A NOTE ON COUNTABLY BI-QUOTIENT MAPPINGS

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Abstract

In this paper some properties of weakly first countable spaces and sequence-covering images of metric spaces are studied. Strictly Fréchet spaces are characterized as the spaces in which every sequence-covering mapping onto them is strictly countably bi-quotient. Strict accessibility spaces are introduced, in which a T_1-space X is strict accessibility if and only if every quotient mapping onto X is strictly countably bi-quotient. For a T_2, k-space X every quotient mapping onto X is strictly countably bi-quotient or bi-quotient if and only if X is discrete. They partially answer some questions posed by F. Siwiec in [16, 17].

1. Introduction

Topologists obtained many interesting characterizations of spaces by mappings, in particular some images of metric spaces. Fréchet spaces, and sequential spaces belong to the class of weakly first countable spaces. The class of weakly first countable spaces plays an important role in generalized metric spaces and metrization, which has become a striking research subject in general topology. For example,

THEOREM 1.1 [4, 16]. *The following are equivalent for a space X:*
(1) *X is a Fréchet space;*
(2) *Every sequence-covering mapping onto X is pseudo-open;*
(3) *X is a pseudo-open image of a metric space.*

THEOREM 1.2 [18]. *A T_1-space X is an accessibility space if and only if every quotient mapping onto X is pseudo-open.*

THEOREM 1.3 [13]. *Every $\aleph$-space is preserved by a closed and countably bi-quotient mapping.*

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In this paper we study some properties of weakly first countable spaces and sequence-covering images of metric spaces. Every strongly Fréchet space can be characterized as a countably bi-quotient image of a metric space [16]. In 1982, Gerlits and Nagy [5] defined the strictly Fréchet spaces, which are also known as w-spaces in the early [6, 14]. Every first countable space is strictly Fréchet, and every strictly Fréchet space is strongly Fréchet. In the sections 2 and 3, we discuss the strictly Fréchet spaces by the above theorems' inspiring. In 1987, Jianping Zhu [21] defined the w-mappings, and proved that a space X is a w-space if and only if it is a w-image of a metric space. Whether a strictly Fréchet space X can be characterized as every sequence-covering mapping onto X is a w-mapping? On the other hand, how to characterize a space X such that every quotient mapping onto X is a w-mapping? In this paper w-mappings are renamed to "strictly countably bi-quotient mappings". We obtain a new characterization of strictly Fréchet spaces by sequence-covering mappings, and introduce the concept of strictly accessibility spaces, which satisfies the condition that every quotient mapping onto this space is strictly countably bi-quotient.

In 1975, Siwiec [17, Table 22, p. 32] posed the following question: give an intrinsic characterization of the class of spaces Y such that every quotient mapping onto Y is bi-quotient. In the section 4, we discuss some relations of mappings about almost-open mappings, bi-quotient mappings, strictly countably bi-quotient mappings and sequence-covering mappings, and give a positive answer to Sewiec's question.

In this paper all mappings are continuous and onto.

2. Strictly Fréchet spaces

In this section, we discuss the relations among strictly Fréchet spaces, and sequence-covering mappings, strictly countably bi-quotient mappings.

Definition 2.1 [5]. A space X is called *strictly Fréchet* if whenever $\{A_n\}_n$ is a sequence of subsets in X and a point $x \in \bigcap_{n \in \mathbf{N}} \overline{A_n}$, there exists an $x_n \in A_n$ for each $n \in \mathbf{N}$ such that the sequence $x_n \to x$.

A *Fréchet space* [4], by definition, is a space satisfying Definition 2.1 but with all the sets A_n being equal. A *strongly Fréchet space* [16], by definition, is a space satisfying Definition 2.1 but with the sequence $\{A_n\}_n$ being decreasing in X.

It is obvious that, first countable spaces $\Rightarrow$ strictly Fréchet spaces $\Rightarrow$ strongly Fréchet spaces $\Rightarrow$ Fréchet spaces.

In 1987, Jianping Zhu [21] defined w-mappings. In this paper w-mappings are renamed to "strictly countably bi-quotient mappings".

Definition 2.2. A mapping $f : X \to Y$ is called *strictly countably bi-quotient* if for each $y \in Y$ and for each countable cover $\{U_n : n \in \mathbf{N}\}$ of $f^{-1}(y)$ by open subsets of X there exists an $m \in \mathbf{N}$ such that $y \in \operatorname{int}(f(U_m))$.

