows with some simple adjustment. $\square$

LEMMA 3.2 ([2, Lemma 6]). *Let* $x_2 > x_1 > 0$ *and* $q \ge 2$. *Then*

$$
\begin{aligned}
q\int_{x_1}^{x_2} e^{q(x-\ln x)}\,dx &\le I_{x_1<1}[\alpha_1^{-1}(1-(x_1/\tilde{x})^{(q-1)\alpha_1})]\frac{q}{q-1}e^{qx_1-(q-1)\ln x_1} \\
&\quad + I_{x_2>1}[\alpha_2^{-1}(1-e^{-q\alpha_2(x_2-\tilde{x})})]e^{q(x_2-\ln x_2)}
\end{aligned}
$$

where $\tilde{x} = \max(x_1, \min(x_2, 1))$ *with*

$$
\alpha_1 = 1 - \frac{q}{q-1}\tilde{x}\frac{1-x_1/\tilde{x}}{\ln(\tilde{x}/x_1)} \quad \text{and} \quad \alpha_2 = 1 - x_2^{-1}\frac{\ln(x_2/\tilde{x})}{1-\tilde{x}/x_2}.
$$

The indicator function $I_{x<y}$ *is one if* $x < y$ *and zeros otherwise.*

From now on, let $\{\lambda_k\}_{k\in\mathbf{Z}}$ satisfy $\sup_{k\in\mathbf{Z}}|\lambda_k - k| \le L < 1/4$ and $\lambda_1 > 1$, $\lambda_{-1} < -1$, Then by Theorem 2.1, the set

$$
\left\{E_{\lambda_k}^{\theta_a} = e^{i\lambda_k\theta_a(t)} = \left(\frac{z-a}{1-\bar{a}z}\right)^{\lambda_k} : |k| < N\right\}_{k\in\mathbf{Z}}
$$

is a Riesz basis of $L^2((-\pi,\pi), d\theta_a)$, where $\theta_a(t)$ is defined as in (1.2) and $z = e^{it}$. Furthermore, for $f \in L^2((-\pi,\pi), d\theta_a)$, we have the frame operator,

$$Sf(t) = \sum_{k\in\mathbf{Z}} b_k E^{\theta_a}_{\lambda_k} = \sum_{k\in\mathbf{Z}} \langle f, e^{i\lambda_k\theta_a(t)}\rangle_{\theta_a} e^{i\lambda_k\theta_a(t)} \tag{3.4}$$

and the partial sum of right series defined by,

$$(S_N f)(t) = \sum_{|k|<N} b_k E^{\theta_a}_{\lambda_k}, \tag{3.5}$$

where $b_k = \langle f, e^{i\lambda_k\theta_a(t)}\rangle_{\theta_a}$ for $k \in \mathbf{Z}$. The analysis of convergence of the expansion coefficients $a_{\pm k}$ leads to the following theorem.

THEOREM 3.3. *Let $f(t)$ be a 2π-periodic function having a continuous q-th derivative with $q > 2$. Then the coefficients of frame operator Sf defined in* (3.4) *satisfy*

$$|b_0| \le \|f\|_1 \quad \text{and} \quad |b_{\pm k}| \le \left(K_1(c,q)\frac{1}{\ln\hat{\rho}} + K_2\frac{\varepsilon\lambda_k}{q-1}\right)\|f^{(q)}\|_1/(\varepsilon\lambda_k)^q,\ k \ge 1, \tag{3.6}$$

where $\varepsilon = \dfrac{\ln\dfrac{\tilde{\rho}+c}{1+c\tilde{\rho}}}{\ln\tilde{\rho}}$, $c = |a|$ *with*

$$\tilde{\rho}(c,q,k) = \min\left(\left[\left(\frac{-4q}{(k-1)\ln c}\right)^{q/4} + (k/2)^{q/(k-2)}\right] + \left[\frac{1+c}{\sqrt{cq}} + \frac{1}{c(q-1)}\right], e^q\right)$$

and $\hat{\rho} = \lim_{k\to\infty} \tilde{\rho}(c,q,k)$.

