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HELICAL GEODESIC IMMERSIONS OF SEMI-RIEMANNIAN MANIFOLDS

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Abstract

We obtain some basic results on helical geodesic immersions in semi-Riemannian geometry. For example, it is shown that, for an indefinite semi-Riemannian submanifold, if any space-like geodesics of the submanifold are helices of order d, curvatures $\lambda_1, \dots, \lambda_{d-1}$ and signatures $\varepsilon_1, \dots, \varepsilon_d$ in the ambient space, then any time-like geodesics of the submanifold have the same order and curvatures, and signatures $(-1)^1\varepsilon_1, \dots, (-1)^d\varepsilon_d$.

1. Introduction

In Riemannian geometry, an isometric immersion is called a *helical geodesic immersion of order* d if the immersion maps every unit speed geodesic of the submanifold to a helix of order d in the ambient space whose curvatures are independent of the chosen geodesic. Several authors studied helical geodesic immersions, see [2], [6], [9] and [11], for example. We wish to investigate a semi-Riemannian version of helical geodesic immersions. First we consider non-null curves which satisfy Frenet formula (see [10]) and have constant curvatures as helices in a semi-Riemannian manifold. In contrast to the Riemannian case, we must be careful with causal characters, i.e., signatures, of Frenet frame fields of a curve. In fact, helices in a semi-Riemannian manifold of constant sectional curvature are locally determined by not only the curvatures but also the signatures up to isometries.

Let M and $\overline{M}$ be semi-Riemannian manifolds. In this paper, we consider an isometric immersion $f : M \to \overline{M}$ which maps every unit speed space-like geodesic γ of M to a helix of order d in $\overline{M}$ whose curvatures and signatures are independent of γ. We call such an immersion a *helical space-like geodesic immersion*. Hereafter we assume M is indefinite. Thus there also exist time-like and null geodesics on M. Under this situation, we investigate shapes of time-like geodesics of M in $\overline{M}$ and get the following result.

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Theorem A. *Let $f : M \to \overline{M}$ be a helical space-like geodesic immersion of order d, curvatures $\lambda_1, \dots, \lambda_{d-1}$ and signatures $\varepsilon_1, \dots, \varepsilon_d$. If M is indefinite, then f maps any time-like geodesics of M to helices of order d in $\overline{M}$ which have curvatures $\lambda_1, \dots, \lambda_{d-1}$ and signatures $(-1)^1\varepsilon_1, (-1)^2\varepsilon_2, \dots, (-1)^d\varepsilon_d$.*

In this case, the order of a helical space-like geodesic immersion is estimated by the dimension and the index of the ambient space as follows.

Theorem B. *Let M be indefinite. If $f : M \to \overline{M}$ is a helical space-like geodesic immersion of order d, then*

$$d \le \min\left\{ m - \frac{1 + (-1)^{[(m-1)/2]}}{2}, 4l \right\},$$

where $m = \dim \overline{M}$ and $l = \min\{\text{ind } \overline{M}, m - \text{ind } \overline{M}\}$.

Thus, for example, the order of a helical space-like geodesic immersion between Lorentzian manifolds is less than or equal to 4.

When the ambient space is of constant sectional curvature, the order of a helical space-like geodesic immersion is estimated by the dimension and the index of the normal spaces as follows.

Theorem C. *Let M be indefinite and $\overline{M}$ of constant sectional curvature. If $f : M \to \overline{M}$ is a helical space-like geodesic immersion of order d, then*

$$d \le \min\left\{ p + \frac{1 + (-1)^{[p/2]}}{2}, 4l' + 2 \right\},$$

where p is the codimension of f and $l' = \min\{\text{ind } T_qM^\perp, p - \text{ind } T_qM^\perp\}$ $(q \in M)$.

Moreover we obtain the following result on a null geodesic of M.

Theorem D. *Let $f : M \to \overline{M}$ be a helical space-like geodesic immersion of order d. If M is indefinite and $\overline{M}$ has constant sectional curvature, then f maps each null geodesic of M to a curve in an isotropic totally geodesic submanifold of $\overline{M}$. Moreover the proper order in $\overline{M}$ is less than or equal to* $\min\{l' + 1, d\}$, *where $l' = \min\{\text{ind } T_qM^\perp, p - \text{ind } T_qM^\perp\}$ $(q \in M)$.*

In this theorem, we call a submanifold of a semi-Riemannian manifold whose induced metric is vanishing an isotropic submanifold.

We note that the key in the argument of this paper is the fact that a multilinear map of a vector space V to a vector space vanishes if it takes 0 on an open set in V (Lemma 2.2).

In Section 2, we give the fundamental formulas in the theory of submanifolds. We also prepare several algebraic lemmas for later use. In Section 3, we recall the definition of Frenet curves in a semi-Riemannian manifold. Basic results of the curves are also stated. We give the precise definition and some examples of a helical space-like (resp. time-like) geodesic immersion. Theorem A and Theorem B are proved in this section. In Section 4, we study helical space-like geodesic immersions into a semi-Riemannian manifold of constant sectional curvature. To obtain Theorem C, we prove Theorem 4.2, which is a generalization of results on helical geodesic immersions in [9] and [6]. It is also shown that helical space-like geodesic immersions have geodesic normal sections, which is a semi-Riemannian version of a result in [4]. Applying this fact, we prove Theorem D.

2. Preliminaries

We let M and $\overline{M}$ be semi-Riemannian manifolds and $f: M \to \overline{M}$ be an isometric immersion. For all local formulas and computations we may assume f as an imbedding and thus we shall often identify $q \in M$ with $f(q) \in \overline{M}$. The tangent space T_qM is identified with a subspace $f_*(T_qM)$ of $T_q\overline{M}$. Letters X, Y, and Z (resp. ξ and η) will be vector fields tangent (resp. normal) to M. Let $\overline{\nabla}$ (resp. ∇) be the Levi-Civita connection of $\overline{M}$ (resp. M). Then Gauss formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

where B denotes the second fundamental form. Weingarten formula is given by

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where A_ξ denotes the shape tensor corresponding to ξ and $\nabla^\perp$ the normal connection. Clearly A_ξ is related to B as $\langle A_\xi X, Y\rangle = \langle B(X, Y), \xi\rangle$, $\langle\, ,\rangle$ being the induced metric on M from the semi-Riemannian metric of $\overline{M}$. Let R (resp. $R^\perp$) be the curvature tensor of ∇ (resp. $\nabla^\perp$) and D the covariant differentiation with respect to the induced connection in the direct sum of the tangent bundle TM and the normal bundle $TM^\perp$. Let us assume that $\overline{M}$ is a space of constant sectional curvature $\bar{c}$. Then the structure equations of Gauss, Codazzi, and Ricci are written as

$$R(X, Y)Z = \bar{c}(\langle Y, Z\rangle X - \langle X, Z\rangle Y) + A_{B(Y,Z)}X - A_{B(X,Z)}Y, \tag{1}$$

$$(D_X B)(Y, Z) = (D_Y B)(X, Z), \tag{2}$$

$$R^\perp(X, Y)\xi = B(X, A_\xi Y) - B(A_\xi X, Y). \tag{3}$$

The following identity is well known

$$\langle R^\perp(X, Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle. \tag{4}$$

The *mean curvature vector* H of f at q is defined by $H = \sum_{i=1}^n \langle e_i, e_i\rangle B(e_i, e_i)/n$, where $e_1, \dots, e_n$ is an orthonormal basis of T_qM and $n = \dim M$.

We define the covariant differentiation for any $TM^\perp$-valued tensor field T of type $(0, k)$ as follows. For vector fields $X, X_1, \dots, X_k$ tangent to M, we define

$$(D_X T)(X_1, \dots, X_k) = \nabla^{\perp}_X (T(X_1, \dots, X_k)) - \sum_{i=1}^{k} T(X_1, \dots, \nabla_X X_i, \dots, X_k),$$

and DT is defined by $(DT)(X, X_1, \dots, X_k) = (D_X T)(X_1, \dots, X_k)$, which is a $TM^{\perp}$-valued tensor field of type $(0, k+1)$. We denote by $D^2 T$ the covariant derivative of DT. Furthermore, we can inductively define the covariant derivative $D^i T$. An isometric immersion f is said to be *parallel* if $DB = 0$. For convenience' sake, a value $T(v, \dots, v)$ is written as $T(v^k)$ for a tensor field T of type $(0, k)$ and any $v \in TM$. We have Ricci identity for $T(X_1, \dots, X_k)$

$$\begin{aligned}(D^2 T)&(X, Y, X_1, \dots, X_k) - (D^2 T)(Y, X, X_1, \dots, X_k) \\ &= R^{\perp}(X, Y) T(X_1, \dots, X_k) - \sum_{i=1}^{k} T(X_1, \dots, R(X, Y) X_i, \dots, X_k).\end{aligned}$$

The following algebraic Lemmas are often used in this paper. Let V and W be finite-dimensional real vector spaces. It is easy that we prove the following lemma.

Lemma 2.1. *Let T_1, T_2 be r-linear mappings on V to W. Suppose $T_1(v^r) = T_2(v^r)$ for all $v \in V$, then for $v_1, \dots, v_r \in V$,*

$$\sum_{\sigma \in S_r} T_1(v_{\sigma(1)}, \dots, v_{\sigma(r)}) = \sum_{\sigma \in S_r} T_2(v_{\sigma(1)}, \dots, v_{\sigma(r)}),$$

where S_r is the symmetric group on r letters.

Let V be a scalar product space, that is, a real vector space furnished with a scalar product $\langle\, , \rangle$ which is a non-degenerate symmetric bilinear form on V. A vector v of V is said to be *space-like* (resp. *time-like* or *null*) if $\langle v, v\rangle > 0$ or $v = 0$ (resp. if $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$ and $v \neq 0$). Particularly null vectors are also said to be *light-like*. We may simply call a unit space-like vector (resp. time-like) $(+1)$-*vector* (resp. (-1)-). We say that $\varepsilon \in \{-1, +1\}$ is an *admissible signature* to a scalar product space V, if $-\mathrm{ind}\, V \leq \varepsilon \leq \dim V - \mathrm{ind}\, V$. It is easy to prove the following lemma ([1]).

