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KNOT QUANDLES AND INFINITE CYCLIC COVERING SPACES

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Abstract

Let K be an n-dimensional knot ($n \geq 1$), $Q(K)$ the knot quandle of K, $\mathbf{Z}_q[t^{\pm 1}]/J$ an Alexander quandle, and $C_\infty(K)$ the infinite cyclic covering space of $S^{n+2}\backslash K$. We show that the set consisting of homomorphisms $Q(K) \to \mathbf{Z}_q[t^{\pm 1}]/J$ is isomorphic to $\mathbf{Z}_q[t^{\pm 1}]/J \oplus \mathrm{Hom}_{\mathbf{Z}[t^{\pm 1}]}(H_1(C_\infty(K)), \mathbf{Z}_q[t^{\pm 1}]/J)$ as $\mathbf{Z}[t^{\pm 1}]$-modules. Here, $\mathrm{Hom}_{\mathbf{Z}[t^{\pm 1}]}(H_1(C_\infty(K)), \mathbf{Z}_q[t^{\pm 1}]/J)$ denotes the set consisting of $\mathbf{Z}[t^{\pm 1}]$-homomorphisms $H_1(C_\infty(K)) \to \mathbf{Z}_q[t^{\pm 1}]/J$.

1. Introduction

A quandle is an algebraic system having a self-distributive binary operation whose definition is motivated by knot theory. Associated with an n-dimensional knot K ($n \geq 1$), we have the knot quandle $Q(K)$ [5, 7, 9], which is a generalization of the knot group $\pi_1(S^{n+2}\backslash K)$. Here, an n-dimensional knot denotes the image of a locally flat PL embedding of an oriented n-dimensional sphere S^n into S^{n+2}. We are interested in the set $\mathrm{Hom}(Q(K), X)$ consisting of homomorphisms from $Q(K)$ to a quandle X to compute a quandle cocycle invariant of K [1, 2, 3, 4].

Let $\mathbf{Z}_q[t^{\pm 1}]/J$ be a $\mathbf{Z}[t^{\pm 1}]$-module for some $q \geq 2$ and an ideal J of $\mathbf{Z}_q[t^{\pm 1}]$. Here, we denote by $R[t^{\pm 1}]$ the Laurent polynomial ring in the variable t over a ring R. We can provide $\mathbf{Z}_q[t^{\pm 1}]/J$ with a quandle structure called an Alexander quandle. The set $\mathrm{Hom}(Q(K), \mathbf{Z}_q[t^{\pm 1}])$ has a $\mathbf{Z}[t^{\pm 1}]$-module structure. Let $C_\infty(K)$ be the infinite cyclic covering space of $S^{n+2}\backslash K$. We denote by $\mathrm{Hom}_{\mathbf{Z}[t^{\pm 1}]}(H_1(C_\infty(K)), \mathbf{Z}_q[t^{\pm 1}]/J)$ the $\mathbf{Z}[t^{\pm 1}]$-module consisting of $\mathbf{Z}[t^{\pm 1}]$-homomorphisms $H_1(C_\infty(K)) \to \mathbf{Z}_q[t^{\pm 1}]/J$, where we consider $H_1(C_\infty(K))$ as a $\mathbf{Z}[t^{\pm 1}]$-module. We denote by $\Delta_K^{(i)}(t)$ the i-th Alexander polynomial of K. The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let K be an n-dimensional knot ($n \geq 1$), and $Q(K)$ the knot quandle of K. Let $\mathbf{Z}_q[t^{\pm 1}]/J$ be an Alexander quandle. Then*

