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A UNIQUENESS THEOREM FOR MEROMORPHIC MAPPINGS WITH A SMALL SET OF IDENTITY

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Abstract

The purpose of this article is to show a uniqueness theorem for meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$ with truncated multiplicities and a small set of identity.

1. Introduction

In 1983, L. Smiley [7] showed that

Theorem S. *Let f, g be linearly nondegenerate meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$. Let $\{H_j\}_{j=1}^q$ $(q \ge 3n+2)$ be hyperplanes in $\mathbf{C}P^n$ in general position. Asumme that*

(a) $f^{-1}(H_j) = g^{-1}(H_j)$, *for all* $1 \le j \le q$,

(b) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \le m-2$ *for all* $1 \le i < j \le q$, *and*

(c) $f = g$ *on* $\bigcup_{j=1}^q f^{-1}(H_j)$.

Then $f \equiv g$.

In [2]–[5], [8], [10] the authors and others extended the result of L. Smiley to the case where the number of hyperplanes is replaced by a smaller one and multiplicities are truncated by a positive integer bigger than 1. There are now many different results for the uniqueness problem with few hyperplanes. However, so far, in all results on the uniqueness problem of meromorphic mappings into $\mathbf{C}P^n$ with truncated multiplicities, the condition (c) on the identity set in the above theorem occurs. The main purpose of this paper is to give a uniqueness theorem for meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$ with truncated multiplicities and a smaller set of identity, in particular, the number of hyperplanes which appear in the above condition (c) will become to be only $(n+1)$. Our methods are quite different from those used in the proofs of previous unicity theorems. This comes from the fact that with only $(n+1)$ hyperplanes (in the condition (c)), generally, we cannot use any more the Second Main Theorem for meromorphic mappings and these hyperplanes.

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2. Preliminaries

We set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \dots, z_m) \in \mathbf{C}^m$ and define

$$B(r) := \{z \in \mathbf{C}^m : |z| < r\}, \quad S(r) := \{z \in \mathbf{C}^m : |z| = r\} \quad \text{for all } 0 < r < \infty.$$

Define

$$d^c := \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial), \quad \upsilon := (dd^c\|z\|^2)^{m-1} \quad \text{and} \quad \sigma := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1}.$$

Let F be a nonzero holomorphic function on $\mathbf{C}^m$. For each $a \in \mathbf{C}^m$, expanding F as $F = \sum P_i(z - a)$ with homogeneous polynomials P_i of degree i around a, we define

$$\nu_F(a) := \min\{i : P_i \not\equiv 0\}.$$

Let φ be a nonzero meromorphic function on $\mathbf{C}^m$. We define the divisor ν_φ as follows: For each $z \in \mathbf{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of z such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \le m - 2$ and then we put $\nu_\varphi(z) := \nu_F(z)$.

Let ν be a divisor in $\mathbf{C}^m$ and let k, M be positive integers or $+\infty$. Set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$ and

$$^{\le M}\nu^{[k]}(z) = 0 \quad \text{if } \nu(z) > M \quad \text{and} \quad ^{\le M}\nu^{[k]}(z) = \min\{\nu(z), k\} \quad \text{if } \nu(z) \le M,$$

$$^{> M}\nu^{[k]}(z) = 0 \quad \text{if } \nu(z) \le M \quad \text{and} \quad ^{> M}\nu^{[k]}(z) = \min\{\nu(z), k\} \quad \text{if } \nu(z) > M.$$

The counting function is defined by

$$^{\le M}N^{[k]}(r, \nu) := \int_1^r \frac{^{\le M}n(t)}{t^{2m-1}}\, dt$$

and

$$^{> M}N^{[k]}(r, \nu) := \int_1^r \frac{^{> M}n(t)}{t^{2m-1}}\, dt \quad (1 \le r < +\infty)$$

where

$$^{\le M}n(t) := \int_{|\nu| \cap B(r)} {}^{\le M}\nu^{[k]}.\upsilon \quad \text{for } m \ge 2, \quad ^{\le M}n(t) := \sum_{|z| \le t} {}^{\le M}\nu^{[k]}(z) \quad \text{for } m = 1$$

$$^{> M}n(t) := \int_{|\nu| \cap B(r)} {}^{> M}\nu^{[k]}.\upsilon \quad \text{for } m \ge 2, \quad ^{> M}n(t) := \sum_{|z| \le t} {}^{> M}\nu^{[k]}(z) \quad \text{for } m = 1.$$