Proof. The statement for $|b_0| \le \|f\|_1$ is trivial, Next, we only consider b_k with $k > 0$ (the case $k < 0$ can be obtained similarly). Firstly, we use the fact that the function $f(t)$ can be expanded into Fourier series:

$$f(t) = \sum_{n\in\mathbf{Z}} w_n e^{int},$$

where the coefficients satisfy $|w_0| \le \|f\|_1$ and $|w_{\pm k}| \le \|f^{(q)}\|_1/k^q$. So,

$$\begin{aligned}
b_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n\in\mathbf{Z}} w_n e^{int}\overline{e^{i\lambda_k\theta_a(t)}}\, d\theta_a(t) \\
&= \frac{1}{2\pi}\sum_{n\in\mathbf{Z}} w_n \int_{-\pi}^{\pi} e^{int}e^{-i\lambda_k\theta_a(t)}\, d\theta_a(t) \\
&= \frac{1}{2\pi}\sum_{n\in\mathbf{Z}} w_n \int_{-\pi}^{\pi} e^{in\theta_{-a}(t)}e^{-i\lambda_k t}\, dt \\
&= \frac{i}{2\pi}\sum_{n\in\mathbf{Z}} w_n \oint_{|z|=1}\left(\frac{z+\bar{a}}{1+az}\right)^n z^{\lambda_k - 1}\, dz
\end{aligned}$$

with $z = e^{-it}$. Here the fact $\theta_a^{-1}(t) = \theta_{-a}(t)$ and $e^{i\theta_{-a}(t)} = \left(\dfrac{z - \bar{a}}{1 + az}\right)$ are used, see Section 7 and Lemma 7.1 in [14]. By Cauchy's integral formula and $\lambda_k - 1 > 0$, one obtains $\oint_{|z|=1} z^{\lambda_k}\left(\dfrac{z+\bar{a}}{1+az}\right)^n \dfrac{dz}{z} = 0$ for $n \leq 0$. Thus

$$\begin{aligned}
b_k &= \frac{i}{2\pi}\sum_{n=1}^{\infty} w_n \oint_{|z|=1}\left(\frac{z+\bar{a}}{1+az}\right)^n z^{\lambda_k - 1}\, dz \\
&= \frac{i}{2\pi}\sum_{n=1}^{n_1} w_n \oint_{|z|=\rho_1}\left(\frac{z+\bar{a}}{1+az}\right)^n z^{\lambda_k - 1}\, dz + \frac{i}{2\pi}\sum_{n=n_1-1}^{n_2-1} w_n \oint_{|z|=1}\left(\frac{z+\bar{a}}{1+az}\right)^n z^{\lambda_k - 1}\, dz \\
&\quad + \frac{i}{2\pi}\sum_{n\geq n_2} w_n \oint_{|z|=\rho_2}\left(\frac{z+\bar{a}}{1+az}\right)^n z^{\lambda_k - 1}\, dz \\
&\triangleq \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3
\end{aligned}$$

with appropriate $|a| < \rho_1 < 1$ and $\rho_2 > 1$, and integers $n_2 > n_1 > 0$ to be specified below. Next, we estimate the bounds of II_1, II_2 and II_3, separately.

(*First sum*) By Lemma 3.1, we have

$$|\mathrm{II}_1| \leq \|f^{(q)}\|_1 \rho_1^{\lambda_k} \sum_{n=1}^{n_1}\left(\frac{1 - c\rho_1}{\rho_1 - c}\right)^n \Big/ n^q. \tag{3.7}$$

Denote $\tilde{\rho}_1 = \dfrac{1 - c\rho_1}{\rho_1 - c} > 1$. Then $\rho_1 = \dfrac{1 + c\tilde{\rho}_1}{\tilde{\rho}_1 + c}$. Thus (3.7) can be written as

$$|\mathrm{II}_1| \leq \|f^{(q)}\|_1 \left(\frac{1 + c\tilde{\rho}_1}{\tilde{\rho}_1 + c}\right)^{\lambda_k} \sum_{n=1}^{n_1} \tilde{\rho}_1^n / n^q. \tag{3.8}$$

