

A NOTE ON FONTAINE THEORY USING DIFFERENT LUBIN-TATE GROUPS

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Abstract

The starting point of Fontaine theory is the possibility of translating the study of a p -adic representation of the absolute Galois group of a finite extension K of \mathbf{Q}_p into the investigation of a (φ, Γ) -module. This is done by decomposing the Galois group along a totally ramified extension of K , via the theory of the field of norms: the extension used is obtained by means of the cyclotomic tower which, in turn, is associated to the multiplicative Lubin-Tate group. It is known that one can insert different Lubin-Tate groups into the “Fontaine theory” machine to obtain equivalences with new categories of (φ, Γ) -modules (here φ may be iterated). This article uses only (φ, Γ) -theoretical terms to compare the different (φ, Γ) modules arising from various Lubin-Tate groups.

1. Introduction and notations

Let K be a finite extension of \mathbf{Q}_p , \bar{K} an algebraic closure of K , $G_K = \text{Gal}(\bar{K}/K)$ its absolute Galois group. In [2], Fontaine shows how the study of representations of G_K over finite dimensional \mathbf{Q}_p -vector spaces is reduced to the study of an algebraic object, namely, the étale (φ, Γ) -module attached to it. In particular, he shows that there is a functor D between the category $\mathbf{Rep}_{G_K}(\mathbf{Q}_p)$ of p -adic representations of G_K and the category $\Phi_{\varphi, \Gamma}^{et}(\mathbf{B}_K)$ of étale (φ, Γ) -modules over the discretely valued field \mathbf{B}_K , giving an equivalence of categories. The functor D is defined as follows: $D(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B})^{H_K}$, where \mathbf{B} is the completion of an unramified closure of \mathbf{B}_K ; the group H_K is the kernel of the cyclotomic character $\chi_p : G_K \rightarrow \mathbf{Z}_p^\times$.

The field \mathbf{B}_K has as residue field the field of norms \mathbf{E}_K of the extension $\bigcup_n K(\zeta_{p^n})$ of K , obtained from K by adding the p^n -th roots of 1 for every n . The field \mathbf{B}_K is endowed with the action of $\Gamma_K = \text{Im}(\chi_p)$ and a commuting action of the Frobenius which lift those on \mathbf{E}_K .

The p^n -th roots of unity are the p^n -th torsion points of the multiplicative Lubin-Tate formal group on \mathbf{Q}_p , associated to the uniformizer p . The cyclotomic tower is exactly the tower associated by Lubin-Tate theory to the multi-

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plicative Lubin-Tate formal group. It is natural to try to carry out the theory for an arbitrary Lubin-Tate formal group \mathcal{F} over a subfield F of K , not only for \mathbf{Q}_p (Partial results in this direction have been established by Kisin and Ren [6] and Fourquaux [4]).

We now give a more detailed description of our work along with that of [6]. Let F be an extension of \mathbf{Q}_p contained in K : \mathcal{O}_F (resp. \mathcal{O}_K) the ring of integers, k_F (resp. k_K) the residue field, p^r (resp. p^{rs}) the cardinality of k_F (resp. k_K), and π (resp. ϖ) a uniformizer of F (resp. K). We will denote by F_0 (resp. K_0) the maximal unramified extension of \mathbf{Q}_p in F (resp. in K).

Let F_π denote the maximal abelian totally ramified extension of F associated to π by Lubin-Tate theory; denote by $\chi_\pi : G_F \rightarrow \mathcal{O}_F^\times$ its associated character. Let $X_K(K \cdot F_\pi) = \mathbf{E}_{K,\pi}$ be the field of norms of the extension of K which is the compositum of K and F_π , and let its Galois group be denoted by $H_{K,\pi}$, so that $H_{K,\pi} = G_K \cap \text{Ker}(\chi_\pi)$. The field $X_K(K \cdot F_\pi) = \mathbf{E}_{K,\pi}$ is endowed with an action of $\Gamma_{K,\pi} = \text{Gal}(K \cdot F_\pi/K)$ and a commuting action of the Frobenius morphism φ^r (relative to k_F). Kisin and Ren construct (under the hypothesis of the inclusion of K in the compositum $K_0 \cdot F_\pi$ [6] 2.1) a complete discrete valuation ring, $\mathcal{O}_\mathcal{E}$ (in our notation $\mathbf{A}_{F,\pi}(K)$), with residue field $X_K(K \cdot F_\pi) = \mathbf{E}_{K,\pi}$, unramified over \mathcal{O}_F . They are able to lift the commuting actions of $\Gamma_{K,\pi}$ and φ^r on $X_K(K \cdot F_\pi)$ to commuting actions of $\Gamma_{K,\pi}$ and φ^r on $\mathcal{O}_\mathcal{E}$ ($\mathbf{A}_{K,\pi}(K)$ in our notation). Let $\mathbf{Rep}_{G_K}(\mathcal{O}_F)$ be the category of representations of G_K on finite free \mathcal{O}_F -modules and $\Phi_{\varphi^r, \Gamma_{K,\pi}}^{et}(\mathbf{A}_{K,\pi}(K))$ be the category of finite free $\mathcal{O}_\mathcal{E}(= \mathbf{A}_{K,\pi}(K))$ -modules endowed with an étale action of φ^r and a commuting action of $\Gamma_{K,\pi}$. Kisin and Ren prove that the functor $V \mapsto (V \otimes_{\mathcal{O}_{\hat{\mathcal{E}}^{nr}}} \mathcal{O}_{\hat{\mathcal{E}}^{nr}})^{H_{K,\pi}}$ is an equivalence of categories between $\mathbf{Rep}_{G_K}(\mathcal{O}_F)$ and $\Phi_{\varphi^r, \Gamma_{K,\pi}}^{et}(\mathcal{O}_\mathcal{E})$, with as quasi-inverse the functor $M \mapsto (M \otimes_{\mathcal{O}_{\hat{\mathcal{E}}^{nr}}} \mathcal{O}_{\hat{\mathcal{E}}^{nr}})^{\varphi^r=1}$, where $H_{K,\pi}$ is the absolute Galois group of $K \cdot F_\pi$. In this paper we want to give a more general formulation to these results: we won't impose any condition on the extension K/F and we will treat \mathcal{O}_K -representations. Such results are perhaps known in the math common knowledge (see, for example, Fourquaux in 1.4.1 of [4]), but we are not aware of any such explicit statement in the literature. Namely, we prove that starting with a Lubin-Tate group \mathcal{F} over F associated to the uniformizer π and for any finite extension K , one may construct a complete discrete valuation ring $\mathbf{A}_{K,\pi}$ with residue field the field of norms $X_K(K \cdot F_\pi) = \mathbf{E}_{K,\pi}$ and unramified over \mathcal{O}_K ; moreover, $\mathbf{A}_{K,\pi}$ is endowed with commuting actions of G_K and φ^{rs} , lifting those on $X_K(K \cdot F_\pi) = \mathbf{E}_{K,\pi}$. We define as well, two functors D_π and V_π . We prove that the functor D_π is an equivalence between $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$ and $\Phi_{\varphi^{rs}, \Gamma_{K,\pi}}^{et}(\mathbf{A}_{K,\pi})$; and V_π its quasi-inverse going from $\Phi_{\varphi^{rs}, \Gamma_{K,\pi}}^{et}(\mathbf{A}_{K,\pi})$ to $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$.

Naturally one could have started with $F = K$ in the previous setting. Then one would study the Lubin-Tate group in K associated to a choice ϖ of a uniformizer of K . Here the associated totally ramified tower will define a factorization of G_K by $H_{K,\varpi}$ and $\Gamma_{K,\varpi}$ and the theory of the field of norms will yield a field $\mathbf{E}_{K,\varpi}$ together with a lifting $\mathbf{A}_{K,\varpi}$. Finally one will have a new equivalence between $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$ and $\Phi_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{A}_{K,\varpi})$. In particular these two

categories of (φ, Γ) -modules, namely $\Phi_{\varphi^{rs}, \Gamma_{K, \pi}}^{et}(\mathbf{A}_{K, \pi})$ and $\Phi_{\varphi^{rs}, \Gamma_{K, \varpi}}^{et}(\mathbf{A}_{K, \varpi})$ are equivalent: they are equivalent to the same category of representations. In the second part of our present work we will give a proof of the equivalence in purely (φ, Γ) -module theoretic terms.