For a nonzero meromorphic function φ on $\mathbf{C}^m$, we set $^{\le M}N_\varphi^{[k]}(r) := {}^{\le M}N^{[k]}(r, \nu_\varphi)$ and $^{> M}N_\varphi^{[k]}(r) := {}^{> M}N^{[k]}(r, \nu_\varphi)$. For brevity we will omit the character $^{[k]}$ (respectively $^{\le M}$) in the counting function and in the divisor if $k = +\infty$ (respectively $M = +\infty$).

Let $f : \mathbf{C}^m \to \mathbf{C}P^n$ be a meromorphic mapping. For arbitrary fixed homogeneous coordinates $(w_0 : \dots : w_n)$ of $\mathbf{C}P^n$, we take a reduced representation $f = (f_0 : \dots : f_n)$ which means that each f_i is a holomorphic function on $\mathbf{C}^n$ and $f(z) = (f_0(z) : \dots : f_n(z))$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$.

The characteristic function of f is defined by

$$T_f(r) = \int_{S(r)} \log\|f\|\sigma - \int_{S(1)} \log\|f\|\sigma, \quad 1 \leq r < +\infty.$$

For a meromorphic function φ on $\mathbf{C}^m$, the characteristic function $T_\varphi(r)$ of φ is defined by considering φ as a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^1$.

The proximity function $m(r, \varphi)$ is defined by

$$m(r, \varphi) = \int_{S(r)} \log^+|\varphi|\sigma,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$. Then

$$T_\varphi(r) = N_{1/\varphi}(r) + m(r, \varphi) + O(1).$$

We say that φ is "small" with respect to f if $T_\varphi(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgue measure). Denote by $\mathscr{R}_f$ the field of all "small" (with respect to f) functions on $\mathbf{C}^m$.

Let f, a be two meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$ with reduced representations $f = (f_0 : \dots : f_n)$, $a = (a_0 : \dots : a_n)$. Set $(f, a) := a_0 f_0 + \cdots + a_n f_n$. We say that a is "small" with respect to f if $T_a(r) = o(T_f(r))$ as $r \to \infty$ (outside a set of finite Lebesgue measure). We say that f is linearly nondegenerate over $\mathscr{R}_f$ if $f_0, \dots, f_n$ are linearly independant over $\mathscr{R}_f$.

Let $a_1, \dots, a_q$ $(q \geq n+1)$ be meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$ with reduced representations $a_j = (a_{j0} : \dots : a_{jn})$, $j = 1, \dots, q$. We say that $\{a_j\}_{j=1}^q$ are in general position if for any $1 \leq j_0 < \cdots < j_n \leq q$, $\det(a_{j_k i}, 0 \leq k, i \leq n) \not\equiv 0$.

We state the First and Second Main Theorems of Value Distribution Theory:

First Main Theorem. *Let a be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$ such that $(f, a) \not\equiv 0$ then*

$$N_{(f,a)}(r) \leq T_f(r) + T_a(r) \quad \text{for all } r \geq 1.$$

As usual, by the notation "$\|$ P" we mean the assertion P holds for all $r > 1$ outside a set of finite Lebesgue measure. For a hyperplane $H : a_0 w_0 + \cdots + a_n w_n = 0$ in $\mathbf{C}P^n$ with $\operatorname{im} f \not\subseteq H$, we denote $(f, H) = a_0 f_0 + \cdots + a_n f_n$.