Lemma 2.2. *For any r-linear mapping T on V to W and an admissible signature ε_1 to V, the following conditions are equivalent*:

(i) *$T(x, \dots, x) = 0$ for any unit ε_1-vector x of V,*

(ii) *$T(v, \dots, v) = 0$ for any vector v of V.*

Lemma 2.3. *For any $2r$-linear mapping T on V to W and an admissible signature ε_1 to V, the following conditions are equivalent*:

(i) *there exists an element w of W such that $T(x, \dots, x) = w$ for any unit ε_1-vector x of V,*

(ii) *there exists an element w of W such that $T(v,\dots,v)=(\varepsilon_1\langle v,v\rangle)^r w$ for any vector v of V,*

(iii) *there exists an element w of W such that $T(u,\dots,u)=(\varepsilon\varepsilon_1)^r w$ for any unit ε-vector u of V, where $\varepsilon\in\{-1,+1\}$ is admissible to V.*

Proof. Suppose that (i) holds. Because of $\varepsilon_1\langle x,x\rangle=1$ for any unit ε_1-vector x, we have $T(x,\dots,x)-(\varepsilon_1\langle x,x\rangle)^r w=0$. Taking Lemma 2.2 into account, we obtain (ii). The other are rather clear. □

3. Helical space-like geodesic immersions

Let N be a semi-Riemannian manifold. A curve c in N is *space-like* if all of its velocity vectors $c'(s)$ are space-like; similarly for *time-like* and *null*. An arbitrary curve need not have one of these *causal characters*, but a geodesic γ always dose since γ' is parallel, and parallel translation preserves causal character of vectors. A non-null curve c is said to have *unit speed* if $\langle c',c'\rangle=+1$ or -1 along c. We naturally generalize the notation of causal character of vectors to geodesics. Namely, we may denote a unit speed space-like (resp. time-like) geodesic by $(+1)$- (resp. (-1)-)*geodesic*.

We recall the definition of a Frenet curve of order d in a semi-Riemannian manifold. Let c be a unit speed curve in N. The curve c is said to be a *Frenet curve of order d* in N, if it has the orthonormal frame field $c_1,\dots,c_d$ and the following Frenet formulas along c are satisfied for all $1\le i\le d(\le \dim N)$

$$\begin{cases} c_1=c', \\ \langle c_i,c_i\rangle=\varepsilon_i, \\ \nabla_{c'}c_i=-\varepsilon_{i-1}\varepsilon_i\lambda_{i-1}c_{i-1}+\lambda_i c_{i+1}, \end{cases}$$

where ∇ denotes the Levi-Civita connection of N, $\lambda_0=\lambda_d=\varepsilon_0=0$, $c_0=c_{d+1}=0$, λ_i $(1\le i\le d-1)$ is a positive function along σ and $\varepsilon_i\in\{-1,+1\}$ $(1\le i\le d)$. Then λ_i (resp. σ_i and ε_i) is called the *i-th curvature* (resp. *i-th Frenet vector field* and *i-th signature*) of c. We may call such a curve a Frenet curve of $(d;\lambda_1,\dots,\lambda_{d-1};\varepsilon_1,\dots,\varepsilon_d)$ shortly. Furthermore, if the all curvatures are constant along c, then we call the curve a *helix of order d* in N. For example, a space-like (resp. time-like) geodesic is a helix of order 1 with $\varepsilon_1=+1$ (resp. $\varepsilon_1=-1$), and a circle (resp. hyperbola) is a helix of order 2 with $\varepsilon_1\varepsilon_2=+1$ (resp. $\varepsilon_1\varepsilon_2=-1$).

The proofs of the following propositions are essentially the same as that in Riemannian geometry.

Proposition 3.1. *Given a point $q\in N$, positive functions $\lambda_1,\dots,\lambda_{d-1}$ on an interval I $(0\in I)$ and an orthonormal frame $u_1,\dots,u_d$ of T_qN, then there locally exists a unique Frenet curve c of order d in N with $c(0)=q$ and $c'(0)=u_1$ such that its Frenet frame $c_1,\dots,c_d$ satisfies $(c_1(0),\dots,c_d(0))=(u_1,\dots,u_d)$, the signatures $\varepsilon_i=\langle u_i,u_i\rangle$ $(1\le i\le d)$, and $\lambda_1,\dots,\lambda_{d-1}$ are the curvatures of c.*

Proposition 3.2. *Suppose N is of constant sectional curvature. Let $c(s)$ (resp. $\hat{c}(s)$) $(s \in I)$ be a Frenet curve of $(d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$ (resp. $(\hat{d}; \hat{\lambda}_1, \dots, \hat{\lambda}_{d-1}; \hat{\varepsilon}_1, \dots, \hat{\varepsilon}_d)$). Then there exists an isometry g of N such that $\hat{c}(s) = (g \circ c)(s)$ if and only if $d = \hat{d}$, $\lambda_i = \hat{\lambda}_i$ $(1 \leq i \leq d-1)$ and $\varepsilon_i = \hat{\varepsilon}_i$ $(1 \leq i \leq d)$.*

For $\alpha \in \{-1, +1\}^k$ $(k \in \mathbf{N})$, we define $n(\alpha)$ (resp. $p(\alpha)$) by the potency of the negative (resp. positive) members of α. We say that $\alpha \in \{-1, +1\}^k$ is *admissible* to a semi-Riemannian manifold N if $n(\alpha) \leq \operatorname{ind} N$ and $p(\alpha) \leq \dim N - \operatorname{ind} N$. It is easy to see that there exists a Frenet curve of signatures $\varepsilon_1, \dots, \varepsilon_d$ in N if and only if $(\varepsilon_1, \dots, \varepsilon_d)$ is admissible to N.

If c is a Frenet curve on the ordinary n-sphere $S^n(K)$ of constant sectional curvature K in $\mathbf{R}^{n+1}$, then $\imath \circ c$ is always a Frenet curve in $\mathbf{R}^{n+1}$, where $\imath : S^n(K) \hookrightarrow \mathbf{R}^{n+1}$ is the inclusion. However this fact need not hold for Frenet curves on hyperquadrics of semi-Euclidean spaces. Let $Q_t^n(\varepsilon K) = \{p \in \mathbf{R}_{\bar{t}}^{n+1} \mid \langle p, p \rangle = \varepsilon K^{-1}\}$, where $K > 0$, $\varepsilon \in \{-1, +1\}$, and the index $\bar{t} = t$ if $\varepsilon = +1$ and $\bar{t} = t+1$ if $\varepsilon = -1$. We note that $Q_t^n(\varepsilon K)$ is a totally umbilical hypersurface with constant curvature εK in $\mathbf{R}_{\bar{t}}^{n+1}$. For a Frenet curve c of $(d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$, put $\Lambda_1 = \varepsilon K + \varepsilon_2 \lambda_1^2$ and $\Lambda_{2i-1} = \varepsilon_{2i-2} \lambda_{2i-2}^2 + \varepsilon_{2i} \lambda_{2i-1}^2$ $(3 \leq 2i-1 \leq d)$. We define the following polynomials of $\lambda_1^2, \dots, \lambda_{d-1}^2$ by

$$\mathscr{L}_1 = \Lambda_1, \quad \mathscr{L}_3 = \Lambda_1 \Lambda_3 - \lambda_1^2 \lambda_2^2,$$
$$\mathscr{L}_{2i-1} = \Lambda_{2i-1} \mathscr{L}_{2i-3} - \lambda_{2i-3}^2 \lambda_{2i-2}^2 \mathscr{L}_{2i-5},$$

where $5 \leq 2i-1 \leq d$. Let $\imath : Q_t^n(\varepsilon K) \hookrightarrow \mathbf{R}_{\bar{t}}^{n+1}$ be the inclusion. By a straightforward calculation, we have the following proposition.

Proposition 3.3. *Under the above situation, if $\mathscr{L}_{2i-1} \neq 0$ for all $1 \leq 2i-1 \leq d$, then the curve $\imath \circ c$ in $\mathbf{R}_{\bar{t}}^{n+1}$ is a Frenet curve of order d^* in $\mathbf{R}_{\bar{t}}^{n+1}$, where $d^* = d$ if d is even and $d^* = d+1$ if d is odd. The curvatures $\bar{\lambda}_1, \dots, \bar{\lambda}_{d^*-1}$ and signatures $\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_{d^*}$ of $\imath \circ c$ satisfy*

$$\bar{\varepsilon}_2 \bar{\lambda}_1^2 = \varepsilon K + \varepsilon_2 \lambda_1^2,$$
$$\bar{\varepsilon}_{2i-2} \bar{\lambda}_{2i-2}^2 + \bar{\varepsilon}_{2i} \bar{\lambda}_{2i-1}^2 = \varepsilon_{2i-2} \lambda_{2i-2}^2 + \varepsilon_{2i} \lambda_{2i-1}^2,$$
$$\bar{\lambda}_{2i-1} \bar{\lambda}_{2i} = \lambda_{2i-1} \lambda_{2i}, \quad \bar{\varepsilon}_{2i-1} = \varepsilon_{2i-1},$$

where $\bar{\varepsilon}_0 = \bar{\lambda}_0 = 0$ and $2 \leq i \leq [(d+1)/2]$.

Let $f : M \to \overline{M}$ be an isometric immersion between semi-Riemannian manifolds. Suppose that $(+1)$ is an admissible signature to M, let γ be any unit speed space-like geodesic of M. If the curve $f \circ \gamma$ in $\overline{M}$ is a helix of $(d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$ which are independent of the choice of γ, then f is called a *helical space-like geodesic immersion of order d, curvatures $\lambda_1, \dots, \lambda_{d-1}$ and signatures $\varepsilon_1, \dots, \varepsilon_d$*. For convenience' sake, we may call such an immersion a helical $(+1)$-

geodesic immersion of $(d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$, or a helical $(+1)$-geodesic immersion unless we need to specify the order, the curvatures and the signatures. We also define that f is a *helical time-like geodesic immersion* in a similar way.