Let us now review in more detail the content of the various sections. In section 2, we deal with generalities on Lubin-Tate theory. Namely we study maximal abelian ramified extensions of F and K given by Lubin-Tate theories and their relative position. In particular we characterize when the two extensions are nested one inside the other. In section 3, we study the fields of norms attached to two maximal abelian ramified extensions of F and K via Lubin-Tate groups of different heights. We consider their relative position inside $\bar{\mathbf{E}} = \lim \mathbf{C}_p$. Here, the main result is that the two extensions overlap only on the field of constants. In section 4, we study the equivalences in positive characteristic between $\mathbf{Rep}_{G_K}(\mathbf{F}_{p^{rs}})$, $\Phi_{\varphi, \Gamma_{K, \pi}}^{et}(\mathbf{E}_{K, \pi})$ and $\Phi_{\varphi, \Gamma_{K, \varpi}}^{et}(\mathbf{E}_{K, \varpi})$. In particular, we describe the equivalence between the last two of these categories without appealing to the category $\mathbf{Rep}_{G_K}(\mathbf{F}_{p^{rs}})$. In section 5, we begin to lift the theory to characteristic 0. In particular, we recall and adapt to our present case the construction of Kisin and Ren of the equivalence between the category of representations of G_F on finite free \mathcal{O}_F -modules, and the category of étale (φ', Γ_F) -modules on finite free $\mathbf{A}_{F, \pi}$ modules (note that Γ_F is linked to the totally ramified tower over F given by π). In section 6, we construct a complete discrete valuation ring $\mathbf{A}_{K, \pi}$ endowed with actions of G_K and φ^{rs} and prove that the category $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$ is equivalent to the category $\Phi_{\varphi^{rs}, \Gamma_{K, \pi}}^{et}(\mathbf{A}_{K, \pi})$ of étale $(\varphi^{rs}, \Gamma_{K, \pi})$ -modules on finite free $\mathbf{A}_{K, \pi}$ -modules. In section 7, we give our main comparison theorem between two categories of (φ, Γ) -modules. We express these equivalence functors with the help of a system of rings $\mathbf{A}_{\pi, \varpi, \bullet}$ endowed with partial Frobenius maps and an action of G_K .

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2. Lubin-Tate theory of F and K

Lubin-Tate theory gives an explicit construction of the maximal abelian extension of a local field and of the norm residue map of local class field theory ([5], [8]).

PROPOSITION 2.1. *Let $F \subset K$ be an extension of local fields, with K finite over \mathbf{Q}_p . Let $\pi \in \mathcal{O}_F$ be a uniformizer of F , $\varpi \in \mathcal{O}_K$ be a uniformizer of K , let F_π and K_ϖ be the Lubin-Tate extensions of F and K respectively, attached to π and ϖ . Then*

- (i) $F_\pi \subset K_\varpi$ if and only if $N_{K/F}(\varpi) = \pi^l$, where l is the index of inertia of the extension K/F .
- (ii) If K/F is Galois, then K_ϖ is Galois over F if and only if K/F is unramified and $\varpi \in F$.

Proof.

(i) By [8] or [7], it follows that $\text{Gal}(F^{ab}/K_\varpi \cap F^{ab})$ is the completion of $N_{K/F}(\langle \varpi \rangle)$, hence by Galois correspondence $F_\pi \subset K_\varpi$ if and only if $N_{K/F}(\langle \varpi \rangle) \subset \langle \pi \rangle$, which is easily seen to be equivalent to $N_{K/F}(\varpi) = \pi^l$.

(ii) Since K/F is Galois, we have that K^{ab}/F is Galois. By functoriality of the norm residue symbol, it follows that, if $\sigma \in \text{Gal}(\bar{F}/F)$, then $\text{Gal}(K^{ab}/K_\varpi^\sigma)$ is the completion of $\langle \varpi^\sigma \rangle$. \square

Remark. If K/F is ramified, the norm map $N_{K/F}$ from \mathcal{O}_K^\times to \mathcal{O}_F^\times is not surjective. This implies that for a general choice of π there need not be a ϖ such that $F_\pi \subset K_\varpi$; in fact, one can find such K_ϖ if and only if $K \cdot F_\pi$ is totally ramified over K . The fact that the image of the norm map is of finite index in \mathcal{O}_F^\times corresponds to the fact that upon taking a finite unramified extension K' of K it is always possible to find such a ϖ ; the smallest possible choice of K' being the unramified closure of K inside $K \cdot F_\pi$.

3. The fields of norms

Having chosen uniformizers $\pi \in \mathcal{O}_F$ and $\varpi \in \mathcal{O}_K$ such that $F_\pi \subset K_\varpi$, let \mathcal{F}_π be a Lubin-Tate group over \mathcal{O}_F attached to the uniformizer π (see [7]) and let $[\pi]$ be the multiplication by π in \mathcal{F}_π . Let $\Lambda_{\pi,n}$ be the submodule of π^n -torsion elements in the maximal ideal of $\mathcal{O}_{\bar{F}}$, i.e., $\Lambda_{\pi,n} = \{x \in \bar{F} \mid v_F(x) > 0 \text{ and } [\pi^n](x) = 0\}$, and let $K_{\pi,n}$ denote the extension of K generated by $\Lambda_{\pi,n}$. We may define the field of norms, denoted $X_K(K \cdot F_\pi)$ in [9], of the extension $K \cdot F_\pi/K$ as follows:

$$X_K(K \cdot F_\pi) = \varprojlim K_{\pi,n},$$

where the projective limit is taken according to the norm maps.

In the same way, \mathcal{F}_ϖ denotes a Lubin-Tate group over \mathcal{O}_K attached to the uniformizer ϖ , $\Lambda_{\varpi,n} = \{x \in \bar{K} \mid v_K(x) > 0 \text{ and } [\varpi^n](x) = 0\}$ and $K_{\varpi,n} = K[\Lambda_{\varpi,n}]$. We may define, according to [9], the field of norms $X_K(K_\varpi)$, of the extension K_ϖ/K , as follows:

$$X_K(K_\varpi) = \varprojlim K_{\varpi,n},$$

where the projective limit is again taken according to the norm maps.

In what follows, we will denote $\mathbf{E}_{K,\pi}$ the field of norms of the extension $K \cdot F_\pi/K$, and $\mathbf{E}_{K,\varpi}$ the field of norms of the extension K_ϖ/K .

We recall some results of [9], and give the main results concerning the relation between $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$.

THEOREM 3.1 (see [9] Theorem 2.1.3). $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$ have the structure of complete discretely valued fields of characteristic p with residue field isomorphic to k_K , the residue field of K ; i.e. $\mathbf{E}_{K,\pi} \cong k_K((x))$ and $\mathbf{E}_{K,\varpi} \cong k_K((y))$. Moreover, $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$, are endowed with a natural action of $\Gamma_{K,\varpi} = \Gamma_K = \text{Gal}(K_\varpi/K) \cong \mathcal{O}_K^\times$, which on $\mathbf{E}_{K,\pi}$ factors through its image $\Gamma_{K,\pi} \subset \Gamma_F \cong \mathcal{O}_F^\times$ by the norm map $N_{K/F}$.

Let $H_{K,\pi}, H_{K,\varpi}$ be respectively $\text{Gal}(\bar{K}/K \cdot F_\pi)$ and $\text{Gal}(\bar{K}/K_\varpi)$. We observe that $H_{K,\pi}/H_{K,\varpi}$ is isomorphic to the kernel of the norm map from \mathcal{O}_K^\times to \mathcal{O}_F^\times . Let \mathbf{E}_π be a separable algebraic closure of $\mathbf{E}_{K,\pi}$ and $G_{\mathbf{E}_{K,\pi}}$ be the absolute Galois group of $\mathbf{E}_{K,\pi}$; we will use the analogous notation for $\mathbf{E}_{K,\varpi}$.