Second Main Theorem. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{C}P^n$ and $H_1, \dots, H_q$ $(q \geq n+1)$ hyperplanes of $\mathbf{C}P^n$ in general position, then*

$$\| \ (q - n - 1) T_f(r) \leq \sum_{j=1}^{q} N_{(f, H_j)}^{[n]}(r) + o(T_f(r)).$$

3. Uniqueness theorem

Main Theorem. *Let f, g be nonconstant meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$. Let $\{H_j\}_{j=1}^q$ $(q \geq 3n+2)$ be hyperplanes in $\mathbf{C}P^n$ in general position such that*

$$\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \leq m-2 \quad \textit{for all } i,j \ (1 \leq i < j \leq q).$$

Assume that f and g are linearly nondegenerate over $\mathscr{R}_f$ and

(a) $\min\{\nu_{(f,H_j)}, n\} = \min\{\nu_{(g,H_j)}, n\}$, *for all $n+2 \leq j \leq q$, and*

(b) $f = g$ *on* $\bigcup_{j=1}^{n+1}(f^{-1}(H_j) \cup g^{-1}(H_j))$.

Then $f \equiv g$.

In order to prove the above Theorem, we need the following Lemmas.

Lemma 3.1. *Let $f : \mathbf{C}^m \to \mathbf{C}P^n$ be a nonconstant meromorphic mapping and $\{a_j\}_{j=1}^q$ $(q \geq n+2)$ be "small" (with respect to f) meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$ in general position. Assume that f is linearly nondegenerate over $\mathscr{R}_f$. Then*

$$\| \quad \frac{q}{n+2} T_f(r) \leq \sum_{j=1}^{q} N_{(f,a_j)}^{[n]}(r) + o(T_f(r)).$$

Proof. We refer to ([9], Theorem 3.1) ([6], Theorem 2.3). □

Lemma 3.2. *Let $f, g : \mathbf{C}^m \to \mathbf{C}P^n$ be two nonconstant meromorphic mappings. Assume that f and g are linearly nondegenerate over $\mathscr{R}_f$. Let $\{a_i\}_{i=1}^q$, $(q \geq 2n+3)$ be "small" (with respect to f) meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{C}P^n$ in general position. Assume that the followings are satisfied*

(i) $\| \ N_{(f,a_i)}^{[1]}(r) = o(T_f(r))$ *and* $\| \ N_{(g,a_i)}^{[1]}(r) = o(T_f(r))$, *for all* $i \in \{1, \ldots, n+1\}$.

(ii) $\min\{\nu_{(f,a_i)}, n\} = \min\{\nu_{(g,a_i)}, n\}$ *for all* $i \in \{n+2, \ldots, q\}$.

(iii) $\dim\{z \in \mathbf{C}^m : \nu_{(f,a_i)}(z) > 0$ and $\nu_{(f,a_j)}(z) > 0\} \leq m-2$, *for all* $n+2 \leq i < j \leq q$.

Then $f \equiv g$.

Proof. By the assumption i) we have

$$\| \ N_{(f,a_i)}^{[n]}(r) = o(T_f(r)) \quad \text{and} \quad \| \ N_{(g,a_i)}^{[n]}(r) = o(T_f(r)), \quad \text{for all } i \in \{1, \ldots, n+1\}.$$

Thus, by Lemma 3.1 and by the First Main Theorem, we have

$$\begin{aligned} \| \quad \frac{2n+3}{n+2} T_f(r) &\leq \sum_{i=1}^{2n+3} N_{(f,a_i)}^{[n]}(r) + o(T_f(r)) \\ &= \sum_{i=n+2}^{2n+3} N_{(f,a_i)}^{[n]}(r) + o(T_f(r)) \end{aligned}$$

$$= \sum_{i=n+2}^{2n+3} N^{[n]}_{(g,a_i)}(r) + o(T_f(r))$$
$$\leq (n+2)T_g(r) + o(T_f(r)).$$