Using Lemma 3.2, we obtain

$$\begin{aligned}
\sum_{n=1}^{n_1} \tilde{\rho}_1^n / n^q &\leq q(\ln \tilde{\rho}_1)^{q-1} e^{-q\ln q} \int_{\ln \tilde{\rho}_1/q}^{n_1 \ln \tilde{\rho}_1/q} e^{q(x - \ln x)}\, dx \\
&\leq I_{x_1<1}[\alpha_1^{-1}(1 - (x_1/\tilde{x})^{(q-1)\alpha_1})] \frac{\tilde{\rho}_1}{q-1} \\
&\quad + I_{x_2>1}[\alpha_2^{-1}(1 - e^{-q\alpha_2(x_2 - \tilde{x})})] \frac{1}{\ln \tilde{\rho}_1} \frac{\tilde{\rho}_1^{n_1}}{n_1^q} \\
&\triangleq \mathrm{II}_{11} + \mathrm{II}_{12},
\end{aligned}$$

where $x_1 = \ln \tilde{\rho}_1/q$ and $x_2 = n_1 \ln \tilde{\rho}_1/q$. Here the technique in [2, Theorem 1] is used. Denote (as [2, Theorem 1])

$$\tilde{\rho}_1 = \min([(\tilde{\rho}_{11} - 1) + \tilde{\rho}_{12}] + (\tilde{\rho}_{13} - 1), e^q), \tag{3.9}$$

where

$$\tilde{\rho}_{11} = 1 + \left(\frac{q}{k-1}\frac{-4}{\ln c}\right)^{q/4}, \quad \tilde{\rho}_{12} = (k/2)^{q/(k-2)} \quad \text{and} \quad \tilde{\rho}_{13} = 1 + \frac{1+c}{\sqrt{cq}} + \frac{1}{c(q-1)},$$

with k satisfying $|\lambda_k - k| < 1/4$. Then

$$\hat{\rho} = \lim_{k\to\infty} \tilde{\rho}(c,q,k) = \min\left(1 + \frac{1+c}{\sqrt{cq}} + \frac{1}{c(q-1)}, e^q\right).$$

Define $n_1 = \lfloor \varepsilon_1 \lambda_k \rfloor$ with

$$0 < \varepsilon_1 = \frac{\ln \dfrac{\tilde{\rho}_1 + c}{1 + c\tilde{\rho}_1}}{\ln \tilde{\rho}_1} = \frac{\ln \rho_1}{\ln \dfrac{\rho_1 - c}{1 - c\rho_1}}.$$

Specifically if $n_1 = 0$, then it is trivial that $\mathrm{II}_1 = 0$. Since $\dfrac{\rho_1 - c}{1 - c\rho_1} \le \rho_1 < 1$, we get that $\varepsilon_1 \le 1$. As a result it is not difficulty to show that

$$\tilde{\rho}_1^{n_1}\left(\frac{1 + c\tilde{\rho}_1}{\tilde{\rho}_1 + c}\right)^{\lambda_k} < 1.$$

Next we show the boundedness of $\mathrm{II}_{1x_1} := \mathrm{I}_{x_1<1}[\alpha_1^{-1}(1 - (x_1/\tilde{x})^{(q-1)\alpha_1})]$ and $\mathrm{II}_{1x_2} := \mathrm{I}_{x_2>1}[\alpha_2^{-1}(1 - e^{-q\alpha_2(x_2-\tilde{x})})]$. Since $\hat{\rho} \le \tilde{\rho}_1 \le e^q$, one obtains $x_{10} := \dfrac{\ln \hat{\rho}}{q} \le x_1 \le 1$ and $x_{20} := \dfrac{n_1 \ln \hat{\rho}}{q} \le x_2 \le n_1$.