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$$\mathrm{Hom}(Q(K), \mathbf{Z}_q[t^{\pm 1}]/J) \cong \mathbf{Z}_q[t^{\pm 1}]/J \oplus \mathrm{Hom}_{\mathbf{Z}[t^{\pm 1}]}(H_1(C_\infty(K)), \mathbf{Z}_q[t^{\pm 1}]/J)$$

as $\mathbf{Z}[t^{\pm 1}]$*-modules. Further, if* q *is prime,*

$$\mathrm{Hom}(Q(K), \mathbf{Z}_q[t^{\pm 1}]/J) \cong \mathbf{Z}_q[t^{\pm 1}]/J \oplus \bigoplus_{i=0}^{\infty} \mathbf{Z}_q[t^{\pm 1}]/((\Delta_K^{(i)}(t)/\Delta_K^{(i+1)}(t)), J).$$

The second isomorphism in Theorem 1.1 is known for $n = 1$ in [6] and $n = 2$ with $\Delta_K^{(0)}(t) = 1$ in [11]. Theorem 1.1 is a generalization of these results for any dimension.

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2. Preliminaries

A *quandle* is a non-empty set X with a binary operation $*$ satisfying the following properties:

(Q1) For any $x \in X$, $x * x = x$.

(Q2) For any $y \in X$, the map $*y : X \to X$ $(x \mapsto x * y)$ is bijective.

(Q3) For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

The notions of homomorphism and isomorphism are appropriately defined. For any quandles X and Y, we denote by $\mathrm{Hom}(X, Y)$ the set consisting of homomorphisms $X \to Y$.

Let X be a subset of a group closed under conjugations. Then X is a quandle with a binary operation $*$ defined by $x * y = y^{-1}xy$ for any $x, y \in X$. We call it a *conjugation quandle*.

Let $\mathbf{Z}_q[t^{\pm 1}]/J$ be a $\mathbf{Z}[t^{\pm 1}]$-module for some $q \geq 2$ and an ideal J of $\mathbf{Z}_q[t^{\pm 1}]$. Then $\mathbf{Z}_q[t^{\pm 1}]/J$ is a quandle with a binary operation $*$ defined by $x * y = tx + (1-t)y$ for any $x, y \in \mathbf{Z}_q[t^{\pm 1}]/J$. We call it an *Alexander quandle*. Suppose X is another quandle. For any $\varphi, \psi \in \mathrm{Hom}(X, \mathbf{Z}_q[t^{\pm 1}]/J)$, a map $\varphi + \psi : X \to \mathbf{Z}_q[t^{\pm 1}]/J$ defined by $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ for any $x \in X$ is a homomorphism. Further, for any $\varphi \in \mathrm{Hom}(X, \mathbf{Z}_q[t^{\pm 1}]/J)$ and $a \in \mathbf{Z}[t^{\pm 1}]$, a map $a\varphi : X \to \mathbf{Z}_q[t^{\pm 1}]/J$ defined by $(a\varphi)(x) = a(\varphi(x))$ for any $x \in X$ is a homomorphism. Thus, $\mathrm{Hom}(X, \mathbf{Z}_q[t^{\pm 1}]/J)$ has a $\mathbf{Z}[t^{\pm 1}]$-module structure.

For a quandle X, let $\mathscr{F}(X)$ be the free group generated by the elements of X, and $\mathscr{N}(X)$ the subgroup of $\mathscr{F}(X)$ normally generated by $y^{-1}xy(x * y)^{-1}$ for any $x, y \in X$. We call the quotient group $\mathrm{As}(X) = \mathscr{F}(X)/\mathscr{N}(X)$ the *associated group* of X. Consider a natural map $\rho : X \to \mathrm{As}(X)$ which is the composition of the inclusion map $X \to \mathscr{F}(X)$ and the projection map $\mathscr{F}(X) \to \mathrm{As}(X)$. We let

$\mathrm{Red}(X) = \mathrm{Im}\,\rho$. By definition, $\mathrm{Red}(X)$ is closed under conjugations. We consider $\mathrm{Red}(X)$ as a conjugation quandle. We call $\mathrm{Red}(X)$ the *reduced quandle* of X.