A *countably bi-quotient mapping* [16], by definition, is a mapping satisfying Definition 2.2 but $y \in \operatorname{int}(f(\bigcup \mathscr{U}))$ for some finite family $\mathscr{U} \subset \{U_n : n \in \mathbf{N}\}$.

It is obvious that, almost-open mappings[1] $\Rightarrow$ strictly countably bi-quotient mappings $\Rightarrow$ countably bi-quotient mappings $\Rightarrow$ pseudo-open mappings[2].

Lemma 2.3 [11]. *Let X be a strictly Fréchet space and $\{A_n\}_n$ be a sequence of subsets in X. If $x \in \bigcap_{n \in \mathbf{N}} \overline{A_n}$, there exists a sequence $\{b_m\}_m$ in X such that $b_m \to x$ and $\{m \in \mathbf{N} : b_m \in A_n\}$ is infinite for each $n \in \mathbf{N}$.*

A mapping $f : X \to Y$ is called *sequence-covering* [16] if whenever $\{y_n\}_n$ is a sequence in Y converging to a point $y \in Y$, there exists a sequence of points $x_n \in f^{-1}(y_n)$ for $n \in \mathbf{N}$, and $x \in f^{-1}(y)$ such that $x_n \to x$.

Lemma 2.4. *Let $f : X \to Y$ be a sequence-covering mapping. If Y is strictly Fréchet, then f is strictly countably bi-quotient.*

Proof. Assume that $y \in Y$ and $f^{-1}(y) \subset \bigcup_{n \in \mathbf{N}} U_n$, where U_n is open in X for each $n \in \mathbf{N}$. Suppose $y \notin \operatorname{int}(f(U_n))$ for each $n \in \mathbf{N}$, then $y \in \overline{Y - f(U_n)}$. By Lemma 2.3, there exists a sequence $\{y_i\}$ in Y converging to y such that $\{i \in \mathbf{N} : y_i \in Y - f(U_n)\}$ is an infinite set for each $n \in \mathbf{N}$. Since f is sequence-covering, there is a sequence $\{x_i\}$ in X and a point $x \in f^{-1}(y)$ such that each $x_i \in f^{-1}(y_i)$ and $x_i \to x$. Then there exists $k \in \mathbf{N}$ such that $x \in U_k$, thus there is $i_0 \in \mathbf{N}$ such that $x_i \in U_k$ for each $i \geq i_0$, so $y_i \in f(U_k)$, a contradiction. So f is strictly countably bi-quotient. □

Lemma 2.5 [21]. *Strictly Fréchet spaces are preserved by strictly countably bi-quotient mappings.*

Definition 2.6 [20]. A mapping $f : X \to Y$ is called *set-sequence-covering* if whenever $\{A_n\}_n$ is a decreasing sequence of subsets in Y converging to a point $y \in Y$, there exists $x \in f^{-1}(y)$ and a decreasing sequence $\{B_n\}_n$ of subsets in X such that $B_n \to x$[3] and $f(B_n) = A_n$, $\forall n \in \mathbf{N}$.

Lemma 2.7 [20]. *Every set-sequence-covering mapping is a sequence-covering mapping.*

Lemma 2.8 [9, 20]. *Every space is a set-sequence-covering image of a metric space.*

[1] A mapping $f : X \to Y$ is *almost-open* if there is $x \in f^{-1}(y)$ for every $y \in Y$ such that $f(U)$ is a neighborhood at y in Y when U is a neighborhood at x in X.

[2] A mapping $f : X \to Y$ is *pseudo-open* if $f(U)$ is a neighborhood at y in Y for every $y \in Y$ when $f^{-1}(y) \subset U$ with U open in X.

[3] $B_n \to x$ in X means that the set-sequence $\{B_n\}_n$ converges to x in X, i.e., if whenever U is a neighborhood of x in X there exists $m \in \mathbf{N}$ such that $B_n \subset U$ for each $n \geq m$.

Proof. This lemma was proved for a T_1-space in [20]. We show the lemma without any properties of separations.