We shall give examples of helical space-like (resp. time-like) geodesic immersions of order 2, 3 and 4.

Example 3.4. The pseudosphere in $\mathbf{R}_t^{n+1}$

$$S_t^n(K) = Q_t^n(+K) = \{q \in \mathbf{R}_t^{n+1} \mid \langle q, q \rangle = K^{-1}\}$$

and the pseudohyperbolic space in $\mathbf{R}_{t+1}^{n+1}$

$$H_t^n(K) = Q_t^n(-K) = \{q \in \mathbf{R}_{t+1}^{n+1} \mid \langle q, q \rangle = -K^{-1}\}.$$

It is well-known that, just as for the ordinary sphere $S^n(K)$, the geodesics of $S_t^n(K)$ (resp. $H_t^n(K)$) are the curves sliced from $S_t^n(K)$ (resp. $H_t^n(K)$) by 2-planes through the origin of $\mathbf{R}_t^{n+1}$ (resp. $\mathbf{R}_{t+1}^{n+1}$). For instance, when $t \neq n$, the space-like geodesics on $S_t^n(K)$ (resp. $H_t^n(K)$) are the circles (resp. hyperbolas) with the curvature $\sqrt{K}$ in $\mathbf{R}_t^{n+1}$ (resp. $\mathbf{R}_{t+1}^{n+1}$). Thus the inclusion $\iota : S_t^n(K) \hookrightarrow \mathbf{R}_t^{n+1}$ (resp. $\iota : H_t^n(K) \hookrightarrow \mathbf{R}_{t+1}^{n+1}$) is a helical space-like geodesic of $(2; \sqrt{K}; +1, +1)$ (resp. $(2; \sqrt{K}; +1, -1)$). We note that all null geodesics of $S_t^n(K)$ (resp. $H_t^n(K)$) are straight lines in $\mathbf{R}_t^{n+1}$, when $t \neq 0, n$. For details, see [8].

To simplify notation, from now on, we may denote $S_t^n(1)$ (resp. $H_t^n(1)$) by S_t^n (resp. H_t^n). The next example is well-known as the Veronese immersion of S_1^2. See [3], for example.

Example 3.5. We consider the following homogeneous polynomials of degree 2:

$$u_1 = xy, \quad u_2 = zx, \quad u_3 = yz,$$

$$u_4 = \frac{\sqrt{3}}{6}(2x^2 + y^2 + z^2), \quad u_5 = \frac{1}{2}(y^2 - z^2),$$

which satisfy $\Delta u_i = 0$ $(1 \leq i \leq 5)$ where $\Delta = -(-\partial_x^2 + \partial_y^2 + \partial_z^2)$ is the Laplacian on $\mathbf{R}_1^3$. Then the restriction f_i of u_i to S_1^2 is a solution of the equation $\Delta_{S_1^2} h = 6h$, where h is a smooth function on S_1^2, and $\Delta_{S_1^2}$ is the Laplacian on S_1^2. We define a map $S_1^2 \to \mathbf{R}_2^5$ by $q \mapsto (f_1(q), \dots, f_5(q))$ for any $q \in S_1^2$. It is easily seen that $-f_1^2 - f_2^2 + f_3^2 + f_4^2 + f_5^2 = 3^{-1}$ on S_1^2, hence we get $f : S_1^2 \to S_2^4(3)$. A direct computation shows that f is a helical space-like geodesic immersion of $(2; 1; +1, +1)$. By virtue of a semi-Riemannian version of Takahashi's theorem ([7]), the mean curvature vector of f vanishes identically in $S_2^4(3)$. Moreover let $\iota : S_2^4(3) \hookrightarrow \mathbf{R}_2^5$ be the inclusion. In view of Proposition 3.3 the composition $\iota \circ f$ is a helical space-like geodesic immersion of $(2; 2; +1, +1)$.

The following examples are concerned with helical space-like geodesic immersions of order 3 and 4.

Example 3.6. We consider the following homogeneous polynomials of degree 3:

$$u_1 = \frac{1}{4}x(x^2 - y^2 + 4z^2), \quad u_2 = \frac{\sqrt{15}}{12}x(x^2 + 3y^2), \quad u_3 = \frac{\sqrt{10}}{2}xyz,$$
$$u_4 = \frac{1}{4}y(x^2 - y^2 + 4z^2), \quad u_5 = \frac{\sqrt{15}}{12}y(3x^2 + y^2), \quad u_6 = \frac{\sqrt{10}}{4}z(x^2 + y^2),$$
$$u_7 = \frac{\sqrt{6}}{12}z(3x^2 - 3y^2 + 2z^2),$$

which satisfy $\Delta u_i = 0$ $(1 \le i \le 7)$. Then the restriction f_i of u_i to S_1^2 is a solution of the equation $\Delta_{S_1^2}h = 12h$. We define a map $S_1^2 \to \mathbf{R}_3^7$ by $q \mapsto (f_1(q), \dots, f_7(q))$ for any $q \in S_1^2$. Since $-f_1^2 - f_2^2 - f_3^2 + f_4^2 + f_5^2 + f_6^2 + f_7^2 = 6^{-1}$ on S_1^2, we obtain $f : S_1^2 \to S_3^6(6)$. According to a direct computation, f is a helical space-like geodesic immersion of $(3; \lambda_1, \lambda_2; +1, +1, +1)$, where its curvatures are

$$\lambda_1 = \sqrt{\frac{5}{2}}, \quad \lambda_2 = \sqrt{\frac{3}{2}}.$$

f has the vanishing mean curvature vector in $S_3^6(3)$. By Proposition 3.3, the composition $\iota \circ f : S_1^2 \to \mathbf{R}_3^7$ is also a helical space-like geodesic immersion of $(4; \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3; +1, +1, +1, +1)$, where its curvatures are

$$\bar{\lambda}_1 = \sqrt{\frac{17}{2}}, \quad \bar{\lambda}_2 = \sqrt{\frac{15}{34}}, \quad \bar{\lambda}_3 = 3\sqrt{\frac{2}{17}},$$

where $\iota : S_3^6(6) \hookrightarrow \mathbf{R}_3^7$ is the inclusion.

Fix an admissible signature $\varepsilon_1 \in \{-1, +1\}$ to M. Let $f : M \to \overline{M}$ be a helical ε_1-geodesic immersion of order d. Let x be an ε_1-tangent vector of M and γ an ε_1-geodesic of M such that $\gamma'(0) = x$. Put $\sigma = f \circ \gamma$. If $d = 1$, then we have $\overline{\nabla}_x\sigma_1 = B(x, x) = 0$ for any ε_1-vector x. Lemma 2.2 implies that f is totally geodesic. Thus we may assume that $d \ge 2$. So the first curvature λ_1 for σ is a positive constant. Let $\varepsilon_2 = \langle \sigma_2, \sigma_2 \rangle$. Then we obtain $\langle B(x, x), B(x, x) \rangle = \langle \overline{\nabla}_x\sigma_1, \overline{\nabla}_x\sigma_1 \rangle = \varepsilon_2\lambda_1^2$ for any ε_1-tangent vector x. From Lemma 2.3, we see f is constant isotropic, i.e., $\langle B(u, u), B(u, u) \rangle$ is constant for every unit vector u tangent to M.

Proof of Theorem A. Let $f : M \to \overline{M}$ be a helical ε_1-geodesic immersion of $(d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$. It suffices to show that, for each signature $\varepsilon \in \{-1, +1\}$, f maps any ε-geodesic of M to a helix in $\overline{M}$ of $(d; \lambda_1, \dots, \lambda_{d-1}; (\varepsilon\varepsilon_1)^1\varepsilon_1, \dots, (\varepsilon\varepsilon_1)^d\varepsilon_d)$. Let $u \in TM$ be an ε-vector, and γ_u ε-geodesic of M such that $\gamma_u'(0) = u$. Put $\sigma_u = f \circ \gamma_u$ and $U = \sigma_u'$. Since f is an isometric immersion, we define the first Frenet vector $\sigma_{u,1}$ (resp. signature) of σ_u by σ_u' (resp. $(\varepsilon\varepsilon_1)^1\varepsilon_1$). We have $\overline{\nabla}_U\sigma_{u,1} = B(U^2)$. If we put $F_2 = B$, then it is a $T\overline{M}$-valued $(0, 2)$-tensor field on M and $\overline{\nabla}_U\sigma_{u,1} = F_2(U^2)$. When u is an ε_1-vector x, by assumption, $\overline{\nabla}_X \sigma_{x,1} = \lambda_1 \sigma_{x,2}$, where $X = \sigma'_x$. Hence we obtain $\langle F_2(x^2), F_2(x^2) \rangle = \varepsilon_2 \lambda_1^2$ for any ε_1-vector $x \in TM$. Using Lemma 2.3, we find for any $v \in TM$

$$\langle F_2(v^2), F_2(v^2) \rangle = \varepsilon_2 \lambda_1^2 \langle v, v \rangle^2. \tag{5}$$

From (5), we define the second Frenet vector field $\sigma_{u,2}$ (resp. first curvature and second signature) of σ_u by $\lambda_1^{-1} F_2(U^2)$ (resp. λ_1 and $\varepsilon_2 = (\varepsilon\varepsilon_1)^2 \varepsilon_2$). Next, from $\varepsilon = \langle U, U \rangle$, we obtain

$$\overline{\nabla}_U \sigma_{u,2} + ((\varepsilon\varepsilon_1)^1 \varepsilon_1)((\varepsilon\varepsilon_1)^2 \varepsilon_2) \lambda_1 \sigma_{u,1} = \lambda_1^{-1} (\hat{D}_U F_2)(U^2) + \varepsilon_2 \langle U, U \rangle U,$$

where $\hat{D}$ denotes the covariant differentiation with respect to ∇ and $\overline{\nabla}$. Let F_1 be a $T\overline{M}$-valued $(0,1)$-tensor field on M such that $F_1(v) = v$ $(v \in TM)$. If we put $F_3 = \lambda_1^{-1} (\hat{D} F_2) + \varepsilon_2 \langle \, , \rangle F_1$, then it is a $T\overline{M}$-valued $(0,3)$-tensor field on M and $\overline{\nabla}_U \sigma_{u,2} + ((\varepsilon\varepsilon_1)^1 \varepsilon_1)((\varepsilon\varepsilon_1)^2 \varepsilon_2) \lambda_1 \sigma_{u,1} = F_3(U^3)$. When u is an ε_1-vector x, by assumption, $\overline{\nabla}_X \sigma_{x,2} + \varepsilon_1 \varepsilon_2 \lambda_1 \sigma_{x,1} = \lambda_2 \sigma_{x,3}$. Therefore we have, for any ε_1-vector $x \in TM$, $\langle F_3(x^3), F_3(x^3) \rangle = \varepsilon_3 \lambda_2^2$. Using Lemma 2.3, we get for any $v \in TM$

$$\langle F_3(v^3), F_3(v^3) \rangle = \varepsilon_1 \varepsilon_3 \lambda_2^2 \langle v, v \rangle^3. \tag{6}$$

From (6), we define the third Frenet vector field $\sigma_{u,3}$ (resp. second curvature and third signature) of σ_u by $\lambda_2^{-1} F_3(U^3)$ (resp. λ_2 and $(\varepsilon\varepsilon_1)^3 \varepsilon_3$).