Boundedness of II_{1x_1}**.** For $x_2 > 1$, we have $\tilde{x} = 1$, then

$$\alpha_1 = 1 - \frac{q}{q-1}\tilde{x}\frac{1 - x_1/\tilde{x}}{\ln(\tilde{x}/x_1)} = 1 - \frac{q}{q-1}\frac{x_1 - 1}{\ln x_1}.$$

Define $\lambda(x) := 1 - \dfrac{q}{q-1}\dfrac{x-1}{\ln x}$ and $\mu(x) := \dfrac{1 - x^{(q-1)\lambda(x)}}{\lambda(x)}$ with $x \in [0,1]$. It is not difficult to get $1 - \dfrac{q}{q-1} \le \lambda(x) \le 1$ and $\lambda(x)$ is a monotonically decreasing function. Then there exists $x_0 \in (0,1)$ such that $\lambda(x_0) = 0$. Noting that $\lim_{x\to x_0} \mu(x) = (q-1)\ln(1/x_0)$ and $\mu(x)$ is continuous, then there exists $C_{11} > 0$ such that $\mathrm{II}_{1x_1} < C_{11}$. It can be proved similarly when $x_2 \le 1$.

Boundedness of II_{1x_2}**.** For $x_2 \le 1$, it is trivial. For $x_2 > 1$, we have $\tilde{x} = 1$, then

$$\alpha_2^{-1}(1 - e^{-q\alpha_2(x_2-\tilde{x})}) = \frac{1 - e^{-q(x_2-1-\ln x_2)}}{1 - \dfrac{\ln x_2}{x_2 - 1}}.$$

Define $\tilde{\lambda}(x) := \dfrac{1 - e^{-q(x-1-\ln x)}}{1 - \dfrac{\ln x}{x-1}}$ with $x \in (0, \infty)$. Noting that $\lim_{x\to 0} \tilde{\lambda}(x) = \lim_{x\to 1} \tilde{\lambda}(x) = 0$, $\lim_{x\to\infty} \tilde{\lambda}(x) = 1$ and $\tilde{\lambda}(x)$ is continuous, then there exists $C_{12} > 0$ such that $\mathrm{II}_{1x_1} < C_{12}$.

Next we show that

$$\left(\frac{1 + c\tilde{\rho}_1}{\tilde{\rho}_1 + c}\right)^{\lambda_k} \cdot n_1^q = (-\rho_1^{\lambda_k/q} \cdot \ln \rho_1^{\lambda_k})^q \left(\ln \frac{1 - c\rho_1}{\rho_1 - c}\right)^{-q} < C_{13} \tag{3.10}$$

with $\rho_1 = \dfrac{1 + c\tilde{\rho}_1}{\tilde{\rho}_1 + c}$ and $C_{13} = \left(\dfrac{q}{e \ln \hat{\rho}}\right)^q$. Actually (3.10) is clear since that $\max_{x\in[0,1]} - x^{1/q} \ln x = q/e$ and $\ln \dfrac{1 - c\rho_1}{\rho_1 - c} = \ln \tilde{\rho}_1 \ge \ln \hat{\rho}$.

As a result, we obtain from (3.7) that

$$\begin{aligned} |\mathrm{II}_1| &\le \|f^{(q)}\|_1 \left(\frac{1 + c\tilde{\rho}_1}{\tilde{\rho}_1 + c}\right)^{\lambda_k} n_1^q C_{11} \frac{e^q}{q-1} \Big/ n_1^q \\ &\quad + \|f^{(q)}\|_1 \left(\frac{1 + c\tilde{\rho}_1}{\tilde{\rho}_1 + c}\right)^{\lambda_k} \rho_1^{n_1} C_{12} \frac{1}{n_1^q \ln \hat{\rho}} \\ &\le \|f^{(q)}\|_1 C_{13} C_{11} \frac{e^q}{q-1} \Big/ n_1^q + \|f^{(q)}\|_1 C_{12} \frac{1}{n_1^q \ln \hat{\rho}} \\ &= \frac{C_1(c,q)}{\ln \hat{\rho}} \|f^{(q)}\|_1 / n_1^q \end{aligned} \tag{3.11}$$

with $C_1(c,q) = C_{11} \left(\dfrac{q}{e \ln \hat{\rho}}\right)^{q-1} \dfrac{q e^{q-1}}{q-1} + C_{12}$.