THEOREM 3.2 (see [9] Corollary 3.2.3, Proposition 3.4.1). *The group $G_{\mathbf{E}_{K,\pi}}$ is canonically isomorphic to $H_{K,\pi}$ and $G_{\mathbf{E}_{K,\varpi}}$ is canonically isomorphic to $H_{K,\varpi}$; moreover the field of norms $\mathbf{E}_{K,\varpi}$ is isomorphic to the field of norms of the extension $\mathbf{E}_\pi^{H_{K,\varpi}}/\mathbf{E}_{K,\pi}$ and this isomorphism respects the action of $H_{K,\pi}$ (which acts through the quotient $H_{K,\pi}/H_{K,\varpi}$).*

Let \mathbf{C}_p denote the completion of \bar{F} . Let us consider the field $\tilde{\mathbf{E}}$ defined as the projective limit of copies of \mathbf{C}_p by the map $x \mapsto x^p$:

$$\tilde{\mathbf{E}} = \varprojlim \mathbf{C}_p.$$

The basic properties of $\tilde{\mathbf{E}}$ are summarized in the following result (see [3], [9]).

THEOREM 3.3. *The field $\tilde{\mathbf{E}}$ is an algebraically closed complete valued field of characteristic p endowed with an action of $\text{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$. Moreover, the fields of norms $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$ have a natural Galois-equivariant immersion in $\tilde{\mathbf{E}}$ and the separable algebraic closures \mathbf{E}_π and \mathbf{E}_ϖ of $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$ respectively, are dense inside $\tilde{\mathbf{E}}$.*

A consequence of Theorem 3.3 and the Theorem of Ax-Sen-Tate is the following.

COROLLARY 3.4. *The fixed fields $\tilde{\mathbf{E}}_{K,\pi} = \tilde{\mathbf{E}}^{H_{K,\pi}}$ and $\tilde{\mathbf{E}}_{K,\varpi} = \tilde{\mathbf{E}}^{H_{K,\varpi}}$ are respectively the completions of the perfections of $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$.*

Remark. Since $H_{K,\varpi} \subset H_{K,\pi}$ one has $\tilde{\mathbf{E}}_{K,\pi} \subset \tilde{\mathbf{E}}_{K,\varpi}$, i.e., the completions of the perfect closure of the fields of norms are included one in the other in a Galois equivariant way. This does not continue to hold at the level of the fields of norms, even if $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$ are both abstractly isomorphic to the field $k_K((t))$ for an indeterminate t .

We now show that, if one takes into account the action of $H_{K,\pi}$, the only possible common subfield of $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$ is their residue field k_K .

LEMMA 3.5. *If $F \neq K$, we have $\mathbf{E}_{K,\varpi}^{H_{K,\pi}} = k_K$.*

Proof. As stated in Theorem 3.2, the field $\mathbf{E}_{K,\varpi}$ may be identified with the field of norms of the extension $\mathbf{E}_{\pi}^{H_{K,\varpi}}/\mathbf{E}_{K,\pi}$, which is the projective limit along the inverse system of finite separable subextensions of $\mathbf{E}_{\pi}^{H_{K,\varpi}}/\mathbf{E}_{K,\pi}$ by the norm maps. The extension $\mathbf{E}_{\pi}^{H_{K,\varpi}}/\mathbf{E}_{K,\pi}$ is Galois with Galois group $H_{K,\pi}/H_{K,\varpi}$. This group is isomorphic to the kernel of the norm map $N_{K/F} : \mathcal{O}_K^{\times} \rightarrow \mathcal{O}_F^{\times}$, hence it is abelian. This implies that any finite separable subextension of $\mathbf{E}_{\pi}^{H_{K,\varpi}}/\mathbf{E}_{K,\pi}$ is necessarily Galois; which implies that the group $H_{K,\pi}$ acts on the projective limit componentwise. Thus the elements fixed by $H_{K,\pi}$ have all components in $\mathbf{E}_{K,\pi} = \mathbf{E}_{\pi}^{H_{K,\pi}}$ and the norm maps coincide with the elevation to the degree of the extension on $\mathbf{E}_{K,\pi}$.

Observe that the group \mathcal{O}_K^{\times} is a p -adic Lie group of dimension $[K : \mathbf{Q}_p]$ and \mathcal{O}_F^{\times} is a p -adic Lie group of dimension $[F : \mathbf{Q}_p]$. The norm map $N : \mathcal{O}_K^{\times} \rightarrow \mathcal{O}_F^{\times}$ is an open morphism of p -adic Lie groups. This implies that the kernel of N is a p -adic Lie group of dimension $[K : \mathbf{Q}_p] - [F : \mathbf{Q}_p]$. Hence, except when $F = K$, the group $H_{K,\pi}/H_{K,\varpi}$ is infinite.

We use U_1 to denote the subgroup of \mathcal{O}_K^{\times} of those elements congruent to 1 modulo the maximal ideal of \mathcal{O}_K^{\times} , and V_1 to indicate the subgroup of \mathcal{O}_F^{\times} of those elements congruent to 1 modulo the maximal ideal of \mathcal{O}_F^{\times} . It is well known that U_1 and V_1 are pro- p -groups of finite index in \mathcal{O}_K^{\times} and \mathcal{O}_F^{\times} respectively, and that the map N sends U_1 to V_1 . This implies that the kernel of N contains a pro- p -subgroup of finite index, namely the intersection of the kernel of N with U_1 .

This proves that, in the case $F \neq K$, the group $H_{K,\pi}/H_{K,\varpi}$ has a finite index subgroup which is an infinite pro- p -group.

Since the inverse limit does not change upon restricting to a cofinal set of indices (and in particular to a finite extension of the base field $\mathbf{E}_{K,\pi}$): one may take the cofinal set of extensions which is made up of all finite extensions of the extension of $\mathbf{E}_{K,\pi}$ which corresponds to this finite index infinite pro- p -group. In this way the extensions are all p -extensions of one another. So an element of the field of norms of the extension $\mathbf{E}_{\pi}^{H_{K,\varpi}}/\mathbf{E}_{K,\pi}$ is seen to be an infinite sequence (x_i) of elements of $\mathbf{E}_{K,\pi}$ coherent under the Frobenius morphism: $x_i = x_{i+1}^p$. So, $\forall i, j \in \mathbf{N}$, $x_i \in (\mathbf{E}_{K,\pi})^{p^j}$. But the intersection of all $(\mathbf{E}_{K,\pi})^{p^j}$ is the field of constants k_K of $\mathbf{E}_{K,\pi}$. Hence the elements of the field of norms of the extension $\mathbf{E}_{\pi}^{H_{K,\varpi}}/\mathbf{E}_{K,\pi}$ are exactly those given by taking coherent sequences of elements of k_K under the Frobenius, and so the field of norms is isomorphic to k_K . \square

COROLLARY 3.6. *Suppose $F \neq K$. Then inside $\tilde{\mathbf{E}}$ one has that the following holds for the intersection of $\mathbf{E}_{K,\varpi}$ and $\tilde{\mathbf{E}}_{K,\pi}$:*

$$\mathbf{E}_{K,\varpi} \cap \tilde{\mathbf{E}}_{K,\pi} = k_K.$$

The statement continues to hold even after replacing $\mathbf{E}_{K,\varpi}$ by its perfect closure inside $\tilde{\mathbf{E}}$. Moreover, if $x \in \tilde{\mathbf{E}}$ is a generating element of $\mathbf{E}_{K,\pi}$ over k_K and $y \in \tilde{\mathbf{E}}$ is a generating element of $\mathbf{E}_{K,\varpi}$ over k_K , then x and y are algebraically independent over k_K .

Proof. The first assertion is a consequence of Lemma 3.5 because of the fact that $\mathbf{E}_{K,\varpi} \cap \tilde{\mathbf{E}}_{K,\pi}$ is included in $\mathbf{E}_{K,\varpi}^{H_{K,\pi}}$. On the other hand, if α is an element in $\tilde{\mathbf{E}}$ which belongs to $\tilde{\mathbf{E}}_{K,\pi}$ and is purely inseparable over $\mathbf{E}_{K,\varpi}$, then a sufficiently high p -power of α will be in $\mathbf{E}_{K,\varpi} \cap \tilde{\mathbf{E}}_{K,\pi}$. Hence α itself will be in k_K .