Let m be a fixed natural number satisfying $3 \le m \le d - 1$. We assume that the followings hold for $2 \le k \le m$:

$$\overline{\nabla}_U \sigma_{u,k-1} = -((\varepsilon\varepsilon_1)^{k-2} \varepsilon_{k-2})((\varepsilon\varepsilon_1)^{k-1} \varepsilon_{k-1}) \lambda_{k-2} \sigma_{u,k-2} + \lambda_{k-1} \sigma_{u,k},$$

$$\sigma_{u,k} = \lambda_{k-1}^{-1} F_k(U^k),$$

where F_k is a $T\overline{M}$-valued $(0,k)$-tensor field on M. Then we have

$$\begin{aligned} &\overline{\nabla}_U \sigma_{u,m} + ((\varepsilon\varepsilon_1)^{m-1} \varepsilon_{m-1})((\varepsilon\varepsilon_1)^m \varepsilon_m) \lambda_{m-1} \sigma_{u,m-1} \\ &\quad = \lambda_{m-1}^{-1} (\hat{D}_U F_m)(U^m) + \varepsilon_1 \varepsilon_{m-1} \varepsilon_m \lambda_{m-1} \lambda_{m-2}^{-1} \langle U, U \rangle F_{m-1}(U^{m-1}). \end{aligned}$$

If we put $F_{m+1} = \lambda_{m-1}^{-1} (\hat{D} F_m) + \varepsilon_1 \varepsilon_{m-1} \varepsilon_m \lambda_{m-1} \lambda_{m-2}^{-1} \langle \, , \rangle F_{m-1}$, then it is a $T\overline{M}$-valued $(0, m+1)$-tensor field on M and we get

$$\overline{\nabla}_U \sigma_{u,m} + ((\varepsilon\varepsilon_1)^{m-1} \varepsilon_{m-1})((\varepsilon\varepsilon_1)^m \varepsilon_m) \lambda_{m-1} \sigma_{u,m-1} = F_{m+1}(U^{m+1}).$$

When u is an ε_1-vector x, by assumption, $\overline{\nabla}_X \sigma_{x,m} + \varepsilon_{m-1} \varepsilon_m \lambda_{m-1} \sigma_{x,m-1} = \lambda_m \sigma_{x,m+1}$. Therefore we have for any ε_1-vector $x \in TM$

$$\langle F_{m+1}(x^{m+1}), F_{m+1}(x^{m+1}) \rangle = \varepsilon_{m+1} \lambda_m^2.$$

Using Lemma 2.3, we have for any $v \in TM$

$$\langle F_{m+1}(v^{m+1}), F_{m+1}(v^{m+1}) \rangle = \varepsilon_1^{m+1} \varepsilon_{m+1} \lambda_m^2 \langle v, v \rangle^{m+1}. \tag{7}$$

From (7), we define the $(m+1)$-st Frenet vector field $\sigma_{u,m+1}$ (resp. m-th curvature and $(m+1)$-st signature) of σ_u by $\lambda_m^{-1} F_{m+1}(U^{m+1})$ (resp. λ_m and $(\varepsilon\varepsilon_1)^{m+1} \varepsilon_{m+1}$). We complete the induction.

We must prove

$$\overline{\nabla}_U \sigma_{u,d} = -((\varepsilon\varepsilon_1)^{d-1}\varepsilon_{d-1})((\varepsilon\varepsilon_1)^d \varepsilon_d)\lambda_{d-1}\sigma_{u,d-1}. \tag{8}$$

Because of $\sigma_{u,d} = \lambda_{d-1}^{-1} F_d(U^d)$, we have $\overline{\nabla}_U \sigma_{u,d} = \lambda_{d-1}^{-1}(\hat{D}_U F_d)(U^d)$. Since $\overline{\nabla}_X \sigma_{x,d} = -\varepsilon_{d-1}\varepsilon_d \lambda_{d-1}\sigma_{x,d-1}$ for any ε_1-vector $x \in TM$, we find $\lambda_{d-1}^{-1}(\hat{D}_x F_d)(x^d) + \varepsilon_{d-1}\varepsilon_d \lambda_{d-1}\lambda_{d-2}^{-1} F_{d-1}(x^{d-1}) = 0$. Using $\varepsilon_1\langle x, x\rangle = 1$,

$$\lambda_{d-1}^{-1}(\hat{D}_x F_d)(x^d) + \varepsilon_1\varepsilon_{d-1}\varepsilon_d \lambda_{d-1}\lambda_{d-2}^{-1}\langle x, x\rangle F_{d-1}(x^{d-1}) = 0. \tag{9}$$

Using Lemma 2.2, we see that Equation (9) holds if x is replaced by any $v \in TM$. From this, we can get (8). Theorem A was proved. □

Theorem A implies

COROLLARY 3.7. *f is a helical space-like geodesic immersion if and only if f is a helical time-like geodesic immersion.*

Thus, we may simply call the immersions helical geodesic immersions.

From Theorem A, if $f : M \to \overline{M}$ is a helical geodesic immersion of $(d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$, then $(\varepsilon_1, \dots, \varepsilon_d)$ and $((-1)^1\varepsilon_1, \dots, (-1)^d\varepsilon_d)$ are admissible to $\overline{M}$. To show Theorem B, we prepare the following paragraphs.

For convenience' sake, if d is odd, put d' and d'' be d and $(d-1)$ respectively, and if d is even, put d' and d'' be $(d-1)$ and d respectively. For any $\alpha = (\varepsilon_1, \dots, \varepsilon_d) \in \{-1, +1\}^d$, we put $\alpha_o = (\varepsilon_1, \varepsilon_3, \dots, \varepsilon_{d'})$, $\alpha_e = (\varepsilon_2, \varepsilon_4, \dots, \varepsilon_{d''})$ and $\bar{\alpha} = ((-1)^1\varepsilon_1, (-1)^2\varepsilon_2, \dots, (-1)^d\varepsilon_d)$. By definition, we have $n(\alpha) = n(\alpha_e) + n(\alpha_o)$, $n(\bar{\alpha}_e) = n(\alpha_e)$ and $n(\bar{\alpha}_o) = [(d+1)/2] - n(\alpha_o)$. Thus $n(\bar{\alpha}) = [(d+1)/2] - n(\alpha_o) + n(\alpha_e)$.

LEMMA 3.8. *For $\alpha, \bar{\alpha} \in \{-1, +1\}^d$, we have*

$$\min_{\alpha \in \{-1,+1\}^d} \max\{n(\alpha), n(\bar{\alpha})\} = \min_{\alpha \in \{-1,+1\}^d} \max\{p(\alpha), p(\bar{\alpha})\} = \left[\frac{d+3}{4}\right].$$

Proof. Because of $0 \le n(\alpha_o) \le [(d+1)/2]$,

$$\begin{aligned}
&\min_{\alpha \in \{-1,+1\}^d} \max\{n(\alpha), n(\bar{\alpha})\} \\
&\quad = \min_{\substack{\alpha \in \{-1,+1\}^d, \\ n(\alpha_e)=0}} \max\{n(\alpha), n(\bar{\alpha})\} \\
&\quad = \min_{\substack{\alpha \in \{-1,+1\}^d, \\ n(\alpha_e)=0}} \max\left\{n(\alpha_o), \left[\frac{d+1}{2}\right] - n(\alpha_o)\right\} \\
&\quad = \min\left\{\max\left\{i, \left[\frac{d+1}{2}\right] - i\right\} \middle| 0 \le i \le \left[\frac{d+1}{2}\right]\right\} = \left[\frac{d+3}{4}\right].
\end{aligned}$$

From the same argument for $p(\alpha)$, we can show the second equation in this lemma. $\square$

Lemma 3.9. *If there exists* $\alpha \in \{-1, +1\}^d$ *such that both* α *and* $\bar{\alpha}$ *are admissible to an m-dimensional scalar product space* V, *then*

$$d \leq m - \frac{1 + (-1)^{[(m-1)/2]}}{2}.$$

Proof. We can see $n(\alpha) - n(\bar{\alpha}) = 2n(\alpha_o) - [(d+1)/2]$. Hence $|n(\alpha) - n(\bar{\alpha})| \geq 1$ when $d \equiv 1, 2 \pmod 4$. In this case, without loss of generality, we can assume $n(\alpha) - n(\bar{\alpha}) \geq 1$. If both α and $\bar{\alpha}$ are admissible to V, then $n(\alpha) \leq \operatorname{ind} V$ and $p(\bar{\alpha}) \leq \dim V - \operatorname{ind} V$. Hence we have

$$d = p(\bar{\alpha}) + n(\bar{\alpha}) \leq \dim V - \operatorname{ind} V + \operatorname{ind} V - 1 = m - 1.$$