(*Third sum*) By Lemma 3.1, we have

$$\begin{aligned} |\mathrm{II}_3| &\le \|f^{(q)}\|_1 \rho_2^{\lambda_k} \sum_{n \ge n_2} \left(\frac{1 + c\rho_2}{\rho_2 + c}\right)^n \Big/ n^q \\ &\le \|f^{(q)}\|_1 \rho_2^{\lambda_k} \left(\frac{\rho_2 + c}{(\rho_2 - 1)(1 - c)}\right) \left(\frac{1 + c\rho_2}{\rho_2 + c}\right)^{n_2} \Big/ n_2^q. \end{aligned}$$

Define $\rho_2 = 1 + (1-c)^{q-1}$ and $n_2 = \lceil \varepsilon_2 \lambda_k \rceil$, where $\lceil x \rceil$ means the smallest integer not smaller than x and

$$\varepsilon_2 = \ln \rho_2 \Big/ \left(\ln \frac{\rho_2 + c}{1 + c\rho_2}\right) \ge 1,$$

because of $\dfrac{\rho_2 + c}{1 + c\rho_2} \le \rho_2$. Since $x \log(1 + x)$ is increasing on $[0, \infty)$, we obtain

$$\begin{aligned}
\varepsilon_2 &= \ln \rho_2 \Big/ \left(\ln \frac{\rho_2 + c}{1 + c\rho_2} \right) \\
&= \ln(1 + (1-c)^{q-1}) \Big/ \left(\ln \frac{(1 + (1-c)^{q-1}) + c}{1 + c(1 + (1-c)^{q-1})} \right) \\
&= \ln(1 + (1-c)^{q-1}) \Big/ \ln \left[1 + \frac{(1-c)^q}{1 + c + c(1-c)^{q-1}} \right] \\
&\geq \frac{1 + c + c(1-c)^{q-1}}{1-c} \geq \frac{1+c}{1-c}.
\end{aligned}$$

Thus $\rho_2 + c \leq 3$ and

$$\begin{aligned}
|\mathrm{II}_3| &\leq \|f^{(q)}\|_1 \frac{3}{(1-c)^q} \Big/ \lceil \varepsilon_2 \lambda_k \rceil^q \\
&\leq \|f^{(q)}\|_1 \frac{3}{((1-c)\varepsilon_2)^q} \Big/ \lambda_k^q \\
&\leq 3\|f^{(q)}\|_1 / \lambda_k^q.
\end{aligned}$$

(*Second sum*) Partitioning $[n_1 + 1, n_2 - 1]$ into two parts $[n_1 + 1, \lfloor \lambda_k \rfloor]$ and $[\lfloor \lambda_k \rfloor + 1, n_2 - 1]$, we may apply Lemma 3.1 to obtain,

$$\begin{aligned}
|\mathrm{II}_2| &\leq \|f^{(q)}\|_1 \sum_{n=n_1+1}^{n_2-1} \frac{1}{k^q} \\
&\leq \|f^{(q)}\|_1 \frac{C_2}{q-1} \left[\frac{1 - \varepsilon_1^{q-1}}{\varepsilon_1^{q-1}} + \frac{\varepsilon_2^{q-1} - 1}{\varepsilon_2^{q-1}} \right] \frac{1}{\lambda_k^{q-1}} \\
&\leq C_2 \frac{\varepsilon_1 \lambda_k}{q-1} ((1 - \varepsilon_1^{q-1}) + (\varepsilon_1)^{q-1}(1 - \varepsilon_2^{1-q})) \frac{\|f^{(q)}\|_1}{(\varepsilon_1 \lambda_k)^q} \\
&= 2C_2 \frac{\varepsilon_1 \lambda_k}{q-1} \frac{\|f^{(q)}\|_1}{(\varepsilon_1 \lambda_k)^q}.
\end{aligned}$$