This implies that

$$\| \ T_f(r) \leq \frac{(n+2)^2}{2n+3} T_g(r) + o(T_f(r)). \tag{3.1}$$

On the other hand $\| \ T_{a_i}(r) = o(T_f(r))$. Hence, $\| \ T_{a_i}(r) = o(T_g(r))$. So, similarly we have

$$\| \ T_g(r) \leq \frac{(n+2)^2}{2n+3} T_f(r) + o(T_g(r)). \tag{3.2}$$

By (3.1) and (3.2), we have

$$\| \ T_f(r) = O(T_g(r)) \quad \text{and} \quad \| \ T_g(r) = O(T_f(r)).$$

By Lemma 3.1, for each $i \in \{n+2, \dots, 2n+3\}$, we have

$$\begin{aligned} \| \ T_f(r) &\leq \sum_{j=1}^{n+1} N^{[n]}_{(f,a_j)}(r) + N^{[n]}_{(f,a_i)}(r) + o(T_f(r)) \\ &= N^{[n]}_{(f,a_i)}(r) + o(T_f(r)) \leq N_{(f,a_i)}(r) + o(T_f(r)). \end{aligned} \tag{3.3}$$

On the other hand, by the First Main Theorem, we have

$$\| \ N_{(f,a_i)}(r) \leq T_f(r) + o(T_f(r)).$$

Hence,

$$\| \ N_{(f,a_i)}(r) \leq N^{[n]}_{(f,a_i)}(r) + o(T_f(r)).$$

This implies that

$$\| \ N^{[n]}_{(f,a_i)}(r) + \frac{1}{n+1} {}^{>n}N_{(f,a_i)}(r) \leq N_{(f,a_i)}(r) \leq N^{[n]}_{(f,a_i)}(r) + o(T_f(r)). \tag{3.4}$$

By (3.3), (3.4) and by the First Main Theorem, for each $i \in \{n+2, \dots, 2n+3\}$, we have

$$\begin{aligned} &\| \ {}^{>n}N_{(f,a_i)}(r) = o(T_f(r)) \quad \text{and} \\ &\| \ T_f(r) = {}^{\leq n}N^{[n]}_{(f,a_i)}(r) + o(T_f(r)) = N_{(f,a_i)}(r) + o(T_f(r)). \end{aligned} \tag{3.5}$$

Similarly, for each $i \in \{n+2, \dots, 2n+3\}$, we have

$$\begin{aligned} &\| \ {}^{>n}N_{(g,a_i)}(r) = o(T_g(r)) \quad \text{and} \\ &\| \ T_g(r) = {}^{\leq n}N^{[n]}_{(g,a_i)}(r) + o(T_g(r)) = N_{(g,a_i)}(r) + o(T_g(r)). \end{aligned} \tag{3.6}$$

Set $\beta_{ij} = \dfrac{(f,a_i)}{(g,a_i)} \cdot \dfrac{(g,a_j)}{(f,a_j)} \ (n+2 \leq i \leq j \leq q)$.

We now show that $\beta_{ij} \in \mathscr{R}_f$. We have

$$\| \ m\left(r, \frac{(f, a_i)}{(f, a_j)}\right) = T_{(f,a_i)/(f,a_j)}(r) - N_{(f,a_j)/(f,a_i)}(r) + O(1)$$
$$\leq T_f(r) - N_{(f,a_j)}(r) + O(1) = o(T_f(r))$$

and

$$\| \ m\left(r, \frac{(g, a_i)}{(g, a_j)}\right) \leq T_g(r) - N_{(g,a_j)}(r) + O(1) = o(T_g(r)),$$

for all $n+2 \leq i \leq j \leq q$. This implies that

$$\| \ m(r, \beta_{ij}) \leq m\left(r, \frac{(f, a_i)}{(f, a_j)}\right) + m\left(r, \frac{(g, a_j)}{(g, a_i)}\right) + O(1) = o(T_f(r)).$$

On the other hand by (3.5), (3.6) and by the assumption ii) we have

$$\| \ N_{1/\beta_{ij}}(r) \leq {}^{>n}N_{(g,a_i)}(r) + {}^{>n}N_{(f,a_j)}(r) = o(T_f(r)), \quad \text{for all } n+2 \leq i \leq j \leq q.$$

Hence,

$$\| \ T_{\beta_{ij}}(r) = m(r, \beta_{ij}) + N_{1/\beta_{ij}}(r) + O(1) = o(T_f(r)), \quad \text{for all } n+2 \leq i < j \leq q.$$

This means that $\beta_{ij} \in \mathscr{R}_f$.