To prove the second, one may argue as follows. If x and y were to satisfy an algebraic relation with coefficients in k_K , then y would be algebraic over $\tilde{\mathbf{E}}_{K,\pi}$ and the group of the continuous automorphisms of $\tilde{\mathbf{E}}$ which fix y and $\tilde{\mathbf{E}}_{K,\pi}$ would have finite index in the group of the continuous automorphisms of $\tilde{\mathbf{E}}$ which fix $\tilde{\mathbf{E}}_{K,\pi}$. By Corollary 3.4, the first group is $H_{K,\varpi}$ and the second is $H_{K,\pi}$. Since $H_{K,\varpi}$ is not of finite index in $H_{K,\pi}$, x and y are algebraically independent over k_K . \square

COROLLARY 3.7. *In the case $F \neq K$, there is no G_K -equivariant inclusion of $\mathbf{E}_{K,\pi}$ in $\mathbf{E}_{K,\varpi}$, nor is there one between their perfect closures.*

A basic consequence of Corollary 3.6 is the following.

LEMMA 3.8. *In the case $F \neq K$, in $\tilde{\mathbf{E}}$, we have $\mathbf{E}_\pi \cap \mathbf{E}_\varpi = \bar{k}_K$, and the two extensions are algebraically independent over \bar{k}_K .*

Proof. By Corollary 3.6, we know that $\mathbf{E}_{K,\pi} = k_K((x))$ and $\mathbf{E}_{K,\varpi} = k_K((y))$, with x and y algebraically independent. Suppose that $\alpha \in \mathbf{E}_\pi \cap \mathbf{E}_\varpi$. The element α has a minimal polynomial $P(T) \in \mathbf{E}_{K,\pi}[T]$ over $\mathbf{E}_{K,\pi}$ and a minimal polynomial $Q(T) \in \mathbf{E}_{K,\varpi}[T]$ over $\mathbf{E}_{K,\varpi}$. These polynomials will still be irreducible over the composite field of $\mathbf{E}_{K,\pi}$ and $\mathbf{E}_{K,\varpi}$ inside $\tilde{\mathbf{E}}$ since x and y are algebraically independent variables over k_K . Hence the polynomials must coincide, and thus have coefficients in k_K . \square

4. The system of fields $\mathbf{E}_{\pi,\varpi,\bullet}$

We first develop the theory in characteristic p . Let W be a representation of G_K over \mathbf{F}_p . To W we may associate $D_\pi(W) = (W \otimes_{\mathbf{F}_p} \mathbf{E}_\pi)^{H_{K,\pi}}$, which is a $(\varphi, \Gamma_{K,\pi})$ -module over the field of norms $\mathbf{E}_{K,\pi}$. A fundamental result of [2] is that the functor D_π gives an equivalence of categories between $\mathbf{Rep}_{G_K}(\mathbf{F}_p)$ and $\Phi_{\varphi, \Gamma_{K,\pi}}^{et}(\mathbf{E}_{K,\pi})$. Analogously, the functor $D_\varpi(W) = (W \otimes_{\mathbf{F}_p} \mathbf{E}_\varpi)^{H_{K,\varpi}}$ gives an equivalence of categories between $\mathbf{Rep}_{G_K}(\mathbf{F}_p)$ and $\Phi_{\varphi, \Gamma_{K,\varpi}}^{et}(\mathbf{E}_{K,\varpi})$.

We show that the argument in [2] may be applied to prove that there is an equivalence between the category $\mathbf{Rep}_{G_K}(k_K)$ and the category $\Phi_{\varphi^{fs}, \Gamma_{K,\pi}}^{et}(\mathbf{E}_{K,\pi})$.

THEOREM 4.1. *The functor $D_\pi(W) = (W \otimes_{k_K} \mathbf{E}_\pi)^{H_{K,\pi}}$ is an equivalence of categories between the category $\mathbf{Rep}_{G_K}(k_K)$ and the category $\Phi_{\varphi^{fs}, \Gamma_{K,\pi}}^{et}(\mathbf{E}_{K,\pi})$. The functor $V_\pi(D) = (D \otimes_{\mathbf{E}_{K,\pi}} \mathbf{E}_\pi)^{\varphi^{fs}=1}$ is quasi-inverse to D .*

Proof. Our argument is an adaptation of Fontaine’s proof [2] (see also [6]). There are natural maps

$$D_\pi(W) \otimes_{\mathbf{E}_{K,\pi}} \mathbf{E}_\pi \rightarrow W \otimes_{k_K} \mathbf{E}_\pi,$$

and

$$V_\pi(D) \otimes_{k_K} \mathbf{E}_\pi \rightarrow D \otimes_{\mathbf{E}_{K,\pi}} \mathbf{E}_\pi.$$

These natural maps commute with the action of G_K and φ^{rs} , so to prove the statement of the theorem it is sufficient to prove that these two natural maps are isomorphisms.

Since $H_{K,\pi}$ is the Galois group of the extension \mathbf{E}_π over $\mathbf{E}_{K,\pi}$, the first map is an isomorphism by Hilbert’s Theorem 90, i.e. $H^1(H_{K,\pi}, \mathrm{GL}(n, \mathbf{E}_\pi)) = \{1\}$.

For the second map, observe that it is injective. In fact, a family of elements of $V_\pi(D)$ which is linearly independent over k_K will remain linearly independent over \mathbf{E}_π . Indeed, let us take a family $\eta_1, \dots, \eta_t \in V_\pi(D)$ of elements which are independent over k_K and such that any strict subfamily of this family remains linearly independent over \mathbf{E}_π . Suppose that the elements $x_1, \dots, x_t \in \mathbf{E}_\pi$ are all nonzero and such that $x_1\eta_1 + \dots + x_t\eta_t = 0$. Dividing by x_1 , we may suppose $x_1 = 1$. Now, applying φ^{rs} to the relation we get $\eta_1 + x_2^{p^{rs}}\eta_2 + \dots + x_t^{p^{rs}}\eta_t = 0$; so, by subtracting one relation from the other one gets $(x_2^{p^{rs}} - x_2)\eta_2 + \dots + (x_t^{p^{rs}} - x_t)\eta_t = 0$. This implies $x_j^{p^{rs}} - x_j = 0$, i.e. all coefficients x_1, \dots, x_t are in k_K .

Hence to prove the second map bijective, it is sufficient to prove that the dimension of $V_\pi(D)$ over k_K is the same as the dimension of D over $\mathbf{E}_{K,\pi}$.

Let us choose a basis of the $\mathbf{E}_{K,\pi}$ -vector space D^* , dual of D , so that D^* is identified with the $\mathbf{E}_{K,\pi}$ -vector space $\mathbf{E}_{K,\pi}^d$. The action of φ^{rs} is identified with an invertible matrix $(a_{j,l})$ in $\mathbf{E}_{K,\pi}$; i.e.,

$$\varphi^{rs}u_j = \sum_{1 \leq l \leq d} a_{j,l}u_l.$$

The k_K -vector space $V_\pi(D)$ is then identified with the k_K -vector space of the $\mathbf{E}_{K,\pi}$ -linear application from D^* to \mathbf{E}_π which commute with φ^{rs} , hence with the d -uples $(x_j)_{1 \leq j \leq d}$ in \mathbf{E}_π verifying

$$x_j^{p^{rs}} = \sum a_{j,l}x_l.$$

So $V_\pi(D)$ is the set of \mathbf{E}_π -points of the étale $\mathbf{E}_{K,\pi}$ -algebra $C = \mathbf{E}_{K,\pi}[X_1, \dots, X_d] / (X_j^{p^{rs}} - \sum a_{j,l}X_l)_{1 \leq j \leq d}$. Since C has rank p^{rsd} , the dimension of $V_\pi(D)$ over k_K is d . \square

Naturally, the same arguments work in the case of D_ϖ and V_ϖ . Thus we have also the following.