In [9, Theorem 4.4] Michael proved the following theorem. Let Y be a space. There are a metric space X and a mapping $f : X \to Y$ such that if $\{C_n\}_n$ is a decreasing sequence of subsets of Y which is a network at a point y in Y[4], there is $x \in f^{-1}(y)$ and a decreasing local base $\{D_n\}_n$ at x in X such that $f(D_n) = C_n$, $\forall n \in \mathbf{N}$. We will show that the mapping $f : X \to Y$ is set-sequence-covering. Let $\{A_n\}_n$ be a decreasing sequence of subsets in Y converging to a point $y \in Y$. Then $\{A_n \cup \{y\}\}_n$ is a decreasing network at y in Y. By Michael's theorem above, there is $x \in f^{-1}(y)$ and a decreasing local base $\{D_n\}_n$ at x in X such that $f(D_n) = A_n \cup \{y\}$, $\forall n \in \mathbf{N}$. We may assume that $y \notin A_n$, and put $B_n = D_n - f^{-1}(y)$. Then $\{B_n\}_n$ is a decreasing sequence of subsets in X, $B_n \to x$ and $f(B_n) = A_n$, $\forall n \in \mathbf{N}$. $\square$

THEOREM 2.9. *The following are equivalent for a space X*:
(1) *X is a strictly Fréchet space*;
(2) *Every sequence-covering mapping onto X is strictly countably bi-quotient*;
(3) *Every set-sequence-covering mapping onto X is strictly countably bi-quotient*;
(4) *X is a strictly countably bi-quotient image of a metric space.*

Proof. (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3). It is obvious by Lemmas 2.4, 2.5 and 2.7.

(3) $\Rightarrow$ (4). Suppose that a space X satisfies the condition (3). There is a metric space M and a set-sequence-covering mapping $f : M \to X$ by Lemma 2.8. Then f is strictly countably bi-quotient by the condition (3).

(1) $\Leftrightarrow$ (4) in Theorem 2.9 is proved by Zhu [21]. $\square$

QUESTION 2.10. Is a strictly countably bi-quotient mapping on a metric space sequence-covering?

QUESTION 2.11. Is an almost-open mapping on a strictly Fréchet space sequence-covering?

3. Strict accessibility spaces

By Theorem 1.2 we are interesting in the following question: under what condition for a space X in which every quotient mapping onto X is strictly countably bi-quotient? The following concept is introduced.

DEFINITION 3.1. A space X is called a *strict accessibility space* if whenever $\{A_n\}_n$ is a sequence of subsets in X and x is an accumulation point of A_n for each $n \in \mathbf{N}$, there exists a closed set C in X such that x is an accumulation point of C, but not of $C - A_n$ for each $n \in \mathbf{N}$.

[4] A family $\mathscr{P}$ of subsets of a space Y is a *network* at $y \in Y$ if U is a neighborhood of y in Y then $P \subset U$ for some $P \in \mathscr{P}$, and $y \in \bigcap \mathscr{P}$.

A strong accessibility space [16], by definition, is a space satisfying Definition 3.1 but with the sequence $\{A_n\}_n$ being decreasing in X.

Obviously, every strict accessibility space is strong accessibility.

THEOREM 3.2. *A T_1-space Y is a strict accessibility space if and only if every quotient mapping onto Y is strictly countably bi-quotient.*

Proof. Necessity. Let Y be a strict accessibility space. If there is a quotient mapping f from a space X onto the space Y such that f is not a strictly countably bi-quotient mapping, then there exists a point $y \in Y$ and a sequence $\{U_n\}_n$ of open subsets in X such that $\{U_n\}_n$ covers $f^{-1}(y)$ and $y \in f(U_n) - \mathrm{int}(f(U_n))$ for each $n \in \mathbf{N}$. Let $A_n = Y - f(U_n)$ for each $n \in \mathbf{N}$. Then y is an accumulation point of A_n. By the strict accessibility of Y, there exists a closed set C in Y such that y is an accumulation point of C, but not of $C - A_n = C \cap f(U_n)$ for each $n \in \mathbf{N}$. There exists an open neighborhood V_n of y in Y such that $V_n \cap C \cap f(U_n) = \{y\}$, thus $C \cap \overline{f(U_n)} - \{y\} = C \cap \overline{f(U_n)} - V_n$ is closed. Put $D = C - \{y\}$. Then D is a non-closed set in Y, thus $f^{-1}(D)$ is a non-closed set in X because f is quotient. Take a point $x \in \overline{f^{-1}(D)} - f^{-1}(D)$, then $f(x) \in \bar{D} - D = \{y\}$, so $x \in f^{-1}(y) \subset \bigcup_{n \in \mathbf{N}} U_n$. Hence $x \in U_m$ for some $m \in \mathbf{N}$. Set $G = U_m - f^{-1}(D)$. Then $x \in G \cap \overline{f^{-1}(D)}$, and

$$G = U_m - f^{-1}(D) = U_m - f^{-1}(C \cap \overline{f(U_m)} - \{y\})$$

is open in X, thus $G \cap f^{-1}(D) \neq \emptyset$, a contradiction. Hence, every quotient mapping onto Y is strictly countably bi-quotient.