Combing II_1, II_2 and II_3, we have

$$\begin{aligned}
|b_k| &\leq \left[\left(\frac{C_1(c,q)}{\ln \hat{\rho}} + 3\varepsilon_1^q \right) + 2C_2 \frac{\varepsilon_1 \lambda_k}{q-1} \right] \frac{\|f^{(q)}\|_1}{(\varepsilon_1 \lambda_k)^q} \\
&\leq \left[\frac{K_1(c,q)}{\ln \hat{\rho}} + K_2 \frac{\varepsilon \lambda_k}{q-1} \right] \frac{\|f^{(q)}\|_1}{(\varepsilon \lambda_k)^q}
\end{aligned}$$

with $\varepsilon = \varepsilon_1$, $K_1(c,q) = C_1(c,q) + 3 \ln \hat{\rho}$ and $K_2 = 2C_2$. The proof is completed. □

Corollary 3.4. *Let $f(t)$ satisfy the condition of Theorem* 3.3. *The frame operator $Sf(t)$ and the partial sum $(S_N f)(t)$ are defined by* (3.4) *and* (3.5) *respectively. Then*

$$\lim_{N\to\infty}(S_N f)(t) = Sf(t) \tag{3.12}$$

uniformly in $t \in [-\pi,\pi]$, and the convergence rate at least as fast as $1/\lambda_N^{q-2}$.

Proof. For N large enough, we have

$$\begin{aligned} |e_N(t)| = |(Sf)(t) - (S_N f)(t)| &= \left|\sum_{k\geq N}(b_k e^{-\lambda_k\theta_a(t)} + a_{-k}e^{\lambda_k\theta_a(t)})\right| \\ &\leq 2\sum_{k\geq N}\max(|b_k|,|a_{-k}|)|e^{-\lambda_k\theta_a(t)}| \\ &\leq 2\sum_{k\geq N}\left(K_1\frac{1}{\ln\rho} + K_2(c)\frac{\varepsilon\lambda_k}{q-1}\right)\|f^{(q)}\|_1/(\varepsilon\lambda_k)^q, \end{aligned}$$

where K_1, K_2 and $\tilde{\rho}$ is defined as in Theorem 3.3. Then the partial sum $(S_N f)(t)$ convergence to $Sf(t)$ uniformly in t with convergence rate $1/\lambda_N^{q-2}$ because of $q > 2$. The proof is completed. $\square$

Right now we have four methods of representing $f \in L^2(-\pi,\pi)$. They are Fourier basis $\{e^{ikt}\}_{k\in\mathbf{Z}}$, nonharmonic Fourier frame $\{e^{i\lambda_k t}\}_{k\in\mathbf{Z}}$, nonlinear Fourier basis $\{e^{ik\theta_a(t)}\}_{k\in\mathbf{Z}}$ and nonharmonic nonlinear Fourier frame $\{e^{i\lambda_k\theta_a(t)}\}_{k\in\mathbf{Z}}$. For convergence, it is well known that if a continuous function $f(t)$ is piecewise smooth on $[-\pi,\pi]$, then the traditional Fourier series of $f(t)$ is uniformly convergent.

Young [17, Theorem 15] discussed the equiconvergence of Fourier series and nonharmonic Fourier series of $f \in L^2(-\pi,\pi)$, and showed that nonharmonic Fourier series have, to a large extent, the same convergence and summability properties as traditional Fourier series. Next we shall discuss equiconvergence of two different (nonlinear) Fourier (basis or frame) series of $f \in L^2(-\pi,\pi)$. Recall that two series $\sum a_n$ and $\sum b_n$ are said to be **equiconvergent** if their difference $\sum(a_n - b_n)$ converges to 0.

Theorem 3.5. *Let $\{\lambda_k\}_{k\in\mathbf{Z}}$ be a real sequence. Let $\{e^{i\lambda_k\theta_a(t)}\}_{k\in\mathbf{Z}}$ be a Riesz basis for $L^2((-\pi,\pi),d\theta_a)$, and suppose that*

$$\sup_k|\lambda_k - k| < \infty.$$

Then for each function $f \in L^2((-\pi,\pi),d\theta_a)$, the nonlinear Fourier series and nonharmonic nonlinear Fourier series are uniformly equiconvergent on every compact subset of $(-\pi,\pi)$.