Let K be an n-dimensional knot $(n \geq 1)$. Let $D = \{z \in \mathbf{C} \mid |z| \leq 1\}$ be the oriented closed unit disk, and $R = D \cup \{z \in \mathbf{C} \mid \arg(z) = 0, 1 \leq z \leq 5\}$. A *racket* of K is a continuous map $\mu : (R, \{0\}) \to (S^{n+2}, K)$ satisfying the following conditions:

(1) $\mu(5) = (0, 0, \dots, 0, 1)$, where we identify S^{n+2} with $\mathbf{R}^{n+2} \cup \{\infty\}$.
(2) $\mu(R) \cap K = \mu(0)$.
(3) The restriction $\mu|_D : D \to S^{n+2}$ is an embedding.
(4) The image $\mu(\partial D)$ is a positive meridian of K, where a positive meridian of K denotes an oriented meridian compatible with the orientation of K.

We define a product $*$ of rackets μ and ν by

$$(\mu * \nu)(z) = \begin{cases} \mu(z) & \text{if } |z| \leq 1, \\ \mu(4z - 3) & \text{if } 1 \leq z \leq 2, \\ \nu(13 - 4z) & \text{if } 2 \leq z \leq 3, \\ \nu(e^{2(z-3)\pi i}) & \text{if } 3 \leq z \leq 4, \\ \nu(4z - 15) & \text{if } 4 \leq z \leq 5. \end{cases}$$

Let $Q(K)$ be the set consisting of homotopy classes of rackets of K. Then $Q(K)$ is a quandle with a binary operation $*$ defined by $[\mu] * [\nu] = [\mu * \nu]$ for any $[\mu], [\nu] \in Q(K)$, where $[\mu]$ denotes the homotopy class of a racket μ. We call $Q(K)$ the *knot quandle* of K.

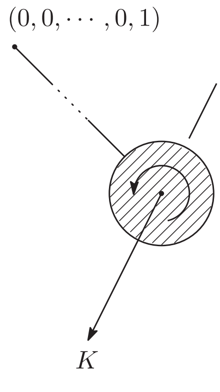

Figure 1. A racket of K

Let $RQ(K)$ be the subset of the knot group $\pi_1(S^{n+2} \backslash K)$ consisting of positive meridians. The set $RQ(K)$ is closed under conjugations. We consider $RQ(K)$ as a conjugation quandle. We call $RQ(K)$ the *reduced knot quandle* of K.

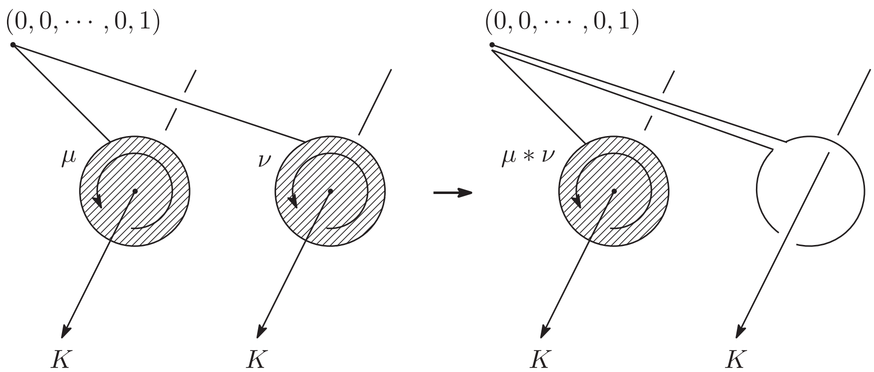

FIGURE 2. Product of rackets

LEMMA 2.1. *The reduced quandle* $\mathrm{Red}(Q(K))$ *is isomorphic to* $RQ(K)$.