THEOREM 4.2. *The functor $D_\varpi(W) = (W \otimes_{k_K} \mathbf{E}_\varpi)^{H_{K,\varpi}}$ is an equivalence of categories between the category $\mathbf{Rep}_{G_K}(k_K)$ and the category $\mathbf{\Phi}_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{E}_{K,\varpi})$. The functor $V_\varpi(D) = (D \otimes_{\mathbf{E}_{K,\varpi}})^{\varphi^{rs}=1}$ is quasi-inverse to D_ϖ .*

Proof. As in 4.1. \square

We now wish to compare the modules $D_\pi(W)$ and $D_\varpi(W)$.

Let σ be the rs -th power of the Frobenius automorphism of \bar{k}_K . We define $\mathbf{E}_{\pi,\varpi,i}$ as the field of fractions of the ring $\mathbf{E}_\pi \otimes_{\bar{k}_K, \sigma^i} \mathbf{E}_\varpi$. By Lemma 3.8, the two fields are algebraically independent over \bar{k}_K , so the tensor product is a domain. The subscript σ^i means that if $\lambda \in \bar{k}_K$, in $\mathbf{E}_\pi \otimes_{\bar{k}_K, \sigma^i} \mathbf{E}_\varpi$, we have $\lambda \otimes 1 = 1 \otimes \sigma^i(\lambda)$. On every field $\mathbf{E}_{\pi,\varpi,i}$ there is an action of G_K , acting diagonally on \mathbf{E}_π and \mathbf{E}_ϖ . Moreover, the rs -th iterate φ_π of the Frobenius morphism of \mathbf{E}_π induces maps $\varphi_\pi : \mathbf{E}_{\pi,\varpi,i} \rightarrow \mathbf{E}_{\pi,\varpi,i-1}$; the rs -th iterate φ_ϖ of the Frobenius morphism of \mathbf{E}_ϖ induces maps $\varphi_\varpi : \mathbf{E}_{\pi,\varpi,i} \rightarrow \mathbf{E}_{\pi,\varpi,i+1}$. These maps commute with the action of G_K , and $\varphi_\pi \circ \varphi_\varpi = \varphi_\varpi \circ \varphi_\pi = \varphi^{rs}$ the rs -th iterate of the absolute Frobenius morphism. So we have a doubly projective system of fields with a G_K -action.

PROPOSITION 4.3. *The projective limit of the system $\mathbf{E}_{\pi,\varpi,\bullet}$ along φ_π is \mathbf{E}_ϖ , and the morphisms φ_ϖ are transformed in the rs -th iterate φ_ϖ of the Frobenius morphism of \mathbf{E}_ϖ . The symmetrical statement holds also.*

Proof. The field \mathbf{E}_ϖ is the union of the finite separable extensions of $\mathbf{E}_{K,\varpi}$. These extensions are all of the type $k'((t))$ with k' a finite extension of k and t a variable. So, any element in $\mathbf{E}_{\pi,\varpi,i}$ is inside the field of quotients of $\mathbf{E}_\pi \otimes_{k', \sigma^i} k'((t))$. This field is inside the field $\mathbf{E}_\pi((t))$; φ_π acts only on \mathbf{E}_π , in the usual way. So, an element is in the projective limit of the system $\mathbf{E}_{\pi,\varpi,i}$ if and only if it is a constant sequence of elements in \mathbf{E}_ϖ . Clearly this can be identified with \mathbf{E}_ϖ , and the morphism φ_ϖ is identified with the usual rs -th iterate of the Frobenius morphism of \mathbf{E}_ϖ . The symmetrical statement is proved symmetrically. \square

We now define the two functors Φ and Ψ between the categories $\mathbf{\Phi}_{\varphi^{rs}, \Gamma_{K,\pi}}^{et}(\mathbf{E}_{K,\pi})$ and $\mathbf{\Phi}_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{E}_{K,\varpi})$ which give the equivalence of these two categories.

Let D_1 be in $\mathbf{\Phi}_{\varphi^{rs}, \Gamma_{K,\pi}}^{et}(\mathbf{E}_{K,\pi})$, we form the system $D_1 \otimes_{\mathbf{E}_{K,\pi}} \mathbf{E}_{\pi,\varpi,\bullet}$. We make φ_π and G_K act diagonally on D_1 and $\mathbf{E}_{\pi,\varpi,\bullet}$. Analogously, if D_2 is in $\mathbf{\Phi}_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{E}_{K,\varpi})$, then we may form the system $D_2 \otimes_{\mathbf{E}_{K,\varpi}} \mathbf{E}_{\pi,\varpi,\bullet}$, making φ_ϖ and G_K act diagonally on D_2 and $\mathbf{E}_{\pi,\varpi,\bullet}$. Let $\Phi(D_1) = (\text{projlim}_{\varphi_2} D_1 \otimes_{\mathbf{E}_{K,\pi}} \mathbf{E}_{\pi,\varpi,\bullet})^{H_{K,\varpi}}$; it is endowed with an action of $\Gamma_{K,\varpi}$ and φ_ϖ . Analogously, let $\Psi(D_2) = (\text{projlim}_{\varphi_\pi} D_2 \otimes_{\mathbf{E}_{K,\varpi}} \mathbf{E}_{\pi,\varpi,\bullet})^{H_{K,\pi}}$; it is endowed with an action of $\Gamma_{K,\pi}$ and φ_π .

PROPOSITION 4.4. *If D_1 is in $\Phi_{\varphi^{rs}, \Gamma_{K, \pi}}^{et}(\mathbf{E}_{K, \pi})$, then $\Phi(D_1)$ is in $\Phi_{\varphi^{rs}, \Gamma_{K, \varpi}}^{et}(\mathbf{E}_{K, \varpi})$. If D_2 is in $\Phi_{\varphi^{rs}, \Gamma_{K, \varpi}}^{et}(\mathbf{E}_{K, \varpi})$, then $\Psi(D_2)$ is in $\Phi_{\varphi^{rs}, \Gamma_{K, \pi}}^{et}(\mathbf{E}_{K, \pi})$. These functors are quasi-inverses of each other.*

Proof. We have $D_1 \otimes_{\mathbf{E}_{K, \pi}} \mathbf{E}_{\pi, \varpi, i} = D_1 \otimes_{\mathbf{E}_{K, \pi}} \mathbf{E}_{\pi} \otimes_{\mathbf{E}_{\pi}} \mathbf{E}_{\pi, \varpi, i}$. By Theorem 4.2, we have that D_1 comes from a representation W of G_K over k_K , and that $D_1 \otimes_{\mathbf{E}_{K, \pi}} \mathbf{E}_{\pi} = W \otimes_{k_K} \mathbf{E}_{\pi}$. So $D_1 \otimes_{\mathbf{E}_{K, \pi}} \mathbf{E}_{\pi, \varpi, i} = W \otimes_{k_K} \mathbf{E}_{\pi, \varpi, i}$. Hence $\Phi(D_1) = (\text{pro} \lim_{\varphi_{\pi}} D_1 \otimes_{\mathbf{E}_{K, \pi}} \mathbf{E}_{\pi, \varpi, \bullet})^{H_{K, \varpi}}$ is equal to $(\lim_{\leftarrow, \varphi_{\pi}} W \otimes_{k_K} \mathbf{E}_{\pi, \varpi, \bullet})^{H_{K, \varpi}}$, which by Proposition 4.3 is equal to $(W \otimes_{k_K} \mathbf{E}_{\varpi})^{H_{K, \varpi}}$ which is the $(\varphi^{rs}, \Gamma_{K, \varpi})$ -module étale associated to W . This shows that $\Phi(D_{\pi}(W)) = D_{\varpi}(W)$. Analogously one proves that $\Psi(D_{\varpi}(W)) = D_{\pi}(W)$. The two functors are thus quasi-inverses and commute the functors D_{π} and D_{ϖ} . \square

5. The ring $\mathbf{A}_{F, \pi}$

In this paragraph we want to lift to characteristic zero the action of Galois and Frobenius on the field of norms. In this paragraph nothing is new. We just adapt to our notations and setting results presented in [6] in the case $F = K$ ($L = K$ in notation of [6]). Let F be a finite extension of \mathbf{Q}_p , \mathcal{O}_F its ring of integers, k_F its residue field, p^r the cardinality of k_F , π a uniformizer of F , \bar{F} an algebraic closure of F and G_F its absolute Galois group. Lubin-Tate theory allows to construct a maximal abelian ramified extension F_{π} of F ([7]). Let $\chi_{\pi} : G_F \rightarrow \mathcal{O}_F^{\times}$ be the associated character. Let us consider the field of norms $X_F(F_{\pi}) = \mathbf{E}_{F, \pi}$ of the extension F_{π} of F (see [3], [9] or [1]). It is a local field of characteristic p , with residue field k_F , such that its absolute Galois group $H_F = \text{Ker } \chi_{\pi}$ is naturally isomorphic to the absolute Galois group of the field F_{π} . Moreover, $\mathbf{E}_{F, \pi}$ comes equipped with an action of the group $\Gamma_F = G_F/H_F \cong \mathcal{O}_F^{\times}$.