Hence we have $d \leq m - 1$, $d \equiv 1, 2 \pmod 4$. When $d \equiv 0, 3 \pmod 4$, at least, $d \leq m$. These inequalities imply

$$d \leq \begin{cases} m - 1 & m \equiv 1, 2 \pmod 4, \\ m & m \equiv 0, 3 \pmod 4. \end{cases}$$

Therefore this lemma is proved. $\square$

Proof of Theorem B. Let $\varepsilon_1, \ldots, \varepsilon_d$ be the Frenet signatures of ε_1-geodesics of M in $\overline{M}$. Put $\alpha = (\varepsilon_1, \ldots, \varepsilon_d)$. Since M is indefinite, from Theorem A, both α and $\bar{\alpha}$ must be admissible to $\overline{M}$. By Lemma 3.9,

$$d \leq m - \frac{1 + (-1)^{[(m-1)/2]}}{2},$$

where $m = \dim \overline{M}$. On the other hand, $\max\{n(\alpha), n(\bar{\alpha})\} \leq \operatorname{ind} \overline{M}$ and $\max\{p(\alpha), p(\bar{\alpha})\} \leq m - \operatorname{ind} \overline{M}$ must also hold. Using Lemma 3.8, we have

$$\left[\frac{d+3}{4}\right] \leq \min\{\operatorname{ind} \overline{M}, m - \operatorname{ind} \overline{M}\} = l.$$

Therefore we have $d \leq 4l$. Theorem B was proved. $\square$

Remark 3.10. We note

$$\min\left\{m - \frac{1 + (-1)^{[(m-1)/2]}}{2}, 4l\right\} = \begin{cases} 4l & (0 \leq l \leq [m/4]), \\ m - \dfrac{1 + (-1)^{[(m-1)/2]}}{2} & ([m/4] < l \leq [m/2]). \end{cases}$$

Moreover we can prove the following: There exists $\alpha = (\varepsilon_1, \ldots, \varepsilon_k)$ such that both α and $\bar{\alpha}$ are admissible to $\overline{M}$, where k is equal to the right hand side of the inequality in Theorem B.

Corollary 3.11. *Let $f : M \to \overline{M}$ be a helical geodesic immersion of order d. If M and $\overline{M}$ are Lorentzian manifolds, then d is less than and equal to 4.*

4. Helical geodesic immersions into a space-form

Let $f : M \to \overline{M}$ be an isometric immersion between semi-Riemannian manifolds. We put for integers k, l, h such that $k, l \geq 2$ and $0 \leq h \leq k-1$ and $v, w \in T_qM$ $(q \in M)$

$$\nu_{k,l}(v) = \langle (D^{k-2}B)(v^k), (D^{l-2}B)(v^l) \rangle,$$
$$\nu^h_{k,l}(v,w) = \langle (D^{k-2}B)(v^h, w, v^{k-h-1}), (D^{l-2}B)(v^l) \rangle.$$

Let ε_1 be an admissible signature to M. We consider the following equations for any natural number $i \geq 2$

$$(\mathscr{B}_i) \qquad \nu^h_{k,l}(v,w) = \begin{cases} (-1)^{(k-l)/2}\varepsilon_1^i \nu_i \langle v, v\rangle^{i-1}\langle v, w\rangle & k+l = 2i, \\ 0 & k+l = 2i+1, \end{cases}$$

where ν_i is constant, $k, l \geq 2$, $0 \leq h \leq k-1$ and $v, w \in T_qM$ $(q \in M)$.

Hereafter we assume that $\overline{M}$ is a semi-Riemannian manifold of constant sectional curvature and M is a (not necessarily indefinite) semi-Riemannian manifold. We prepare the following Lemma.

Lemma 4.1. *Let m be a fixed natural number satisfying $m \geq 2$. If $(\mathscr{B}_2), \ldots, (\mathscr{B}_m)$ and*

$$(10) \qquad \nu_{k,l}(v) = \begin{cases} (-1)^{(k-l)/2}\varepsilon_1^{m+1}\nu_{m+1}\langle v, v\rangle^{m+1} & k+l = 2m+2, \\ 0 & k+l = 2m+3, \end{cases}$$

for any $v \in TM$ hold, where ν_{m+1} is constant, then we have $(\mathscr{B}_{m+1})$.

Proof. Taking account of (10), we need only to consider the case $\langle v, w\rangle = 0$. Because of $(\mathscr{B}_2), \ldots, (\mathscr{B}_m)$, for any $v \in TM$,

$$(11) \qquad A_{(D^{i-2}B)(v^i)}v \wedge v = 0 \quad (2 \leq i \leq 2m-1).$$

Applying Ricci identity to $(D^{k-2}B)(v^k)$, $(2 \leq k \leq 2m-1)$, we have

$$\begin{aligned}(D^kB)&(v, w, v^k) - (D^kB)(w, v^{k+1}) \\ &= R^\perp(v,w)(D^{k-2}B)(v^k) - \sum_{h=0}^{k-1}(D^{k-2}B)(v^h, R(v,w)v, v^{k-h-1}).\end{aligned}$$

By (4), (11) and $\langle v, w\rangle = 0$, we obtain for $2 \leq k, l \leq 2m-1$

$$\langle R^\perp(v,w)(D^{k-2}B)(v^k), (D^{l-2}B)(v^l)\rangle = \langle [A_{(D^{k-2}B)(v^k)}, A_{(D^{l-2}B)(v^l)}]v, w\rangle = 0.$$

Moreover $\nu^h_{k,l}(v, R(v,w)v) = 0$ for $k+l = 2m, 2m+1$ because of $\langle v, R(v,w)v\rangle = 0$ and $(\mathscr{B}_m)$. Therefore we have for $k+l = 2m+2, 2m+3$

$$v^0_{k,l}(v,w) = v^1_{k,l}(v,w). \tag{12}$$

Differentiating $(\mathscr{B}_m)$ for $k+l=2m+1$ in the direction of v, we have for $1 \le h \le 2m-2$

$$\begin{aligned} v^h_{2m,2}(v,w) &= v \cdot v^{h-1}_{2m-1,2}(V,W) - v^{h-1}_{2m-1,3}(v,w) = -v^{h-1}_{2m-1,3}(v,w) \\ &= \cdots = (-1)^h v^0_{2m-h,h+2}(v,w), \end{aligned}$$

where V and W are local vector fields extending v and w such that the covariant derivative $\nabla_v V = \nabla_v W = 0$. Using (12) for $k+l=2m+2$,

$$\begin{aligned} v^0_{2m-h,h+2}(v,w) &= v^1_{2m-h,h+2}(v,w) = -v^0_{2m-h-1,h+3}(v,w) \\ &= -v^1_{2m-h-1,h+3}(v,w) = \cdots = (-1)^{2m-h-2} v^0_{2,2m}(v,w). \end{aligned}$$

Thus we have $v^h_{2m,2}(v,w) = v^0_{2,2m}(v,w)$ for $0 \le h \le 2m-1$. On the other hand using Lemma 2.1 for (10) in the case of $(k,l) = (2m,2)$, we have for $\langle v,w\rangle = 0$

$$\sum_{h=0}^{2m-1} v^h_{2m,2}(v,w) + 2v^0_{2,2m}(v,w) = 0.$$

Because of $v^h_{2m,2}(v,w) = v^0_{2,2m}(v,w)$, we have for $0 \le h \le 2m-1$

$$v^h_{2m,2}(v,w) = v^0_{2,2m}(v,w) = 0. \tag{13}$$

Furthermore, using $(\mathscr{B}_m)$, we get for $k+l=2m+2$ and $0 \le h \le k-1$

$$\begin{aligned} v^h_{k,l}(v,w) &= v \cdot v^h_{k,l-1}(V,W) - v^{h+1}_{k+1,l-1}(v,w) = -v^{h+1}_{k+1,l-1}(v,w) \\ &= \cdots = (-1)^{l-2} v^{h+l-2}_{2m,2}(v,w). \end{aligned}$$

The above equation and (13) imply $(\mathscr{B}_{m+1})$ for $k+l=2m+2$. To obtain $(\mathscr{B}_{m+1})$ for $k+l=2m+3$, differentiating $(\mathscr{B}_{m+1})$ for $k+l=2m+2$ in the direction of v, we have for $1 \le h \le 2m-1$

$$v^h_{2m+1,2}(v,w) = (-1)^h v^0_{2m+1-h,h+2}(v,w).$$

Using (12) for $k+l=2m+3$,

$$v^0_{2m+1-h,h+2}(v,w) = (-1)^{2m+1-h-2} v^0_{2,2m+1}(v,w).$$

Thus we have $v^h_{2m+1,2}(v,w) = -v^0_{2,2m+1}(v,w)$ for $0 \le h \le 2m$. By Lemma 2.1 for (10) in the case of $(k,l) = (2m+1,2)$,

$$\sum_{h=0}^{2m} v^h_{2m+1,2}(v,w) + 2v^0_{2,2m+1}(v,w) = 0.$$

Because of $v^h_{2m+1,2}(v,w) = -v^0_{2,2m+1}(v,w)$ we have for $0 \le h \le 2m$

$$v^h_{2m+1,2}(v,w) = v^0_{2,2m+1}(v,w) = 0. \tag{14}$$

Furthermore, using $(\mathscr{B}_{m+1})$ in the case of $k+l=2m+2$, we have $v^h_{k,l}(v,w) = (-1)^{l-2}v^{h+l-2}_{2m+1,2}(v,w)$, where $k+l=2m+3$ and $0 \le h \le k-1$. This equation and (14) imply $(\mathscr{B}_{m+1})$ for $k+l=2m+3$. Thus, we get $(\mathscr{B}_{m+1})$. $\square$

The following theorem is a semi-Riemannian version of a result on helical geodesic immersions, which is proved by Sakamoto in [9] (see [6] also).