Set $\dfrac{(f, a_{n+2})}{(g, a_{n+2})} = h$. We have

$$(f, a_j) = \frac{h}{\beta_{(n+2)j}} \cdot (g, a_j) \quad (n+2 \leq j \leq 2n+3). \tag{3.7}$$

Set

$$P := \begin{pmatrix} a_{(n+2)0} & \cdots & a_{(2n+2)0} \\ \vdots & \ddots & \vdots \\ a_{(n+2)n} & \cdots & a_{(2n+2)n} \end{pmatrix},$$

and matrices P_i $(i \in \{n+2, \dots, 2n+2\})$ which are defined from P after changing the $(i-n-1)^{th}$ column by $\begin{pmatrix} a_{(2n+3)0} \\ \vdots \\ a_{(2n+3)n} \end{pmatrix}$. Put $u_i = \det(P_i)$, $u = \det(P)$.

It is easy to see that

$$(f, a_{2n+3}) = \sum_{i=n+2}^{2n+2} \frac{u_i}{u}(f, a_i) \quad \text{and} \quad (g, a_{2n+3}) = \sum_{i=n+2}^{2n+2} \frac{u_i}{u}(g, a_i).$$

Combining with (3.7), we get

$$\sum_{i=n+2}^{2n+2} \frac{u_i}{u}(f,a_i) = (f,a_{2n+3}) = \frac{h}{\beta_{(n+2)(2n+3)}} \cdot (g,a_{2n+3}) = \frac{h}{\beta_{(n+2)(2n+3)}} \sum_{i=n+2}^{2n+2} \frac{u_i}{u}(g,a_i)$$
$$= \sum_{i=n+2}^{2n+2} \frac{u_i \cdot \beta_{(n+2)i}}{u \cdot \beta_{(n+2)(2n+3)}}(f,a_i).$$

Thus,

$$\sum_{i=n+2}^{2n+2} \left(1 - \frac{\beta_{(n+2)i}}{\beta_{(n+2)(2n+3)}}\right) \cdot \frac{u_i \cdot (f,a_i)}{\prod_{j=n+2}^{2n+3} a_{jt_j}} = 0,$$

where a_{jt_j} is the first element of $a_{j0},\dots,a_{jn}$ not identically equal to zero ($j \in \{n+2,\dots,2n+3\}$). It is clear that $\frac{a_{ji}}{a_{jt_j}} \in \mathcal{R}_f$ ($i \in \{0,\dots,n\}$, $j \in \{n+2,\dots,2n+3\}$). Thus, since f is linearly nondegenerate over $\mathcal{R}_f$ and $\{a_i\}_{i=1}^q$ are general position, we have $\beta_{(n+2)(n+2)} = \cdots = \beta_{(n+2)(2n+3)}$. On the other hand $\beta_{(n+2)(n+2)} = 1$. Hence, $\frac{(f,a_{n+2})}{(g,a_{n+2})} = \cdots = \frac{(f,a_{2n+3})}{(g,a_{2n+3})}$. This implies that $f \equiv g$. $\square$

Proof of Main Theorem. Assume that $f \not\equiv g$.