Let $\tilde{\mathbf{E}}^+$ be the inverse limit of $\mathcal{O}_{\bar{F}}/p\mathcal{O}_{\bar{F}}$ with transition maps given by the Frobenius map. Its field of fractions, denoted by $\tilde{\mathbf{E}}$, is an algebraically closed field of characteristic p . Moreover there is a natural map $\mathbf{E}_{F, \pi} \rightarrow \tilde{\mathbf{E}}$ (see [9] 4.2) compatible with the Frobenius and the action of G_F (through Γ_F). Let $\tilde{\mathbf{A}} = W(\tilde{\mathbf{E}})$ be the ring of Witt vectors with coefficients in the perfect field $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}} = \text{Frac}(\tilde{\mathbf{A}})$.

LEMMA 5.1. *There exists a complete discrete valuation ring, $\mathbf{A}_{F, \pi}$ with uniformizer π and residue field $\mathbf{E}_{F, \pi}$, endowed with commuting actions of Γ_F and φ^r , lifting the actions of Γ_F and φ^r on $\mathbf{E}_{F, \pi}$.*

Proof. This is the construction in [6] (1.2 and 1.3). Let us recall briefly the main steps. It is possible to find an element $u \in \tilde{\mathbf{A}} \otimes_{\mathcal{O}_{\bar{F}_0}} \mathcal{O}_F$ and to consider the ring $\mathcal{O}_F[[u]][u^{-1}]$. The ring $\mathbf{A}_{F, \pi}$ will be the π -adic completion of this ring. The action of Γ_F will be trivial on \mathcal{O}_F ; on u it is given by $\gamma u = [\chi_F](\gamma)(u)$, and the r -th iterate of the Frobenius is trivial on \mathcal{O}_F and on u is given by $\varphi^r(u) = [\pi](u)$.

The residue field of $\mathbf{A}_{F,\pi}$ is $k_F((u))$ which is just $\mathbf{E}_{F,\pi}$, and the action of Γ_F and φ^r lifts the natural action of Γ_F and φ^r on $\mathbf{E}_{F,\pi}$. \square

Let $\mathbf{A}_{F,\pi}^{ur}$ be the ring of integers of the unramified closure $\mathbf{B}_{F,\pi}^{ur}$ of the fraction field $\mathbf{B}_{F,\pi}$ of the ring $\mathbf{A}_{F,\pi}$. We denote by $\hat{\mathbf{A}}_{F,\pi}^{ur}$ the completion of $\mathbf{A}_{F,\pi}^{ur}$.

LEMMA 5.2. *The residue field of $\mathbf{A}_{F,\pi}^{ur}$ is a separable algebraic closure \mathbf{E}_π of $\mathbf{E}_{F,\pi}$. Moreover, there is a natural isomorphism between the Galois group of $\mathbf{B}_{F,\pi}^{ur}$ over $\mathbf{B}_{F,\pi}$ and H_F .*

Proof. See [6], Lemma (1.4). \square

We may now define the functors V and $D_{\mathbf{A}_{F,\pi}}$. Let A be a p -adic representation of G_F over \mathcal{O}_F , then we define $D_{\mathbf{A}_{F,\pi}}(A) := (A \otimes_{\mathcal{O}_F} \hat{\mathbf{A}}_{F,\pi}^{ur})^{H_F}$. Let B be an étale (φ^r, Γ_F) -module over $\mathbf{A}_{F,\pi}$, then we define $D_{\mathbf{A}_{F,\pi}}(B) := (B \otimes_{\mathbf{A}_{F,\pi}} \hat{\mathbf{A}}_{F,\pi}^{ur})^{\varphi^r=1}$.

THEOREM 5.3. *The functors V and $D_{\mathbf{A}_{F,\pi}}$ are quasi-inverse equivalences between the categories $\mathbf{Rep}_{G_F}(\mathcal{O}_F)$ and $\Phi_{\varphi^r, \Gamma_F}^{et}(\mathbf{A}_{F,\pi})$.*

Proof. See [6], Theorem (1.6). \square

6. The ring $\mathbf{A}_{K,\pi}$ for K a finite extension of F

We now consider a finite extension K of F : unlike the situation considered by Kisin and Ren [6] we place no hypothesis on the extension. Let F_π be the maximal abelian ramified extension of F attached to the uniformizer π by Lubin-Tate theory. By [9] 3.2, the field of norms $\mathbf{E}_{K,\pi}$ of the extension $K \cdot F_\pi$ of K is a finite separable extension of $\mathbf{E}_{F,\pi}$. We remark again that the uniformizer of $\mathbf{A}_{F,\pi}$ is π . There exists a unique subring of $\mathbf{A}_{F,\pi}^{ur}$, which is a discrete valuation ring containing $\mathbf{A}_{F,\pi}$ and whose residue field is $\mathbf{E}_{K,\pi}$. Let us denote this ring by $\mathbf{A}_{F,\pi}(K)$. Our approach follows that given in [1] §4,6.

PROPOSITION 6.1. *The ring $\mathbf{A}_{F,\pi}(K)$ is stable under G_K and φ .*

Proof. On $\mathbf{A}_{F,\pi}^{ur}$, the action of G_F lifts the action of G_F on the fixed separable algebraic closure of $\mathbf{E}_{F,\pi}$, and the action of φ lifts the action of the Frobenius. Since G_K and φ leave $\mathbf{E}_{K,\pi}$ stable, also $\mathbf{A}_{F,\pi}(K)$ will be stable under G_K and φ . \square

Remark. Note that, if we put ourselves under the hypotheses of [6] 2.1 (which are in force throughout that article), our $\mathbf{A}_{F,\pi}(K)$ coincides with \mathcal{O}_ϑ of [6].

The ring $\mathbf{A}_{F,\pi}(K)$ contains the ring $\mathcal{O}_{K_0} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$ (we recall that \mathcal{O}_{F_0} and \mathcal{O}_{K_0} denote respectively the ring of Witt vectors on k_F and k_K). The ring $\mathbf{A}_{F,\pi}(K)$ is unramified over the discrete valuation ring $\mathcal{O}_{K_0} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$. Moreover \mathcal{O}_K is totally

ramified over $\mathcal{O}_{K_0} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F$. Hence the tensor product $\mathbf{A}_{F,\pi}(K) \otimes_{\mathcal{O}_{K_0} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F} \mathcal{O}_K$ is again a discrete valuation ring. Let us indicate it by $\mathbf{A}_{K,\pi}$. It has uniformizer ϖ and residue field $\mathbf{E}_{K,\pi}$. We summarize the properties of $\mathbf{A}_{K,\pi}$: it arises from the field of norms of the maximal totally ramified extension of K related to the Lubin-Tate group on F given by π and we have added to its Cohen lifting the ramification of K . Moreover, one may extend the commuting action of G_K and of φ^{rs} on $\mathbf{A}_{F,\pi}(K)$ to (commuting) actions on $\mathbf{A}_{K,\pi}$ by imposing that the rs -iterate of Frobenius act trivially on \mathcal{O}_K .

We have just proved the following.