THEOREM 4.2. *Let ε_1 be a fixed admissible signature to M and $f : M \to \overline{M}$ a helical geodesic immersion of $(d;\lambda_1,\dots,\lambda_{d-1};\varepsilon_1,\dots,\varepsilon_d)$. Then, for $\sigma = f\circ\gamma$ (γ is any ε-geodesic of M), the Frenet frame field $\sigma_1,\dots,\sigma_d$ of σ is given by*

$(\mathscr{F}_1)$ $\sigma_1 = \sigma'(=: U)$,

$(\mathscr{F}_i)$ $\sigma_i = (\lambda_1\cdots\lambda_{i-1})^{-1}\sum_{j=2}^{i} a^{\varepsilon}_{i,j}(D^{j-2}B)(U^j)$,

where $2\le i\le d$. The coefficients $a^{\varepsilon}_{i,j}$ satisfy

(i) $a^{\varepsilon}_{i,i}=1$ $(1\le i\le d)$,

(ii) $a^{\varepsilon}_{j-1,j}=0$ $(0\le j-1\le d)$,

(iii) $a^{\varepsilon}_{i,1}=0$ $(2\le i\le d)$,

(iv) $a^{\varepsilon}_{i,j}=E^{\varepsilon}_{i-2}E^{\varepsilon}_{i-1}\lambda^2_{i-2}a^{\varepsilon}_{i-2,j}+a^{\varepsilon}_{i-1,j-1}$ $(3\le j+1\le i\le d)$,

where $E^{\varepsilon}_i = (\varepsilon\varepsilon_1)^i\varepsilon_i$ and $\lambda_0=0$. Moreover we have $(\mathscr{B}_i)$ for $2\le i\le d$.

Proof. Let γ be an ε-geodesic of M. Put $\sigma = f\circ\gamma$. From Theorem A, σ satisfies for $1\le i\le d$

$$\overline{\nabla}_U\sigma_i = -E^{\varepsilon}_{i-1}E^{\varepsilon}_i\lambda_{i-1}\sigma_{i-1}+\lambda_i\sigma_{i+1}, \tag{15}$$

where $E^{\varepsilon}_i=(\varepsilon\varepsilon_1)^i\varepsilon_i=\langle\sigma_i,\sigma_i\rangle$ $(0\le i\le d)$, $\varepsilon_0=\lambda_0=\lambda_d=0$ and $\sigma_0=\sigma_{d+1}=0$. Because of $\overline{\nabla}_U\sigma_1=B(U^2)$ and $(15)_{i=1}$, we have $(\mathscr{F}_2)$ and $\langle B(U^2),B(U^2)\rangle = \varepsilon_2\lambda_1^2\langle U,U\rangle^2$. The latter equation and Lemma 2.1 imply for any $v\in TM$,

$$\langle B(v^2),B(v^2)\rangle = \varepsilon_2\lambda_1^2\langle v,v\rangle^2. \tag{16}$$

Differentiating (16) in the directions of v and w respectively,

$$v^2_{3,2}(v,w) = v\cdot v^1_{2,2}(V,W) - v^1_{2,3}(v,w) = -v^1_{2,3}(v,w),$$
$$v^0_{3,2}(v,w) = \frac{1}{2}w\cdot v_{2,2}(V') = 0,$$

where V, W and V' are local vector fields on M extending v, w and v respectively such that their covariant derivatives $\nabla_v V$, $\nabla_v W$ and $\nabla_w V'$ vanish. Codazzi equation (2), the above equations and (16) imply $(\mathscr{B}_2)$.

Let m be a fixed natural number satisfying $2\le m\le d-1$. We assume that $(\mathscr{F}_i)$ and $(\mathscr{B}_i)$ hold for $2\le i\le m$ and every $\sigma=f\circ\gamma$. Because of $(\mathscr{B}_2),\dots,(\mathscr{B}_m)$, for any $v\in TM$,

$$A_{(D^{i-2}B)(v^i)}v\wedge v = 0 \quad (2\le i\le 2m-1). \tag{17}$$

Differentiating $(\mathscr{F}_m)$ along the direction of U,

$$\overline{\nabla}_U\sigma_m = (\lambda_1\cdots\lambda_{m-1})^{-1}\sum_{j=2}^{m} a^{\varepsilon}_{m,j}(-A_{(D^{j-2}B)(U^j)}U + (D^{j-1}B)(U^{j+1})). \tag{18}$$

When $m = 2$, we have $\overline{\nabla}_U \sigma_2 = \lambda_1^{-1}(-\varepsilon\langle B(U^2), B(U^2)\rangle U + (DB)(U^3)) = -\varepsilon\varepsilon_2\lambda_1 U + \lambda_1^{-1}(DB)(U^3)$. On the other hand, $\overline{\nabla}_U \sigma_2 = -\varepsilon\varepsilon_2\lambda_1 U + \lambda_2\sigma_3$ from Frenet formulas. Thus $(\mathscr{F}_3)$ holds. In the case where $m \geq 3$, we will see that $(\mathscr{F}_{m+1})$ holds. Because of $m \geq 3$, $\overline{\nabla}_U \sigma_m$ is orthogonal to U. Hence (17) and (18) imply that the tangential part of $\overline{\nabla}_U \sigma_m$ vanishes, that is,

$$\sum_{j=2}^{m} a_{m,j}^{\varepsilon} A_{(D^{j-2}B)(U^j)} U = 0. \tag{19}$$

Therefore, taking account of $(15)_{i=m}$ and $(\mathscr{F}_{m-1})$, we obtain $(\mathscr{F}_{m+1})$, that is to say,

$$\lambda_m \sigma_{m+1} = (\lambda_1 \cdots \lambda_{m-1})^{-1} \sum_{j=2}^{m+1} a_{m+1,j}^{\varepsilon} (D^{j-2}B)(U^j),$$

where we put

$$a_{m+1,m+1}^{\varepsilon} = 1, \quad a_{m+1,m+2}^{\varepsilon} = a_{m+1,1}^{\varepsilon} = 0,$$
$$a_{m+1,j}^{\varepsilon} = E_{m-1}^{\varepsilon} E_m^{\varepsilon} \lambda_{m-1}^2 a_{m-1,j}^{\varepsilon} + a_{m,j-1}^{\varepsilon} \quad (2 \leq j \leq m).$$

We shall prove $(\mathscr{B}_{m+1})$. Let $x \in TM$ be an ε_1-vector and γ_x an ε_1-geodesic of M, and put $\sigma_x = f \circ \gamma_x$. Because of $(\mathscr{B}_2), \ldots, (\mathscr{B}_m)$ and $(\mathscr{F}_{m+1})$ for σ_x,

$$v_{m+1,m+1}(x) = \varepsilon_{m+1}(\lambda_1 \cdots \lambda_m)^2 - \sum_{(l,n) \neq (m+1,m+1)} a_{m+1,l}^{\varepsilon_1} a_{m+1,n}^{\varepsilon_1} v_{l,n}(x).$$

This equation and $(\mathscr{B}_2), \ldots, (\mathscr{B}_m)$ imply that $v_{m+1,m+1}(x)$ is independent of the choice of the ε_1-vectors x. Thus we put $v_{m+1,m+1}(x) = v_{m+1} \in \mathbf{R}$. By Lemma 2.3, we get for any $v \in TM$

$$v_{m+1,m+1}(v) = \varepsilon_1^{m+1} v_{m+1} \langle v, v\rangle^{m+1}. \tag{20}$$

Differentiating $(\mathscr{B}_m)$ for $k + l = 2m + 1$, we get $v_{k+1,l}(v) + v_{k,l+1}(v) = v \cdot v_{k,l}(V) = 0$, where V is a local vector field on M extending v such that the covariant derivative $\nabla_v V$ vanishes. So, by (20), we have

$$v_{k,l}(v) = (-1)^{(k-l)/2} \varepsilon_1^{m+1} v_{m+1} \langle v, v\rangle^{m+1} \quad (k + l = 2m + 2). \tag{21}$$

From (21) and the constancy of v_{m+1},

$$v_{m+2,m+1}(v) = \frac{1}{2} v \cdot v_{m+1,m+1}(V) = 0. \tag{22}$$

Differentiating (21) in the direction of v, we have $v_{k+1,l}(v) + v_{k,l+1}(v) = v \cdot v_{k,l}(V) = 0$ for $k + l = 2m + 2$. This relation and (22) imply

$$v_{k,l}(v) = 0 \quad (k + l = 2m + 3). \tag{23}$$

Using Lemma 4.1, the inductive hypothsis, (21) and (23) imply $(\mathscr{B}_{m+1})$. This completes the proof of the theorem. □

The coefficients $a^{\varepsilon}_{i,j}$ of $(\mathscr{F}_i)$ have the following properties.