Sufficiency. Let Y be a T_1-space which is not strict accessibility. Then there exists a sequence $\{A_n\}_n$ of subsets in Y and a point $y \in Y$ such that y is an accumulation point of A_n for each $n \in \mathbf{N}$, and if C is a closed set in Y and y is an accumulation point of C, then y is an accumulation point of $C - A_n$ for some $n \in \mathbf{N}$.

Assume that $y \notin A_n$ and let $B_n = Y - (A_n \cup \{y\})$ for each $n \in \mathbf{N}$. Then $Y = A_n \cup B_n \cup \{y\}$. Put $X = \bigcup_{n \in \mathbf{N}} X_n$, where each

$$X_n = ((Y - \{y\}) \times \{0\} \times \{n\}) \cup ((B_n \cup \{y\}) \times \{1\} \times \{n\}).$$

And X is endowed with the subspace topology of the product space $Y \times \{0,1\} \times \mathbf{N}$. Define a mapping $f : X \to Y$ by $f(x,t,n) = x$, $\forall (x,t,n) \in X$. Then f is continuous and onto. For each $n \in \mathbf{N}$, put $U_n = (B_n \cup \{y\}) \times \{1\} \times \{n\}$, then U_n is open in X, and $f(U_n) = B_n \cup \{y\}$ is not a neighborhood of y in Y. Since

$$f^{-1}(y) = \{(y,1,n) : n \in \mathbf{N}\} \subset \bigcup_{n \in \mathbf{N}} U_n,$$

f is not strictly countably bi-quotient. Next, we will show that f is quotient.

Let $E \subset Y$ and $f^{-1}(E)$ be a closed subset of X. Since $(Y - \{y\}) \times \{0\} \times \{0\}$ is a homeomorphic to $Y - \{y\} \subset Y$, assume that $y \notin E$, thus we have left only to show that $y \notin \bar{E}$. Since $f^{-1}(E) \cap ((Y - \{y\}) \times \{0\} \times \{0\})$ is closed in X, $E - \{y\} = E$ is closed in $Y - \{y\}$, thus $E = \bar{E} - \{y\}$, i.e., $\bar{E} \subset E \cup \{y\}$. Since

$f^{-1}(E)$ is closed in X and $(y, 1, n) \notin f^{-1}(E)$ for each $n \in \mathbf{N}$, then there exists an open neighborhood V_n of y in Y such that $V_n \cap B_n \cap E = \emptyset$, so $E \cap V_n \subset A_n$. Put $C = \overline{E} \cap \overline{\bigcup_{n\in\mathbf{N}} V_n}$. Then C is closed in Y and for each $n \in \mathbf{N}$, $C \cap V_n = \overline{E} \cap V_n \subset A_n \cup \{y\}$, so y is not an accumulation point of $C - A_n$. Then y is not an accumulation point of C, thus $y \notin \overline{E}$. Therefore, E is closed in Y, so f is quotient. □

Let X be a space. Denote $A^\alpha = \{x \in X : x \text{ is an accumulation point of } A\}$. A point $x \in X$ is an accumulation point of a family $\mathscr{F}$ of subsets of X if $x \in F^\alpha$ for each $F \in \mathscr{F}$.