LEMMA 6.2. *There exists a complete discrete valuation ring $\mathbf{A}_{K,\pi}$, with uniformizer ϖ and residue field $\mathbf{E}_{K,\pi}$, endowed with commuting actions of G_K and φ^{rs} lifting the natural actions of G_K and rs -th iterate of the Frobenius on $\mathbf{E}_{K,\pi}$.*

Proof. The construction above. Observe that k_K is fixed under the rs -th iterate of the Frobenius, as well as K . \square

We may then take the unramified closure of $\text{Frac}(\mathbf{A}_{K,\pi})$ in the extension of $\tilde{\mathbf{B}}$ by K (denote it by $\tilde{\mathbf{B}}_K$). Note that while K is totally ramified over the K_0 , $\tilde{\mathbf{B}}$ is unramified over K_0 . We will indicate its valuation ring as $\mathbf{A}_{K,\pi}^{ur}$ and by $\hat{\mathbf{A}}_{K,\pi}^{ur}$ its completion. We then have:

LEMMA 6.3. *The ring $\hat{\mathbf{A}}_{K,\pi}^{ur}$ is a complete discrete valuation ring with residue field a separable algebraic closure of $\mathbf{E}_{K,\pi}$; it is equipped with commuting actions of G_K and φ^{rs} , lifting the actions on the separable algebraic closure \mathbf{E}_π of $\mathbf{E}_{K,\pi}$.*

Proof. The ring $\mathbf{A}_{K,\pi}^{ur}$ may be viewed as $\mathbf{A}_{F,\pi}^{ur} \otimes_{\mathcal{O}_{K_0} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F} \mathcal{O}_K$ since K is totally ramified over the $F \cdot K_0$ and $\text{Frac}(\mathbf{A}_{F,\pi}(K))$ is unramified over $F \cdot K_0$. The action of G_K extends that on $\mathbf{A}_{F,\pi}^{ur}$ by the trivial action on \mathcal{O}_K , and analogously for the action of φ^{rs} in the completion: $\hat{\mathbf{A}}_{K,\pi}^{ur} = \hat{\mathbf{A}}_{F,\pi}^{ur} \otimes_{\mathcal{O}_{K_0} \otimes_{\mathcal{O}_{F_0}} \mathcal{O}_F} \mathcal{O}_K$. \square

In particular $\text{Gal}(\hat{\mathbf{A}}_{K,\pi}^{ur}/\mathbf{A}_{K,\pi}) = \text{Gal}(\mathbf{E}_\pi/\mathbf{E}_{K,\pi}) = H_{K,\pi}$ (we consider continuous automorphisms). From now on, in order to lighten the notation, we will denote $\hat{\mathbf{A}}_{K,\pi}^{ur}$ by \mathbf{A}_π . Note that G_K acts on $\mathbf{A}_{K,\pi}$ through $\Gamma_{K,\pi} = \chi_\pi(G_K)$. We may now define the functors V_π and D_π . Let A be p -adic representation of G_K over \mathcal{O}_K , then we define $D_\pi(A) := (A \otimes_{\mathcal{O}_K} \mathbf{A}_\pi)^{H_{K,\pi}}$. Let B be an étale $(\varphi^{rs}, \Gamma_{K,\pi})$ -module over $\mathbf{A}_{K,\pi}$, then we define $V_\pi(B) := (B \otimes_{\mathbf{A}_{K,\pi}} \mathbf{A}_\pi)^{\varphi^{rs}=1}$.

THEOREM 6.4. *The functors V_π and D_π are quasi-inverse equivalences between the categories $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$ and $\Phi_{\varphi^{rs}, \Gamma_{K,\pi}}^{ét}(\mathbf{A}_{K,\pi})$.*

Proof. To prove the statement we proceed as in [2] 1.2.4, 1.2.6. It is enough to prove that the maps

$$D_\pi(V) \otimes_{\mathbf{A}_{K,\pi}} \mathbf{A}_\pi \rightarrow V \otimes_{\mathcal{O}_K} \mathbf{A}_\pi$$

and

$$V_\pi(B) \otimes_{\mathcal{O}_K} \mathbf{A}_\pi \rightarrow B \otimes_{\mathbf{A}_{K,\pi}} \mathbf{A}_\pi$$

are isomorphisms. This follows in the limit from the analogous statement on torsion modules, i.e., modules killed by a power of ϖ . The statement on torsion modules follows by dévissage from the statement on modules killed by ϖ . On modules killed by ϖ , the statement was proved in the proof of Theorem 4.1. \square

Remark. As we mentioned the results presented in this paragraph are generalizations of the constructions given in [6]. Throughout that article the two authors made the assumption ([6] 2.1) that, if we denote by $\{F_n\}$ the totally ramified tower over F associated to the Lubin-Tate group given by π and by $K_{0,L}$ the field $K_0 \otimes_{L_0} L$ then there exists n such that $K \subset F_n \cdot K_{0,L}$. This condition will imply that the field of norms $\mathbf{E}_{K,\pi}$ is unramified over $\mathbf{E}_{F,\pi}$ and the degree of the extension is $[K_0 : L_0]$. Our general setting will allow ramification and the residue degree will be the degree over F of the maximal unramified extension of F in the maximal totally ramified extension of K given by the union of the $F_n \cdot K$.

7. Comparison between the two different equivalences

In the preceding two sections we have seen two different equivalences involving the category $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$ and two categories of (φ, Γ) -modules. The first equivalence is obtained via the method explained in Section 6, hence we start with a Lubin-Tate group on F , associated to a uniformizer π and then we can consider the totally ramified tower attached to it over K . But we could have started directly from K , chosen a uniformizer ϖ in it, and then built another totally ramified tower: then we would be in the case of Section 5 (but with F replaced by K in that section). In the notation of Section 5, then we would then have an equivalence between $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$ and $(\varphi^{rs}, \Gamma_{K,\varpi})$ -modules over $\mathbf{A}_{K,\varpi}$. Theorem 5.3 now would read as follows.

THEOREM 7.1. *The two functors*

$$D_\varpi : \mathbf{Rep}_{G_K}(\mathcal{O}_K) \rightarrow \Phi_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{A}_{K,\varpi})$$

and

$$V_\varpi : \Phi_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{A}_{K,\varpi}) \rightarrow \mathbf{Rep}_{G_K}(\mathcal{O}_K),$$

defined by $D_\varpi(A) = (A \otimes_{\mathcal{O}_K} \hat{\mathbf{A}}_{K,\varpi}^{ur})^{H_{K,\varpi}}$ for $A \in \mathbf{Rep}_{G_K}(\mathcal{O}_K)$, and $V_\varpi(B) = (B \otimes_{\mathbf{A}_{K,\varpi}} \hat{\mathbf{A}}_{K,\varpi}^{ur})^{\varphi^{rs}=1}$, for $B \in \Phi_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{A}_{K,\varpi})$, induce quasi-inverse equivalences of categories.

Again in order to lighten the notation we will indicate $\hat{\mathbf{A}}_{K,\varpi}^{ur}$ by \mathbf{A}_ϖ . So we have that the category $\Phi_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{A}_{K,\varpi})$ is equivalent to $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$. But, by Theorem 6.4, the category $\Phi_{\varphi^{rs}, \Gamma_{K,\pi}}^{et}(\mathbf{A}_{K,\pi})$ is equivalent to $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$. By transitivity, we have that the category $\Phi_{\varphi^{rs}, \Gamma_{K,\varpi}}^{et}(\mathbf{A}_{K,\varpi})$ is equivalent to

$\Phi_{\varphi^s, \Gamma_{K, \pi}}^{et}(\mathbf{A}_{K, \pi})$. Recall again that the first category uses the Lubin-Tate group defined by the uniformizer ϖ of K , while the second uses the extension to K of the Lubin-Tate group of F associated to π .

We wish to describe the two functors which give the equivalence between these two categories without appealing to the category $\mathbf{Rep}_{G_K}(\mathcal{O}_K)$.