Proof. Suppose $\delta > 0$ and $f(t) \in L^2(-\pi, \pi)$ have two norm-convergent expansions: $\sum c_k e^{ik\theta_a(t)}$ and $\sum b_k e^{i\lambda_k \theta_a(t)}$. It is to be show that the difference $\sum_{k=-n}^{n}(c_k e^{ik\theta_a(t)} - b_k e^{i\lambda_k\theta_a(t)})$ convergence to 0 as $n \to \infty$, uniformly on $[-\pi+\delta, \pi-\delta]$.

Since $\|f(t)\|_{\theta_a} = \|f(\theta_{-a}(t))\|$, we have that $g(t) := f(\theta_{-a}(t)) \in L^2(-\pi, \pi)$, and has two expansion: $\sum c_k e^{ikt}$ and $\sum b_k e^{i\lambda_k t}$. Note that $\theta_a(t)$ is a strictly increasing bijective function with $\theta_a(\pm\pi) = \pm\pi$, by [17, Theorem 15], we have

$$\sum_{k=-n}^{n}(c_k e^{ikt} - b_k e^{i\lambda_k t}) \tag{3.13}$$

convergent to 0 uniformly on the interval $[\theta_{-a}(-\pi+\delta), \theta_{-a}(\pi-\delta)]$. If we replace t by $\theta_a(t)$ in (3.13), then the desired conclusion follows. □

Unfortunately, other versions of Theorem 3.5 probably fail when we consider the equiconvergence of traditional Fourier series and nonlinear Fourier series, or the equiconvergence of traditional Fourier series and nonharmonic nonlinear Fourier series of $f \in L^2(-\pi, \pi)$. In the end, we give two results for illustrating that argument.

Proposition 3.6. *Suppose $\{e^{ik\theta_a(t)}\}_{k\in\mathbf{Z}}$ be a orthonormal basis of $L^2((-\pi,\pi),\theta_a)$ with $a \in (-1, -\frac{1}{8}(\sqrt{35}-\sqrt{3}))$. Let a 2π periodic function be defined as*

$$f(t) = \begin{cases} 0, & -\pi \le t < 0, \\ 1, & 0 \le t \le \pi. \end{cases} \tag{3.14}$$

Then its traditional Fourier series and nonlinear Fourier series are not uniformly equiconvergent on $[-\pi+\delta, \pi-\delta]$ with $\delta < 5\pi/6$.

Proof. It is easy to get that $f(t) \in L^2(-\pi, \pi)$ and $f(t) \in L^2((-\pi,\pi),\theta_a)$. By calculation we have its traditional Fourier partial sum, $f_n(t) = 1/2 + \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1}\sin(2k-1)t$, and

$$\lim_{n\to\infty} f_n(t) = \begin{cases} f(t), & 0 < |t| < \pi, \\ 1/2, & t = -\pi, 0, \pi. \end{cases}$$

Next we discuss the corresponding non-linear Fourier partial sum,

$$f_n^{\theta_a}(t) = \sum_{k=-n}^{n}\langle f, e^{ik\theta_a(t)}\rangle_{\theta_a} e^{ik\theta_a(t)} = \sum_{k=-n}^{n}\langle f(\theta_{-a}(t)), e^{ikt}\rangle e^{ik\theta_a(t)}, \tag{3.15}$$

where $\theta_a^{-1}(t) = \theta_{-a}(t)$ is used. Note that $\theta_a(t) = t - 2\arctan\dfrac{|a|\sin t}{1+|a|\cos t}$, we have $\theta_a(t)$ is an increasing function with $\theta_a(-\pi) = -\pi$, $\theta_a(0) = 0$ and $\theta_a(\pi) = \pi$.