We may assume that, after a suitable change of indices in $\{n+2,\dots,q\}$, we have

$$\underbrace{\frac{(f,H_{n+2})}{(g,H_{n+2})} \equiv \frac{(f,H_{k+2})}{(g,H_{k+2})} \equiv \cdots \equiv \frac{(f,H_{k_1})}{(g,H_{k_1})}}_{\text{group 1}} \not\equiv \underbrace{\frac{(f,H_{k_1+1})}{(g,H_{k_1+1})} \equiv \cdots \equiv \frac{(f,H_{k_2})}{(g,H_{k_2})}}_{\text{group 2}}$$
$$\not\equiv \underbrace{\frac{(f,H_{k_2+1})}{(g,H_{k_2+1})} \equiv \cdots \equiv \frac{(f,H_{k_3})}{(g,H_{k_3})}}_{\text{group 3}} \not\equiv \cdots \not\equiv \underbrace{\frac{(f,H_{k_{s-1}+1})}{(g,H_{k_{s-1}+1})} \equiv \cdots \equiv \frac{(f,H_{k_s})}{(g,H_{k_s})}}_{\text{group } s},$$

where $k_s = q$. Since $f \not\equiv g$, the number of elements of each group is at most n. For each $i \in \{n+2,\dots,q\}$, we set

$$\sigma(i) = \begin{cases} i+n & \text{if } i+n \le q, \\ i+2n+1-q & \text{if } i+n > q. \end{cases}$$

It is easy to see that $\sigma : \{n+2,\dots,q\} \to \{n+2,\dots,q\}$ is bijective and $|\sigma(i) - i| \ge n$. This implies that $\frac{(f,H_i)}{(g,H_i)}$ and $\frac{(f,H_{\sigma(i)})}{(g,H_{\sigma(i)})}$ belong to distinct groups. Hence, we have

$$P_i = \frac{(f,H_i)}{(f,H_{\sigma(i)})} - \frac{(g,H_i)}{(g,H_{\sigma(i)})} \not\equiv 0 \quad (n+2 \le i \le q).$$

Fix an arbitrary index i with $n+2 \le i \le q$. By the assumptions (a) and (b), we have

$$v_{P_i} \ge v^{[n]}_{(f,H_i)} + \sum_{v=1}^{n+1} v^{[1]}_{(f,H_v)},$$

outside a finite union of analytic sets of dimension $\le m-2$. It implies that

$$\begin{aligned} N_{P_i}(r) &\ge N^{[n]}_{(f,H_i)}(r) + \sum_{v=1}^{n+1} N^{[1]}_{(f,H_v)}(r) \\ &= N^{[n]}_{(f,H_i)}(r) + \frac{1}{2}\sum_{v=1}^{n+1}(N^{[1]}_{(f,H_v)}(r) + N^{[1]}_{(g,H_v)}(r)). \end{aligned} \tag{3.8}$$

Set $v_i := \max\{v_{(f,H_{\sigma(i)})}, v_{(g,H_{\sigma(i)})}\}$. By the definition of P_i, it is clear that $v_{1/P_i} \le v_i$. This implies that

$$N_{1/P_i}(r) \le N(r, v_i). \tag{3.9}$$

By the First Main Theorem, we have

$$\begin{aligned} m\left(r, \frac{(f,H_i)}{(f,H_{\sigma(i)})}\right) &= T_{(f,H_i)/(f,H_{\sigma(i)})}(r) - N_{(f,H_{\sigma(i)})/(f,H_i)}(r) + O(1) \\ &\le T_f(r) - N_{(f,H_{\sigma(i)})}(r) + O(1). \end{aligned}$$

Similarly,

$$m\left(r, \frac{(g,H_i)}{(g,H_{\sigma(i)})}\right) \le T_g(r) - N_{(g,H_{\sigma(i)})}(r) + O(1).$$

Then

$$\begin{aligned} m(r,P_i) &\le m\left(r, \frac{(f,H_i)}{(f,H_{\sigma(i)})}\right) + m\left(r, \frac{(g,H_i)}{(g,H_{\sigma(i)})}\right) + O(1) \\ &\le T_f(r) + T_g(r) - N_{(f,H_{\sigma(i)})}(r) - N_{(g,H_{\sigma(i)})}(r) + O(1). \end{aligned}$$