Proof. Kamada showed in [8] that $\pi_1(S^{n+2} \backslash K)$ has a finite presentation. The argument in [8] also shows that the associated group $\mathrm{As}(Q(K))$ has a same finite presentation with $\pi_1(S^{n+2} \backslash K)$. Thus, $\mathrm{As}(Q(K))$ is isomorphic to $\pi_1(S^{n+2} \backslash K)$. A map illustrated in Figure 3 denotes an isomorphism $\mathrm{As}(Q(K)) \to \pi_1(S^{n+2} \backslash K)$. Since the isomorphism maps the rackets surjectively onto the positive meridians, $\mathrm{Red}(Q(K))$ is isomorphic to $RQ(K)$. $\square$

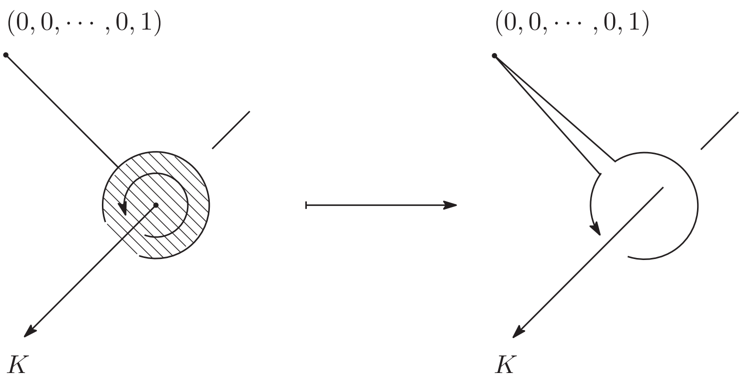

FIGURE 3. An isomorphism $\mathrm{As}(Q(K)) \to \pi_1(S^{n+2} \backslash K)$.

3. Proofs

We first show the following theorem for the reduced knot quandle $RQ(K)$ instead of the knot quandle $Q(K)$.

Theorem 3.1. *Let K be an n-dimensional knot $(n \geq 1)$, and $RQ(K)$ the reduced knot quandle of K. Let $\mathbf{Z}_q[t^{\pm 1}]/J$ be an Alexander quandle. Then*

$$\mathrm{Hom}(RQ(K), \mathbf{Z}_q[t^{\pm 1}]/J) \cong \mathbf{Z}_q[t^{\pm 1}]/J \oplus \mathrm{Hom}_{\mathbf{Z}[t^{\pm 1}]}(H_1(C_\infty(K)), \mathbf{Z}_q[t^{\pm 1}]/J)$$

as $\mathbf{Z}[t^{\pm 1}]$-modules. Further, if q is prime,

$$\mathrm{Hom}(RQ(K), \mathbf{Z}_q[t^{\pm 1}]/J) \cong \mathbf{Z}_q[t^{\pm 1}]/J \oplus \bigoplus_{i=0}^{\infty} \mathbf{Z}_q[t^{\pm 1}]/((\Delta_K^{(i)}(t)/\Delta_K^{(i+1)}(t)), J).$$

Proof. Let $G' = [\pi_1(S^{n+2}\backslash K), \pi_1(S^{n+2}\backslash K)]$ be the commutator subgroup of the knot group $\pi_1(S^{n+2}\backslash K)$. Choose and fix a positive meridian $m \in \pi_1(S^{n+2}\backslash K)$. We define a map $f : RQ(K) \to G'$ by $f(x) = xm^{-1}$ for any $x \in RQ(K)$. We recall that $H_1(C_\infty(K)) = G'/[G', G']$. We thus have a map $f_* : RQ(K) \to H_1(C_\infty(K))$ induced by f. Since $\pi_1(S^{n+2}\backslash K)$ has a finite presentation whose generators are positive meridians [8], $H_1(C_\infty(K))$ has a finite $\mathbf{Z}[t^{\pm 1}]$-module presentation $\langle x_1', \dots, x_u' \,|\, r_1', \dots, r_v' \rangle$ (See Section 7.D of [10]). We may assume that each x_i' is an element of $\mathrm{Im}\, f_*$, and each r_i' has a form $tf_*(x) + (1-t)f_*(y) - f_*(x * y)$ with some $x, y \in RQ(K)$. For each $\varphi \in \mathrm{Hom}(RQ(K), \mathbf{Z}_q[t^{\pm 1}]/J)$ satisfying $\varphi(m) = 0$, there is thus a unique $\mathbf{Z}[t^{\pm 1}]$-homomorphism $\Phi : H_1(C_\infty(K)) \to \mathbf{Z}_q[t^{\pm 1}]/J$ such that $\Phi \circ f_* = \varphi$. Conversely, since $f_*(x * y) = tf_*(x) + (1-t)f_*(y)$ for any $x, y \in RQ(K)$, for each $\mathbf{Z}[t^{\pm 1}]$-homomorphism $\Psi : H_1(C_\infty(K)) \to \mathbf{Z}_q[t^{\pm 1}]/J$, the composition $\Psi \circ f_* : RQ(K) \to \mathbf{Z}_q[t^{\pm 1}]/J$ is a homomorphism satisfying $\Psi \circ f_*(m) = 0$. We thus have a bijection