Lemma 4.3. *The coefficients $a^{\varepsilon}_{i,j}$ of $(\mathscr{F}_i)$ in Theorem 4.2 satisfy*

$$a^{\varepsilon}_{i,j} = 0 \quad (i + j\text{: } odd), \tag{24}$$

$$a^{\varepsilon}_{i,j} = (\varepsilon\varepsilon_1)^{(i-j)/2} a^{\varepsilon_1}_{i,j} \quad (i + j\text{: } even). \tag{25}$$

Proof. Because of (ii) and (iii), for $i + j = 1, 3$, $a^{\varepsilon}_{0,1} = a^{\varepsilon}_{1,2} = a^{\varepsilon}_{2,1} = 0$. Using (iv), we can see that (24) holds for every natural numbers i, j such that $i + j$ is odd, by the induction. We shall prove that $(25)_{i,j}$ is true for even $i + j$. Equation (i) says that $(25)_{i,i}$ holds for $1 \le i \le d$ and (iii) implies that $(25)_{i,1}$ holds for any odd natural number suth that $3 \le i \le d$. Hence we obtain $(25)_{i,j}$ for $1 \le j \le i \le 3$ hold. Let $3 \le k < d$. Suppose that $(25)_{i,j}$ holds for $1 \le j \le i \le k$. Since we already showed that $(25)_{k+1,1}$ and $(25)_{k+1,k+1}$ hold, it suffices to show that $(25)_{k+1,l}$ for $2 \le l \le k-1$. Using the assumption of induction, we have for even $(k+1)+l$

$$\begin{aligned}
a^{\varepsilon}_{k+1,l} &= E^{\varepsilon}_{k-1} E^{\varepsilon}_{k} \lambda^2_{k-1} a^{\varepsilon}_{k-1,l} + a^{\varepsilon}_{k,l-1} \\
&= (\varepsilon\varepsilon_1)\varepsilon_{k-1}\varepsilon_k \lambda^2_{k-1} (\varepsilon\varepsilon_1)^{((k-1)-l)/2} a^{\varepsilon_1}_{k-1,l} + (\varepsilon\varepsilon_1)^{(k-(l-1))/2} a^{\varepsilon_1}_{k,l-1} \\
&= (\varepsilon\varepsilon_1)^{((k+1)-l)/2} (\varepsilon_{k-1}\varepsilon_k \lambda^2_{k-1} a^{\varepsilon_1}_{k-1,l} + a^{\varepsilon_1}_{k,l-1}) \\
&= (\varepsilon\varepsilon_1)^{((k+1)-l)/2} a^{\varepsilon_1}_{k+1,l}.
\end{aligned}$$

We have complete the proof. □

Proposition 4.4. *Let ε_1 be a fixed admissible signature to M and $f : M \to \overline{M}$ a helical geodesic immersion of $(d; \lambda_1, \dots, \lambda_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$. Then we have for any $v \in TM$ and m $(3 \le m \le d)$,*

$$\sum_{j=2}^{m} a^{\varepsilon_1}_{m,j} (\varepsilon_1 \langle v, v \rangle)^{(m-j)/2} A_{(D^{j-2}B)(v^j)} v = 0. \tag{26}$$

If $d = 2$, then f is parallel. If $d \ge 3$, then, for any $v \in TM$,

$$\sum_{j=2}^{d+1} a^{\varepsilon_1}_{d+1,j} (\varepsilon_1 \langle v, v \rangle)^{(d+1-j)/2} (D^{j-2}B)(v^j) = 0, \tag{27}$$

where we put

$$\begin{aligned}
&a^{\varepsilon_1}_{d+1,d+1} = 1, \quad a^{\varepsilon_1}_{d+1,1} = a^{\varepsilon_1}_{d+1,d+2} = 0, \\
&a^{\varepsilon_1}_{d+1,j} = \varepsilon_{d-1}\varepsilon_d \lambda^2_{d-1} a^{\varepsilon_1}_{d-1,j} + a^{\varepsilon_1}_{d,j-1} \quad (2 \le j \le d).
\end{aligned}$$

Proof. Let $x \in TM$ be an ε_1-vector. Using $\varepsilon_1\langle x,x\rangle = 1$, we rewrite Equation (19) in the proof of Theorem 4.2 into the form

$$\sum_{j=2}^{m} a_{m,j}^{\varepsilon_1}(\varepsilon_1\langle x,x\rangle)^{(m-j)/2} A_{(D^{j-2}B)(x^j)}x = 0.$$

From (24), each non-vanishing term of the left hand side is a tensor field of degree $(m+1)$. Taking account of Lemma 2.2, the above equation holds if x is replaced by any $v \in TM$. Thus we have (26). Let γ_x be an ε_1-geodesic of M such that $\gamma_x'(0) = x$, and put $\sigma_x = f \circ \gamma_x$ and $X = \sigma_x'$. If $d = 2$, then $\overline{\nabla}_X \sigma_{x,2} = -\varepsilon_1\varepsilon_2\lambda_1\sigma_{x,1}$. Hence, $(\mathscr{F}_2)$ in Theorem 4.2 implies $(DB)(x^3) = 0$ for any ε_1-vector $x \in TM$. By Lemma 2.2 and Codazzi equation (2), we can see that f is parallel. When $d \geq 3$, because of $\overline{\nabla}_X \sigma_{x,d} = -\varepsilon_{d-1}\varepsilon_d\lambda_{d-1}\sigma_{x,d-1}$, $(\mathscr{F}_{d-1})$ and $(\mathscr{F}_d)$, we have

$$\sum_{j=2}^{d}(\varepsilon_{d-1}\varepsilon_d\lambda_{d-1}^2 a_{d-1,j}^{\varepsilon_1} + a_{d,j-1}^{\varepsilon_1})(D^{j-2}B)(x^j) + (D^{d-1}B)(x^{d+1}) = 0. \tag{28}$$

Using $\varepsilon_1\langle x,x\rangle = 1$ again, we can rewrite (28) into (27). □

Corollary 4.5. *For any null tangent vector z of M, $A_{(D^{i-2}B)(z^i)}z = 0$ $(i \geq 2)$ and $(D^{j-2}B)(z^j) = 0$ $(j \geq d+1)$.*

Proof. By $a_{d+1,d+1}^{\varepsilon} = 1$ and (27), we can see that $(D^{k-2}B)(v^k)$ $(k \geq d+1)$ is a linear combination of $\langle v,v\rangle^{(d+1-i)/2}(D^{i-2}B)(v^i)$ $(2 \leq i \leq d)$. If we replace v by any null vector $z \in TM$ in (26), (27) and the above linear combination, then we obtain $A_{(D^{i-2}B)(z^i)}z = 0$ $(i \geq 3)$ and $(D^{i-2}B)(z^i)$ $(i \geq d+1)$ because of $a_{2,2}^{\varepsilon} = \cdots = a_{d+1,d+1}^{\varepsilon} = 1$. We must prove $A_{B(z^2)}z = 0$. By virtue of $(\mathscr{B}_2)$, we have $A_{B(v^2)}v = \nu_2\langle v,v\rangle v$ for any v, hence the proof is complete. □

Corollary 4.6. *$(\mathscr{B}_i)$ holds for any natural number $i \geq 2$.*

Proof. Let $x \in TM$ be an ε_1-vector. Then, from $a_{d+1,d+1}^{\varepsilon} = 1$, (24) and (27), we see that $(D^{k-2}B)(x^k)$ $(k \geq d+1)$ is a linear combination of $(D^{i-2}B)(x^i)$, $i = 3,5,\ldots,d'$ or $i = 2,4,\ldots,d''$, according as k is odd or even, where we note that the coefficients are independent on the choice of ε_1-vector x. Thus, from $(\mathscr{B}_2),\ldots,(\mathscr{B}_d)$, $\nu_{k,l}(x)$ is constant for any $k,l \geq 2$ and ε_1-vectors x, and $\nu_{k,l}(x) = 0$ in the case that $k+l$ is odd. Put $\nu_i = \nu_{i,i}(x)$ $(i \geq d+1)$. By the same way of showing $(\mathscr{B}_{m+1})$ in the proof of Theorem 4.2, we can prove that $(\mathscr{B}_i)$ holds for $i \geq d+1$ inductively. □

From Theorem 4.2, we see that i-th Frenet vectors of σ $(i \geq 2)$ are normal to M. Hence we get $d \leq p+1$, where p is the codimension of f. However, when M is indefinite, the order of helical geodesic immersion depends on not only the codimension but also the index of the normal spaces (Theorem C). To see this fact, we prepare the following paragraphs. For any $\alpha = (\varepsilon_2,\ldots,\varepsilon_d) \in$

$\{-1,+1\}^{d-1}$, we put $\alpha^o = (\varepsilon_3, \varepsilon_5, \dots, \varepsilon_{d'})$, $\alpha^e = (\varepsilon_2, \varepsilon_4, \dots, \varepsilon_{d''})$ and $\hat{\alpha} = ((-1)^2\varepsilon_2, (-1)^3\varepsilon_3, \dots, (-1)^d\varepsilon_d)$. By definition, we have $n(\alpha) = n(\alpha^e) + n(\alpha^o)$, $n(\hat{\alpha}^e) = n(\alpha^e)$ and $n(\hat{\alpha}^o) = [(d-1)/2] - n(\alpha^o)$. Thus $n(\hat{\alpha}) = n(\alpha^e) + [(d-1)/2] - n(\alpha^o)$.

Lemma 4.7. *For* $\alpha, \hat{\alpha} \in \{-1,+1\}^{d-1}$, *we have*

$$\min_{\alpha\in\{-1,+1\}^{d-1}} \max\{n(\alpha), n(\hat{\alpha})\} = \min_{\alpha\in\{-1,+1\}^{d-1}} \max\{p(\alpha), p(\hat{\alpha})\} = \left[\frac{d+1}{4}\right].$$

Proof. Because of $0 \le N(\alpha^o) \le [d/2]$,

$$\begin{aligned}
&\min_{\alpha\in\{-1,+1\}^{d-1}} \max\{n(\alpha), n(\hat{\alpha})\}\\
&\quad= \min_{\substack{\alpha\in\{-1,+1\}^{d-1},\\ n(\alpha^e)=0}} \max\{n(\alpha), n(\hat{\alpha})\}\\
&\quad= \min_{\substack{\alpha\in\{-1,+1\}^{d-1},\\ n(\alpha^e)=0}} \max\left\{n(\alpha^o), \left[\frac{d-1}{2}\right] - n(\alpha^o)\right\}\\
&\quad= \min\left\{\max\left\{i, \left[\frac{d-1}{2}\right] - i\right\}\,\middle|\, 0 \le i \le \left[\frac{d}{2}\right]\right\} = \left[\frac{d+1}{4}\right].
\end{aligned}$$

From the same argument for $p(\alpha)$, we can show the second equation in this lemma. □

Lemma 4.8. *If there exists* $\alpha \in \{-1,+1\}^{d-1}$ *such that both* α *and* $\hat{\alpha}$ *are admissible to a p-dimensional scalar product space* V, *then*

$$d \le p + \frac{1 + (-1)^{[p/2]}}{2}.$$

Proof. We can easily get $n(\alpha) - n(\hat{\alpha}) = 2n(\alpha^o) - [(d-1)/2]$. Hence $|n(\alpha) - n(\hat{\alpha})| \ge 1$ when $d \equiv 0, 3 \pmod 4$. In this case, without loss of generality, we can assume $n(\alpha) - n(\hat{\alpha}) \ge 1$. If both α and $\hat{\alpha}$ are admissible to V, then $n(\alpha) \le \operatorname{ind} V$ and $p(\hat{\alpha}) \le \dim V - \operatorname{ind} V$. Therefore we have

$$d - 1 = p(\hat{\alpha}) + n(\hat{\alpha}) \le \dim V - \operatorname{ind} V + \operatorname{ind} V - 1 = p - 1.$$