Hence $f(\theta_{-a}(t)) = f(t)$. Then

$$f_n^{\theta_a}(t) = \sum_{k=-n}^{n} \langle f, e^{ikt} \rangle e^{ik\theta_a(t)} = f_n(\theta_a(t)) = 1/2 + \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1}\sin(2k-1)\theta_a(t).$$

We define

$$g_n(t) := f_n(t) - f_n^{\theta_a}(t) = \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1}(\sin(2k-1)t - \sin(2k-1)\theta_a(t)).$$

In the following we show that, $g_n(t)$ is not uniformly convergent on $[-\pi+\delta, \pi-\delta]$. In fact for any $n > 0$, select $t_n = \dfrac{\pi/2}{4n-1}$, then

$$\sin((2k-1)t_n) \geq \sin((2n-1/2)\pi/(8n-2)) = \sqrt{2}/2, \quad \forall k = n+1\cdots 2n.$$

Note that $a \in (-1, -\frac{1}{8}(\sqrt{35}-\sqrt{3}))$, by direct calculation we have

$$\theta_a'(t) = \frac{1-a^2}{1+a^2-2a\cos(t)} \leq \frac{1-a^2}{1+a^2-2a\cos(\pi/6)} \leq 1/3,$$

and

$$\theta_a(t) \leq t/3, \quad \forall t \in (0, \pi/6).$$

Thus

$$\sin((2k-1)\theta_a(t_n)) \leq \sin((2k-1)t_n/3) \leq \sin(\pi/6) = 1/2, \quad \forall k = n+1\cdots 2n.$$

So we have

$$|g_{2n}(t_n) - g_n(t_n)| \geq \frac{2}{\pi}\sum_{k=n+1}^{2n}\frac{(\sqrt{2}-1)/2}{2k-1} \geq \frac{\sqrt{2}-1}{4\pi}. \tag{3.16}$$

Thus $g_n(t)$ is not uniformly convergent on $[-\pi+\delta, \pi-\delta]$ with $\delta < 5\pi/6$. The proof is completed. $\square$

Corollary 3.7. *Under the conditions of Theorem* 3.5, *the traditional Fourier series and nonharmonic nonlinear Fourier series of* $f(t)$, *defined in* (3.14), *are not uniformly equiconvergent on* $[-\pi+\delta, \pi-\delta]$ *with* $\delta < 5\pi/6$.

Proof. Suppose that $f(t)$ have three norm-convergent representations: $\sum d_n e^{ikt}$, $\sum c_n e^{ik\theta_a(t)}$ and $\sum b_n e^{i\lambda_k\theta_a(t)}$, where d_n and c_n have explicit expression as in Proposition 3.6. By Theorem 3.5, for $\varepsilon_0 = \dfrac{\sqrt{2}-1}{16\pi}$, there exists $N > 0$ such that for any $n > N$

$$|h_n(t)| = \left|\sum_{k=-n}^{n}(c_k e^{ik\theta_a(t)} - b_k e^{i\lambda_k\theta_a(t)})\right| < \varepsilon_0$$

with $t \in [-\pi+\delta, \pi-\delta]$ and $\delta \in (0,\pi)$. Then by the proof of Proposition 3.6, we have

$$\begin{aligned}\left|\sum_{|k|=n+1}^{2n}\left(d_k e^{ikt_n} - b_k e^{i\lambda_k\theta_a(t_n)}\right)\right| &\geq \left|\sum_{|k|=n+1}^{2n}\left(d_k e^{ikt_n} - c_k e^{i\lambda_k\theta_a(t_n)}\right)\right| \\ &\quad - \left|\sum_{|k|=n+1}^{2n}\left(c_k e^{ik\theta_a(t_n)} - b_k e^{i\lambda_k\theta_a(t_n)}\right)\right| \\ &\geq |g_{2n}(t_n) - g_n(t_n)| - |h_{2n}(t_n)| - |h_n(t_n)| \\ &\geq \frac{2}{\pi}\sum_{k=n+1}^{2n}\frac{(\sqrt{2}-1)/2}{2k-1} - 2\varepsilon_0 = \frac{\sqrt{2}-1}{8\pi},\end{aligned}$$

where $t_n = \dfrac{\pi/2}{4n-1}$. Thus $\sum_{k=-n}^{n}(d_k e^{ikt} - b_k e^{i\lambda_k\theta_a(t)})$ is not uniformly convergent on $[-\pi+\delta, \pi-\delta]$ with $\delta < 5\pi/6$. The proof is completed. □

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