$$\{\varphi \in \mathrm{Hom}(RQ(K), \mathbf{Z}_q[t^{\pm 1}]/J) \,|\, \varphi(m) = 0\} \to \mathrm{Hom}_{\mathbf{Z}[t^{\pm 1}]}(H_1(C_\infty(K)), \mathbf{Z}_q[t^{\pm 1}]/J).$$

It is easy to see that the map is also a $\mathbf{Z}[t^{\pm 1}]$-isomorphism. For any $a \in \mathbf{Z}[t^{\pm 1}]/J$, we define a homomorphism $\tau_a : RQ(K) \to \mathbf{Z}_q[t^{\pm 1}]/J$ by $\tau_a(x) = a$ for any $x \in RQ(K)$. For any $\varphi \in \mathrm{Hom}(RQ(K), \mathbf{Z}_q[t^{\pm 1}]/J)$, the sum $\varphi + \tau_{-\varphi(m)} : RQ(K) \to \mathbf{Z}_q[t^{\pm 1}]/J$ satisfies $(\varphi + \tau_{-\varphi(m)})(m) = 0$. We thus have the first isomorphism.

If q is prime, since $\mathbf{Z}_q[t^{\pm 1}]$ is a principal ideal domain,

$$H_1(C_\infty(K)) \otimes \mathbf{Z}_q \cong \bigoplus_{i=0}^{\infty} \mathbf{Z}_q[t^{\pm 1}]/(\Delta_K^{(i)}(t)/\Delta_K^{(i+1)}(t)).$$

We thus have the second isomorphism. □

We next show that $\mathrm{Hom}(RQ(K), \mathbf{Z}_q[t^{\pm 1}]/J) \cong \mathrm{Hom}(Q(K), \mathbf{Z}_q[t^{\pm 1}]/J)$. Let $\mathbf{Z}_q[t^{\pm 1}]/J$ be a $\mathbf{Z}[t^{\pm 1}]$-module for some $q \geq 2$ and an ideal J of $\mathbf{Z}_q[t^{\pm 1}]$. Consider a semidirect product $\mathbf{Z}_q[t^{\pm 1}]/J \rtimes \mathbf{Z}$ with respect to an action of $\mathbf{Z}$ on $\mathbf{Z}_q[t^{\pm 1}]/J$ defined by $ka = t^{-k}a$ for any $a \in \mathbf{Z}_q[t^{\pm 1}]/J$ and $k \in \mathbf{Z}$. Let $\pi : \mathbf{Z}_q[t^{\pm 1}]/J \rtimes \mathbf{Z} \to \mathbf{Z}$ be the projection map of the second component. We remark that the preimage $\pi^{-1}(1)$ is closed under conjugations.

LEMMA 3.2. *The conjugation quandle* $\pi^{-1}(1)$ *is isomorphic to the Alexander quandle* $\mathbf{Z}_q[t^{\pm 1}]/J$.