Hence we have $d \le p$, $d \equiv 0, 3 \pmod 4$. When $d \equiv 1, 2 \pmod 4$, at least, $d \le p + 1$. These inequalities imply

$$d \le \begin{cases} p+1 & p \equiv 0, 1 \pmod 4, \\ p & p \equiv 2, 3 \pmod 4. \end{cases}$$

Therefore the proof is finished. □

Proof of Theorem C. Let $\varepsilon_1, \dots, \varepsilon_d$ be the Frenet signatures in $\overline{M}$ of ε_1-geodesic of M. Put $\alpha = (\varepsilon_2, \dots, \varepsilon_d)$. Since M is indefinite, from Theorem A and Theorem 4.2, both α and $\hat{\alpha}$ must be admissible to the normal spaces of M. By Lemma 4.8,

$$d \leq p + \frac{1 + (-1)^{[p/2]}}{2}.$$

On the other hand, $\max\{n(\alpha), n(\hat{\alpha})\} \leq \mathrm{ind}\ T_qM^\perp$ and $\max\{p(\alpha), p(\hat{\alpha})\} \leq p - \mathrm{ind}\ T_qM^\perp$ must also hold ($q \in M$). Using Lemma 4.7, we have

$$\left[\frac{d+1}{4}\right] \leq \min\{\mathrm{ind}\ T_qM^\perp, p - \mathrm{ind}\ T_qM^\perp\} = l'.$$

Therefore we have $d \leq 4l' + 2$. Theorem C was proved. □

Remark 4.9. We note

$$\min\left\{p + \frac{1 + (-1)^{[p/2]}}{2}, 4l' + 2\right\}$$
$$= \begin{cases} 4l' + 2 & (0 \leq l' \leq [(p-1)/4]), \\ p + \dfrac{1 + (-1)^{[p/2]}}{2} & ([(p-1)/4] < l' \leq [(p-1)/2]). \end{cases}$$

Moreover we can prove the following: For a p-dimensional scalar product space V, there exist $\alpha = (\varepsilon_2, \ldots, \varepsilon_k)$ such that both α and $\hat{\alpha}$ are admissible to V, where k is equal to the right hand side of the inequality in Theorem D.

Corollary 4.10. *Let $f : M \to \overline{M}$ be a helical geodesic immersion of order d. If M is indefinite and* $\mathrm{ind}\ M = \mathrm{ind}\ \overline{M}$, *then $d \leq 2$. Hence f is parallel.*

Here we recall several notions which is proper in semi-Riemannian geometry. Let V be a scalar product space. A subspace W of V is called an *isotropic subspace*, if all non-zero vectors of W are light-like. The dimension of a maximal isotropic subspace of V is equal to $\min\{\mathrm{ind}\ V - \dim\ V, \mathrm{ind}\ V\}$. Let L be a merely submanifold of a semi-Riemannian manifold N. In general, the induced tensor field g on L from the semi-Riemannian metric on N is not necessarily non-degenerate. If $g = 0$ on L, then we call L an isotropic submanifold, following [5]. Thus, for an isotropic submanifold L, T_qL ($q \in L$) is an isotropic subspace of T_qN.

Let N be a semi-Riemannian manifold of constant sectional curvature. Hereafter, we say that L is *totally geodesic* in N, if $\nabla_X Y$ is tangent to L for any tangent vector fields X, Y of L, where ∇ is the Levi-Civita connection of N. We note that a semi-Riemannian manifold N of constant sectional curvature satisfies that if, for each point $q \in N$ and any subspace V of the tangent space T_qN, there exists a totally geodesic submanifold L containing q such that the tangent space of L at q is V.

We recall the definition of a normal section for an n-dimensional semi-Riemannian submanifold M immersed in an $(n+p)$-dimensional semi-

Riemannian manifold $\overline{M}$ of constant sectional curvature. For a point q in M and a non-zero vector v in T_qM, the vector v and the normal space $T_qM^\perp$ determine a $(p+1)$-dimensional subspace $E(q,v)$ of $T_q\overline{M}$, which determines a $(p+1)$-dimensional totally geodesic submanifold L satisfying $q \in L$ and $T_qL = E(q,v)$. Then the intersection of M and L gives rise to a regular curve β_v in a neighborhoods of q such that $\beta_v'(0) = v$, which is called the normal section of M at q in the direction v. Without loss of generality, we may assume that β_v is parametrized by arc-length whenever v is non-null.

In this paper, we say that an isometric immersion $f : M \to \overline{M}$ *has geodesic normal sections*, if any geodesic of M is locally a normal section. Similarly to the Riemannian case, the next proposition holds.

PROPOSITION 4.11. *A helical geodesic immersion has geodesic normal sections.*

Proof. Equation (27) and the continuity of $(D^{i-2}B)$ imply, for any $v \in TM$,

$$\operatorname{span}\{(D^{i-2}B)(v^i) \mid 2 \le i \le d\} = \operatorname{span}\{(D^{i-2}B)(v^i) \mid i \ge 2\}, \tag{29}$$

and the dimension $\le (d-1)$ because of Theorem 4.2. By Proposition 4.4, $A_{(D^{i-2}B)(v^i)}v \wedge v = 0$ for any $v \in TM$ and $i \ge 2$. Let q be an arbitrary point in M and $\gamma : I \to M$ a geodesic of M satisfying $\gamma'(0) = v \in T_qM$, put $\sigma = f \circ \gamma$. Then, we have for $1 \le i \le d$

$$\overline{\nabla}_V^{i-1}\sigma' = \sum_{j=1}^{i} c_j^i (D^{j-2}B)(V^j), \tag{30}$$

where $(D^{-1}B)(V^1) = V = \sigma'$ and c_j^i is a function on I. So we have $\sigma^{(i)}(0) \in E(q,v)$, $(1 \le i \le d)$. Let L be a totally geodesic submanifold of $\overline{M}$ determined by $E(q,v)$. Then L has the induced connection ∇^L i.e., $\nabla_X^L Y = \overline{\nabla}_X Y$ for any tangent vector fields X, Y of L. Let β be a curve on L satisfies the following equation with an initial condition $\beta'(0) = v$

$$\nabla_V^{L(i-1)}\beta' = \sum_{j=1}^{i} c_j^i (D^{j-2}B)(V^j). \tag{31}$$

By virtue of $\nabla_X^L Y = \overline{\nabla}_X Y$ and the uniqueness for solutions of Equations (30) and (31), we see that σ coincides with β locally (hence β is a normal section of M at q in the direction v). Thus all geodesics of M are normal sections of M. This proposition is proved. □

Remark 4.12. Chen and Verheyen [4] proved that a Euclidean helical submanifolds has geodesic normal sections. In the Riemannian case, its inverse was proved by Verheyen [11]. Moreover Hong and Houh [6] showed it in case the ambient space is of constant sectional curvature. However, in the proper semi-Riemannian case, space-like geodesics of a submanifold with geodesic normal sections are not necessarily helices (in the sense of this paper) in the ambient space. See Example 2.3 in Blomstrom [3], for example.

We say that a curve c in a semi-Riemannian manifold N of constant sectional curvature is of *proper order* d if the image of c is contained in a d-dimensional totally geodesic submanifold of N and not in any $(d-1)$-dimensional totally geodesic submanifold of N.

As can be seen from Example 3.4, under the assumption that $f : M \to \overline{M}$ is a helical geodesic immersion, the proper order of a null geodesic of M in $\overline{M}$ need not coincide with one of a space-like geodesic of M. A direct computation shows that f and $\imath \circ f$ in Example 3.5, and f in Example 3.6 map each null geodesics of S_1^2 to curves whose proper order is equal to the helical order in $S_2^4(3)$, $\mathbf{R}_2^5$ and $S_3^6(6)$ respectively. On the other hand, though $\imath \circ f$ in Example 3.6 is a helical geodesic immersion of order 4, each null geodesics of S_1^2 are mapped to curves of proper order 3 in $\mathbf{R}_3^7$ by $\imath \circ f$. In general, for the behavior of null geodesics mapped by a helical geodesic immersion, we can obtain Theorem D.

Proof of Theorem D. Let z a null vector of T_qM $(q \in M)$ and γ a null geodesic of M satisfying $\gamma'(0) = z$. We have for $\sigma = f \circ \gamma$

$$\sigma^{(i)} = \sum_{j=1}^{i} c_j^i (D^{j-2}B)(Z^j),$$

where $(D^{-1}B)(Z^1) = Z = \sigma'$. On the other hand, $(\mathscr{B}_i)$ $(i \geq 2)$ imply for any null vector z tangent to M

$$\langle (D^{k-2}B)(z^k), (D^{l-2}B)(z^l) \rangle = 0,$$

for any $k, l \geq 1$. Therefore we have $\langle \sigma^{(k)}, \sigma^{(l)} \rangle = 0$ along σ for any $k, l \geq 1$ and see that $\sigma'(0), \ldots, \sigma^{(d)}(0)$ are contained in an isotropic subspace V of $E(q, z)$, where $\dim V \leq l' + 1$. By virtue of Proposition 4.11 and (29), σ is a normal section of M and its proper order $\leq d$. Since M is of constant sectional curvature, there exists an isotropic totally geodesic submanifold L_V determined by V. Since, from (29), we can see that $\mathrm{span}\{\sigma', \ldots, \sigma^{(d)}\}$ is parallel along σ, the image of $\sigma \subset L_V$ and $\dim L_V = \dim V = l' + 1$. So, the proper order of $\sigma \leq \min\{l' + 1, d\}$. Theorem D was proved. □

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