Theorem 3.3. *The following are equivalent for a strong accessibility space X*:
(1) *X is a strict accessibility space*;
(2) *If a sequence $\{A_n\}_n$ of subsets of X has an accumulation point x, then x is also an accumulation point of the sequence $\{\bigcap_{n\le m} A_n\}_m$*;
(3) *$(A \cap B)^\alpha = A^\alpha \cap B^\alpha$ for each $A, B \subset X$.*

Proof. (1) ⇒ (2). Suppose that a sequence $\{A_n\}_n$ of subsets of X has an accumulation point x. Since X is strict accessibility, there exists a closed subset C such that x is an accumulation point of C, but not of $C - A_n$ for each $n \in \mathbf{N}$. For each $m \in \mathbf{N}$ and an open neighborhood U at x in X, if $n \le m$, there exists an open neighborhood V_n at x in X such that $V_n \cap (C - A_n) \subset \{x\}$, i.e., $V_n \cap C \subset A_n \cup \{x\}$. Put $V = \bigcap_{n\le m} V_n$. Then $V \cap C \subset \bigcap_{n\le m} A_n \cup \{x\}$, thus $U \cap V \cap C - \{x\} \subset U \cap (\bigcap_{n\le m} A_n)$. Since x is an accumulation of C, x is an accumulation point of $\bigcap_{n\le m} A_n$.

(2) ⇒ (3) is obvious.

(3) ⇒ (1). If a sequence $\{A_n\}_n$ of subsets of X has an accumulation point $x \in X$, then x is also an accumulation point of the sequence $\{\bigcap_{n\le m} A_n\}_m$ by (3). By the strong accessibility of X, there is a closed subset C of X such that x is an accumulation of C, but not of $C - \bigcap_{n\le m} A_n$ for each $m \in \mathbf{N}$, thus x not of $C - A_n$ for each $n \in \mathbf{N}$. Hence, X is strict accessibility. □

Corollary 3.4. *Strict accessibility is a hereditary property.*

Proof. Since strong accessibility is hereditary [16], and the condition (3) in Theorem 3.3 is also hereditary, strict accessibility is hereditary by Theorem 3.3. □

A space X is a *k-space* [3] if U is open in X whenever $U \cap K$ is open in K for every compact subset K of X. Every sequential space is a k-space.

Lemma 3.5 [1]. *Let X be a T_2-space. Then X is a Fréchet space if and only if X is a k-space and every quotient mapping onto X is pseudo-open.*

Corollary 3.6. *Every compact subset is finite in a strict accessibility T_2-space.*

Proof. Let X be a strict accessibility T_2-space. If X contains an infinite compact subset K, then K is Fréchet by Corollary 3.4, Lemma 3.5, and Theorem 3.2, thus there exists a non-trivial convergent sequence $\{x_k\}$ in K. Put $A = \{x_{2k} : k \in \mathbf{N}\}$, $B = \{x_{2k+1} : k \in \mathbf{N}\}$, then $(A \cap B)^\alpha \neq A^\alpha \cap B^\alpha$, a contradiction by Theorem 3.3. Hence, every compact subset is finite in X. □

COROLLARY 3.7. *There are no non-trivial convergent sequences in a strict accessibility space.*

Every strongly Fréchet T_2-space is a strong accessibility space [16]. Is a strictly Fréchet T_2-space a strict accessibility space? The answer is negative. The real line $\mathbf{R}$ is not a strict accessibility space by Corollary 3.7.

4. Related mappings

A mapping $f : X \to Y$ is *bi-quotient* [8] if for each $y \in Y$ and for each cover $\mathscr{U}$ of $f^{-1}(y)$ by open subsets of X, $y \in \mathrm{int}(f(\bigcup \mathscr{U}'))$ for some finite family $\mathscr{U}' \subset \mathscr{U}$.

Zhu [22] proved that an almost-open mapping is equivalent to a mapping satisfying the definition of bi-quotient mappings but with $y \in \mathrm{int}(f(U))$ for some $U \in \mathscr{U}$. It is easy to see that, almost-open mappings $\Rightarrow$ bi-quotient mappings and strictly countably bi-quotient mappings.

A space X is *bi-sequential* [10] if whenever $\mathscr{F}$ is a filter base in X with a cluster point $x \in X$[5], there exists a decreasing sequence $\{A_n\}_n$ of subsets in X such that each A_n intersects each element of $\mathscr{F}$, and $A_n \to x$.

Zhu [22] proved that a first countable space is equivalent to a space satisfying the definition of bi-sequential spaces but with the family $\mathscr{F}$ of subsets of X having a cluster point $x \in X$. It is easy to see that, first countable spaces $\Rightarrow$ bi-sequential spaces and strictly Fréchet spaces.

F. Siwiec [17, Table 22, p. 32] posed the following question in 1975: give an intrinsic characterization of the class of spaces Y such that every quotient mapping onto Y is bi-quotient. In this section the question is answered and some related mappings are discussed.