First we define a system of rings $\mathbf{A}_{\pi, \varpi, i}$, together with a G_K -action on each of these rings, two partial Frobenius morphisms $\varphi_{\pi} : \mathbf{A}_{\pi, \varpi, i} \rightarrow \mathbf{A}_{\pi, \varpi, i-1}$ and $\varphi_{\varpi} : \mathbf{A}_{\pi, \varpi, i} \rightarrow \mathbf{A}_{\pi, \varpi, i+1}$. These rings will be complete discretely valued rings with uniformizer ϖ and the residue field of $\mathbf{A}_{\pi, \varpi, i}$ being $\mathbf{E}_{\pi, \varpi, i}$; the G_K -action and the partial Frobenius morphisms are compatible with reduction mod ϖ . The system of fields $\mathbf{B}_{\pi, \varpi, i}$ is obtained from the system of rings $\mathbf{A}_{\pi, \varpi, i}$ by inverting ϖ ; of course it inherits the G_K -action and the partial Frobenius morphisms.

Let us now define the rings $\mathbf{A}_{\pi, \varpi, i}$. Consider the rings $\mathbf{A}_{\pi} = \hat{\mathbf{A}}_{K, \pi}^{ur}$ and $\mathbf{A}_{\varpi} = \hat{\mathbf{A}}_{K, \varpi}^{ur}$ of the previous sections. Consider the tensor product $\mathbf{A}_{\pi} \otimes_{W(\bar{k}_K), \sigma^i} \mathbf{A}_{\varpi}$, where $W(\bar{k}_K)$ is the ring of Witt vectors with coefficients in the field \bar{k}_K , and the tensor product is twisted by the rsi -th power of the Frobenius automorphism of $W(\bar{k}_K)$. In this tensor product, the ideal $\varpi(\mathbf{A}_{\pi} \otimes_{W(\bar{k}_K), \sigma^i} \mathbf{A}_{\varpi})$ is prime, since the quotient is $\mathbf{E}_{\pi} \otimes_{\bar{k}_K, \sigma^i} \mathbf{E}_{\varpi}$, which is a domain by Lemma 3.8. Let T be the localization of $\mathbf{A}_{\pi} \otimes_{W(\bar{k}_K), \sigma^i} \mathbf{A}_{\varpi}$ with respect to the prime ideal $\varpi(\mathbf{A}_{\pi} \otimes_{W(\bar{k}_K), \sigma^i} \mathbf{A}_{\varpi})$. It is then a local ring with residue field $\mathbf{E}_{\pi, \varpi, i}$ and its maximal ideal is generated by ϖ . The ring $\mathbf{A}_{\pi, \varpi, i}$ is defined as the ϖ -adic completion of the ring T ; and $\mathbf{B}_{\pi, \varpi, i}$ is the quotient field $\mathbf{A}_{\pi, \varpi, i}[1/\varpi]$ of $\mathbf{A}_{\pi, \varpi, i}$.

The G_K -action and the partial Frobenius maps of $\mathbf{A}_{\pi} \otimes_{W(\bar{k}_K), \sigma^i} \mathbf{A}_{\varpi}$ induce a G_K -action and partial Frobenius morphisms on the system $\mathbf{A}_{\pi, \varpi, i}$, and hence also on the system $\mathbf{B}_{\pi, \varpi, i}$. Clearly, on reducing $\mathbf{A}_{\pi, \varpi, i}$ mod ϖ one obtains the system $\mathbf{E}_{\pi, \varpi, i}$.

The following proposition is the lifting of Proposition 4.3 to characteristic 0.

PROPOSITION 7.2. *The projective limit of the system $\mathbf{A}_{\pi, \varpi, \bullet}$ along φ_{π} is \mathbf{A}_{ϖ} , and the morphisms φ_{ϖ} are transformed into the Frobenius morphism φ_{ϖ} of \mathbf{A}_{ϖ} . The symmetrical statement holds also and the same holds true for the system $\mathbf{B}_{\pi, \varpi, \bullet}$.*

Proof. An element (x_i) is in the projective limit of the system $\mathbf{B}_{\pi, \varpi, i}$ for the morphism φ_{π} if and only if, there exists a $t \in \mathbf{N}$, such that $(\varpi^t x_i)$ is an element of the projective limit of the system $\mathbf{A}_{\pi, \varpi, i}$ for the morphism φ_{π} . So one is reduced to the computation of the projective limit of the system $\mathbf{A}_{\pi, \varpi, i}$ for the morphism φ_{π} . Of course, any element of the type (x_i) , with $x_i = x \in \mathbf{A}_{\varpi}$ is in this projective limit; so one must just show that any (x_i) in this projective limit is of this type.

Consider the reduction mod ϖ of the element (x_i) , it is an element of the projective limit of the system $\mathbf{E}_{\pi, \varpi, i}$ along the morphism φ_{π} ; so by Proposition 4.3, there exists an element $x^{(0)} \in \mathbf{A}_{\varpi}$ such that for every i , $x_i \cong x^{(0)} \pmod{\varpi}$. It follows that $(x_i - x^{(0)})$ is an element of the projective limit divisible by ϖ ; so also $(x_i - x^{(0)})/\varpi$ is an element of the projective limit. One may apply the

same argument to get an element $x^{(1)} \in \mathbf{A}_{\varpi}$ such that, for every i , $(x_i - x^{(0)})/\varpi$ is congruent to $x^{(1)} \bmod \varpi$. So $x^{(0)} + \varpi x^{(1)}$ is congruent to $x_i \bmod \varpi^2$. By iterating this process, one obtains a convergent sequence of elements of \mathbf{A}_{ϖ} , converging to x_i for every index i . So this projective limit can be identified with \mathbf{A}_{ϖ} ; and in this way, the morphism φ_{ϖ} is identified with the usual Frobenius morphism φ^{rs} of \mathbf{A}_{ϖ} .

The symmetrical statement is dealt with symmetrically. \square

We can now state our main result:

THEOREM 7.3. *There is an equivalence of categories between the categories $\Phi_{\varphi^{rs}, \Gamma_{K, \pi}}^{et}(\mathbf{A}_{K, \pi})$ and $\Phi_{\varphi^{rs}, \Gamma_{K, \varpi}}^{et}(\mathbf{A}_{K, \varpi})$. This equivalence is realized by the following functors*

$$D_1 \mapsto \left(\lim_{\leftarrow, \varphi_{\pi}} D_1 \otimes_{\mathbf{A}_{\varpi, \pi}} \mathbf{A}_{\pi, \varpi, \bullet} \right)^{H_{K, \varpi}};$$

and

$$D_2 \mapsto \left(\lim_{\leftarrow, \varphi_{\varpi}} D_2 \otimes_{\mathbf{A}_{K, \varpi}} \mathbf{A}_{\pi, \varpi, \bullet} \right)^{H_{K, \pi}}.$$

Proof. Consider the étale $(\varphi^{rs}, \Gamma_{K, \pi})$ -module D_1 over the ring $\mathbf{A}_{K, \pi}$, coming from the representation V of G_K over \mathcal{O}_K . Then we have, $D_1 \otimes_{\mathbf{A}_{K, \pi}} \mathbf{A}_{\pi, \varpi, \bullet} = V \otimes_{\mathcal{O}_K} \mathbf{A}_{\pi, \varpi, \bullet}$. Hence, $\lim_{\leftarrow, \varphi_{\pi}} D_1 \otimes_{\mathbf{A}_{K, \pi}} \mathbf{A}_{\pi, \varpi, \bullet} = V \otimes_{\mathcal{O}_K} \lim_{\leftarrow, \varphi_{\pi}} \mathbf{A}_{\pi, \varpi, \bullet}$. By Proposition 7.2, the projective limit is \mathbf{A}_{ϖ} . Thus $(\lim_{\leftarrow, \varphi_{\pi}} D_1 \otimes_{\mathbf{A}_{K, \pi}} \mathbf{A}_{\pi, \varpi, \bullet})^{H_{K, \varpi}} = (V \otimes_{\mathcal{O}_K} \mathbf{A}_{\varpi})^{H_{K, \varpi}}$. This shows that it is the equivalence of categories obtained by composing the two equivalences of categories with the category of representations of G_K over the field \mathcal{O}_K .

The second statement is dealt with analogously. \square

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