Proof. Straightforward. □

For a quandle X, let $\rho : X \to \mathrm{Red}(X)$ be a natural map that is the composition of the inclusion map $X \to \mathscr{F}(X)$ and the projection map $\mathscr{F}(X) \to \mathrm{Red}(X) \subset \mathrm{As}(X)$. It is easy to see that ρ is a surjective homomorphism. We say X is *irreducible* if ρ is an isomorphism.

LEMMA 3.3. *Any conjugation quandle is irreducible.*

Proof. Suppose X is a subset of a group G closed under conjugations, and G_X the minimal subgroup of G containing X. We consider X as a conjugation quandle. Let $\imath : X \to \mathscr{F}(X)$ be the inclusion map. We define a homomorphism $\Phi : \mathscr{F}(X) \to G_X$ by $\Phi(\imath(x)) = x$ for any $x \in X$. Since $\Phi(\imath(x * y)) = y^{-1}xy$ for any $x, y \in X$, Φ sends $\mathscr{N}(X)$ to $\{1\}$. Further, for any elements $x, y \in X$ satisfying $xy^{-1} \neq 1$, $\Phi(\imath(x)\imath(y)^{-1}) \neq 1$. Thus, the natural map $\rho : X \to \mathrm{Red}(X)$ is also injective. □

Combining Lemmas 3.2 and 3.3, we have the following corollary.

COROLLARY 3.4. *Any Alexander quandle is irreducible.*

Let X and Y be quandles, and $\rho_X : X \to \mathrm{Red}(X)$ the natural map. We have an injective map $F : \mathrm{Hom}(\mathrm{Red}(X), Y) \to \mathrm{Hom}(X, Y)$ defined by $F(\varphi) = \varphi \circ \rho_X$ for any $\varphi \in \mathrm{Hom}(\mathrm{Red}(X), Y)$. The following key lemma is proved by Seiichi Kamada.

LEMMA 3.5 (Kamada). *The map* $F : \mathrm{Hom}(\mathrm{Red}(X), Y) \to \mathrm{Hom}(X, Y)$ *is bijective, if* Y *is irreducible.*

Proof. Suppose $\imath_X : X \to \mathscr{F}(X)$ and $\imath_Y : Y \to \mathscr{F}(Y)$ are inclusion maps. For each $\psi \in \mathrm{Hom}(X, Y)$, we define a homomorphism $\Psi : \mathscr{F}(X) \to \mathscr{F}(Y)$ by $\Psi(\imath_X(x)) = \imath_Y(\psi(x))$ for any $x \in X$. Since $\Psi(\imath_X(x * y)) = \imath_Y(\psi(x) * \psi(y))$ for any $x, y \in X$, Ψ sends $\mathscr{N}(X)$ to $\mathscr{N}(Y)$. Thus, Ψ induces a homomorphism $\Psi_* : \mathrm{As}(X) \to \mathrm{As}(Y)$. We define a homomorphism $\psi_* : \mathrm{Red}(X) \to \mathrm{Red}(Y)$ by $\psi_*(x) = \Psi_*(x)$ for any $x \in \mathrm{Red}(X)$. By assumption, we have a homomorphism $\rho_Y^{-1} \circ \psi_* : \mathrm{Red}(X) \to Y$, where $\rho_Y : Y \to \mathrm{Red}(Y)$ denotes the natural map. By construction, $F(\rho_Y^{-1} \circ \psi_*) = \psi$. Therefore, F is also surjective. □

We recall that the reduced knot quandle $RQ(K)$ is isomorphic to the reduced quandle $\mathrm{Red}(Q(K))$ (Lemma 2.1). Combining Corollary 3.4 and Lemma 3.5, we have a bijection $F : \mathrm{Hom}(RQ(K), \mathbf{Z}_q[t^{\pm 1}]/J) \to \mathrm{Hom}(Q(K), \mathbf{Z}_q[t^{\pm 1}]/J)$. It is easy to check that F is also a $\mathbf{Z}[t^{\pm 1}]$-isomorphism. We thus prove Theorem 1.1 by Theorem 3.1.

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