A space X is *determined* by a cover $\mathscr{P}$ of X, or $\mathscr{P}$ *determines* X, if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is relatively open (relatively closed) in P for every $P \in \mathscr{P}$ [7].

THEOREM 4.1. *The following are equivalent for a T_1-space X:*
(1) *Every quotient mapping onto X is almost-open*;
(2) *If X is determined by a cover $\mathscr{P}$, then $\{\mathrm{int}(P) : P \in \mathscr{P}\}$ is a cover of X.*

Proof. (1) $\Rightarrow$ (2). Assume that every quotient mapping onto X is almost-open. If X is determined by a cover $\mathscr{P}$, let $Z = \bigoplus \mathscr{P}$, and f be the natural mapping from Z onto X. Then f is quotient by [7, Lemma 1.8], thus f is

[5] A point $x \in X$ is a *cluster point* of a family $\mathscr{F}$ of subsets of a space X if $x \in \bar{F}$ for each $F \in \mathscr{F}$.

almost-open. For each $x \in X$, there is a $z_x \in f^{-1}(x)$ such that $f(U)$ is a neighborhood at x in X if U is a neighborhood at z_x in Z. Take $P \in \mathscr{P}$ with $z_x \in P$, then P is open in Z, thus $x \in \mathrm{int}(f(P)) = \mathrm{int}(P)$. Hence $\{\mathrm{int}(P) : P \in \mathscr{P}\}$ is a cover of X.

(2) $\Rightarrow$ (1). Let $f : Z \to X$ be quotient, where X satisfies the condition (2). If f is not almost-open, then there is $x_0 \in X$ satisfying that there is an open neighborhood U_z of z in Z such that $f(U_z)$ is not a neighborhood of x_0 in X for each $z \in f^{-1}(x_0)$. Put $\mathscr{U} = \{U_z : z \in f^{-1}(x_0)\} \cup \{Z - f^{-1}(x_0)\}$. Then $\mathscr{U}$ is an open cover of Z, Z is determined by $\mathscr{U}$, thus X is determined by $f(\mathscr{U})$ because f is quotient [7, Lemma 1.7]. Hence $\{\mathrm{int}(P) : P \in f(\mathscr{U})\}$ is a cover of X by the condition (2). There is $z \in f^{-1}(x_0)$ such that $x_0 \in \mathrm{int}(f(U_z))$, a contradiction. So f is almost-open. $\square$

By the similar method in Theorem 4.1 Siwiec's question above has an answer as follow.

Theorem 4.2. *Let X be a T_1-space. Then every quotient mapping onto X is bi-quotient if and only if whenever X is determined by a cover $\mathscr{P}$ then $\{\mathrm{int}(\bigcup \mathscr{P}') :$ a finite $\mathscr{P}' \subset \mathscr{P}\}$ is a cover of X.*

Lemma 4.3. *Every space is a sequence-covering image of a metric space which is the topological sum of some convergent sequences.*

Proof. Let X be a space. Denote the family of all convergent sequences containing its limit in X by $\{S_\alpha : \alpha \in A\}$. For every $\alpha \in A$, set $S_\alpha = \{x_\alpha\} \cup \{x_{\alpha,n} : n \in \mathbf{N}\}$, where $x_{\alpha,n} \to x_\alpha$. Denote S_α endowed with the following new topology by S'_α: the neighborhoods of the point x_α in S'_α are the finite complement subsets of S_α, the other points are isolated. Then S'_α is a compact metric space, the topology on S'_α is finer than the subspace topology on S_α of X. Let M be the disjoint topological sum of the family $\{S'_\alpha : \alpha \in A\}$ [3], and define a function $f : M \to X$ by $f|_{S'_\alpha} : S'_\alpha \to S_\alpha$ is homeomorphic for each $\alpha \in A$. Then M is a metric space which is the topological sum of some convergent sequences, and f is continuous and onto. It is easy to see that f is sequence-covering. $\square$

Theorem 4.4. *The following are equivalent for a T_2, k-space X:*

(1) *X is a discrete space*;

(2) *Every mapping onto X is open*;

(3) *Every quotient mapping onto X is almost-open*;

(4) *Every quotient mapping onto X is bi-quotient*;

(5) *Every quotient mapping onto X is strictly countably bi-quotient.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5). And (5) $\Rightarrow$ (1) by Theorem 3.2 and Corollary 3.6.