Hence, by (3.8), (3.9) and by the First Main Theorem we have

$$\begin{aligned} &N^{[n]}_{(f,H_i)}(r) + \frac{1}{2}\sum_{v=1}^{n+1}(N^{[1]}_{(f,H_v)}(r) + N^{[1]}_{(g,H_v)}(r)) \\ &\quad \le N_{P_i}(r) \le T_{P_i}(r) = N_{1/P_i}(r) + m(r,P_i) + O(1) \\ &\quad \le T_f(r) + T_g(r) + N(r,v_i) - N_{(f,H_{\sigma(i)})}(r) - N_{(g,H_{\sigma(i)})}(r) + O(1). \end{aligned} \tag{3.10}$$

Since $\min\{v_{(f,H_j)}, n\} = \min\{v_{(g,H_j)}, n\}$ $(j \ge n+2)$, we have $v_i(z) - v_{(f,H_{\sigma(i)})} - v_{(g,H_{\sigma(i)})} + v^{[n]}_{(f,H_{\sigma(i)})} \le 0$ on $\mathbf{C}^m$ (note that $\sigma(i) \ge n+2$). It follows that

$$N(r,v_i) - N_{(f,H_{\sigma(i)})}(r) - N_{(g,H_{\sigma(i)})}(r) + N^{[n]}_{(f,H_{\sigma(i)})}(r) \le 0.$$

Hence, by (3.10) we have

$$\frac{1}{2}\sum_{v=1}^{n+1}(N^{[1]}_{(f,H_v)}(r)+N^{[1]}_{(g,H_v)}(r))+N^{[n]}_{(f,H_i)}(r)+N^{[n]}_{(f,H_{\sigma(i)})}(r) \tag{3.11}$$
$$\leq T_f(r)+T_g(r)+O(1),$$

for all $n+2 \leq i \leq q$.

This implies that

$$\frac{q-n-1}{2}\sum_{v=1}^{n+1}(N^{[1]}_{(f,H_v)}(r)+N^{[1]}_{(g,H_v)}(r))+2\sum_{i=n+2}^{q}N^{[n]}_{(f,H_i)}(r) \tag{3.12}$$
$$\leq (q-n-1)(T_f(r)+T_g(r))+O(1),$$

(note that $\sigma:\{n+2,\dots,q\}\to\{n+2,\dots,q\}$ is bijective).

By the Second Main Theorem we have

$$\begin{aligned}
\| \ (q-n-1)(T_f(r)+T_g(r)) &\leq \sum_{i=1}^{q}(N^{[n]}_{(f,H_i)}(r)+N^{[n]}_{(g,H_i)}(r))+o(T_f(r)) \\
&= \sum_{i=1}^{n+1}(N^{[n]}_{(f,H_i)}(r)+N^{[n]}_{(g,H_i)}(r)) \\
&\quad + \sum_{i=n+1}^{q}(N^{[n]}_{(f,H_i)}(r)+N^{[n]}_{(g,H_i)}(r))+o(T_f(r)) \\
&\leq n\sum_{i=1}^{n+1}(N^{[1]}_{(f,H_i)}(r)+N^{[1]}_{(g,H_i)}(r)) \\
&\quad + 2\sum_{i=n+2}^{q}N^{[n]}_{(f,H_i)}(r)+o(T_f(r)).
\end{aligned}$$

Combining with (3.12) we have

$$\| \ \frac{q-n-1}{2}\sum_{v=1}^{n+1}(N^{[1]}_{(f,H_v)}(r)+N^{[1]}_{(g,H_v)}(r))\leq n\sum_{i=1}^{n+1}(N^{[1]}_{(f,H_i)}(r)+N^{[1]}_{(g,H_i)}(r))+o(T_f(r)).$$

On the other hand, $q \geq 3n+2$. Hence, we get

$$\| \ \sum_{i=1}^{n+1}(N^{[1]}_{(f,H_i)}(r)+N^{[1]}_{(g,H_i)}(r))=o(T_f(r)).$$

This implies that $\| \ N^{[1]}_{(f,H_i)}(r)=o(T_f(r))$ and $\| \ N^{[1]}_{(g,H_i)}(r)=o(T_f(r))$, for all $i\in\{1,\dots,n+1\}$. Then, by Lemma 3.2 we have $f\equiv g$. □

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