(4) $\Rightarrow$ (1). Assume that every quotient mapping onto X is bi-quotient. Then X is Fréchet by Lemma 3.5. There is a metric space $M = \bigoplus \mathscr{S}$ and a sequence-covering mapping $f : M \to X$ such that each $S \in \mathscr{S}$ is a convergent

sequence containing its a limit by Lemma 4.3. Then f is quotient because X is a sequence space [16], so f is bi-quotient. For each $x \in X$, $f^{-1}(x)$ is covered by the family $\mathscr{S}$ of open subsets of M, there is a subset C in X which is the union of finite convergent sequences in X such that C is a neighborhood of x in X because f is bi-quotient. If x is not an isolated point in X, assume that $C = \{x\} \cup \{x_n : n \in \mathbf{N}\}$ is an open subset of X, where the sequence $\{x_n\}_n$ is nontrivial and converges to x. So $X = C \oplus (X - C)$. Let $\Psi(\mathbf{N}) = \mathscr{A} \cup \mathbf{N}$ be the Isbell-Mrówka space [2, Example 4.4], and let $Y = \Psi(\mathbf{N}) \oplus (X - C)$. A mapping $f : Y \to X$ is defined by

$$f(y) = \begin{cases} x, & y \in \mathscr{A} \\ x_n, & y = n \in \mathbf{N} \\ y, & y \in X - C. \end{cases}$$

Then f is quotient, but not bi-quotient [15, Theorem 2.2], a contradiction. Thus X is a discrete space. □

LEMMA 4.5. *Let $f : X \to Y$ be strictly countably bi-quotient. If f is a boundary Lindelöf mapping with a T_1-space Y, then f is almost-open.*

Proof. If f is not almost-open, there exists $y \in Y$ such that for each $x \in f^{-1}(y)$ there exists an open neighborhood U_x at x in X satisfying $y \notin \mathrm{int}(f(U_x))$. Then y is not an isolated point in Y. Since $\partial f^{-1}(y)$ is Lindelöf, there is a countable subset $\{x_i : i \in \mathbf{N}\} \subset f^{-1}(y)$ such that $\partial f^{-1}(y) \subset \bigcup\{U_{x_i} : i \in \mathbf{N}\}$, thus $f^{-1}(y) \subset \mathrm{int}(f^{-1}(y)) \cup (\bigcup\{U_{x_i} : i \in \mathbf{N}\})$. Since f is strictly countably bi-quotient, $y \in \mathrm{int}(f(U_{x_i}))$ for some $i \in \mathbf{N}$, a contradiction. □

THEOREM 4.6. *Let $f : X \to Y$ be a closed mapping. The following are equivalent for a metric space X:*
(1) *f is an almost-open mapping*;
(2) *f is a set-sequence-covering mapping*;
(3) *f is a sequence-covering mapping*;
(4) *f is a strictly countably bi-quotient mapping.*

Proof. (1) ⇒ (2) by [20, Proposition 2.4], and (2) ⇒ (3) is obvious.

(3) ⇒ (4). Suppose that f is sequence-covering. Since metric spaces are preserved by sequence-covering and closed mappings [12, 19], Y is metric, thus f is strictly countably bi-quotient by Lemma 2.4.

(4) ⇒ (1). Suppose that f is strictly countably bi-quotient. Since f is countably bi-quotient, f is boundary-compact by [10, Corollary 9.10]. Thus f is almost-open by Lemma 4.5. □

Remark 4.7. (1) There exists a perfect mapping[6] which is not sequence-covering with compact metric domain and range [16, Example 2.6]. Thus a

[6] A mapping $f : X \to Y$ is *perfect* if f is closed and each $f^{-1}(y)$ is compact for each $y \in Y$.

perfect mapping on a metric space can not be strictly countably bi-quotient because it can not be sequence-covering.

(2) A closed and almost-open mapping on a metric space can not be open. For example, let $Y = \{y\} \cup \{y_n : n \in \mathbf{N}\}$, where $\{y_n\}$ is a non-trivial sequence converging to y. Let $X = \{y\} \oplus Y$, and f be the natural mapping from X onto Y. Then X is a metric space, f is closed and almost-open, but not open.

Question 4.8. Give an intrinsic characterization of the class of spaces X satisfying the condition that every sequence-covering mapping onto X is set-sequence-covering.

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