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ON HYPERSURFACES INTO RIEMANNIAN SPACES OF CONSTANT SECTIONAL CURVATURE

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Abstract

In this paper, we compute $L_r(S_r)$ for an isometric immersion $x : M^n \to \bar{M}_c^{n+1}$, from an n-dimensional Riemannian manifold M^n into an $(n+1)$-dimensional Riemannian manifold $\bar{M}_c^{n+1}$, of constant sectional curvature c. Here, by L_r we mean the linearization of the second order differential operator associated to the $(r+1)$-th elementary symmetric function S_{r+1} on the eigenvalues of the second fundamental form A of x. The resulting formulae are then applied to study how the behavior of higher-order mean curvature functions of M^n influence its geometry.

1. Introduction

In a seminal paper ([15]), J. Simons computed the Laplacian of the second fundamental form of isometric immersions in spheres, applying the result to get an integral inequality to be satisfied by the squared norm of the second fundamental form A of a minimal oriented hypersurface of the unit sphere $\mathbf{S}^{n+1}$. More specifically, he proved that

$$\int_M |A|^2(n - |A|^2)\, dM \leq 0,$$

what immediately gives a gap theorem concerning the size of the squared norm of A for minimal hypersurfaces of the sphere. In fact, if $0 \leq |A|^2 \leq n$ for such an immersion, then one has $|A|^2 = 0$ or n.

Following Simons' approach and working independently, S. S. Chern, M. do Carmo and S. Kobayashi in [6], and H. B. Lawson in [12], characterized minimal Clifford tori $\mathbf{S}_{r_1}^{n_1} \times \mathbf{S}_{r_2}^{n_2}$, $n_1 + n_2 = n$, $r_1 = \sqrt{n_1/n}$, $r_2 = \sqrt{n_2/n}$ as the only closed minimal hypersurfaces of the unit sphere $\mathbf{S}^{n+1}$ for which $|A|^2 = n$. The natural immediate generalization, namely, the study of rigidity properties of constant mean curvature hypersurfaces of the sphere under appropriate constraints on $|A|^2$, is due to H. Alencar and M. do Carmo, in [1], still working along the same lines of [15].

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A slight variation of this general method for investigating rigidity properties of hypersurfaces in Riemannian spaces of constant sectional curvature appeared in a work of H. Alencar, M. do Carmo and A. G. Colares (see [2]), where the authors obtained a formula for $L_1(S_1)$ under the additional hypothesis that M had scalar curvature identically equal to that of the ambient space. This formula was later used by H. Alencar, M. do Carmo and W. Santos (see [3]) to prove a gap theorem for compact orientable hypersurfaces of the unit sphere having scalar curvature equal to 1. Here, $x : M^n \to \overline{M}_c^{n+1}$ is an n-dimensional oriented hypersurface of a Riemannian space $\overline{M}$, of constant sectional curvature c, and A denotes the second fundamental form of x with respect to a unit normal vector field N globally defined on M. For $0 \leq r \leq n$, S_r is the r-th elementary symmetric function on the eigenvalues of A, and $P_r : TM \to TM$ is the r-th Newton transformation on M, recursively defined by $P_0 = I$ and $P_r = S_r I - AP_{r-1}$; L_r is the second order differential operator on M, given for a smooth $f : M \to \mathbf{R}$ by

$$L_r(f) = \mathrm{tr}(P_r \, Hess\, f)$$

Observe that $L_0 = \Delta$, the Laplacian of M.

In this paper, we compute $L_r(S_r)$ for isometric immersions of Riemannian manifolds M as hypersurfaces of Riemannian ambient spaces $\overline{M}$, of constant sectional curvature, without additional restrictions (corollary 3.3). We then apply this formula to study how the behavior of higher order mean curvature functions of M influence its shape. We start generalizing (theorem 4.3) the above-mentioned gap theorem of Alencar and do Carmo for hypersurfaces of the unit sphere having constant mean or scalar curvature (not necessarily equal to 1). Then we prove a result (theorem 4.5) generalizing Simons' integral inequality for r-minimal hypersurfaces of the sphere. The above-mentioned results of Chern, do Carmo, Kobayashi and Lawson, as well as a theorem of J. Hounie and M. L. Leite, allows us to characterize $(r-1)$-minimal Clifford tori $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, with $r_1^2 + r_2^2 = 1$ and $n_1 + n_2 = n$, as the only closed oriented hypersurfaces of the unit sphere $\mathbf{S}^{n+1}$ for which $S_r = 0$, $S_{r+1} \neq 0$ and

$$\mathrm{tr}(A^2 P_{r-1})\, \mathrm{tr}(P_{r-1}) \geq \mathrm{tr}(A^2 P_{r-1})^2.$$

We next apply the formula for $L_r(S_r)$, together with a theorem of J. L. Barbosa and A. G. Colares giving sufficient conditions for the ellipticity of the operator L_r, to characterize, in theorem 4.9, geodesic hyperspheres as the only closed orientable hypersurfaces of the standard Riemannian space forms having one constant nonzero higher order mean curvature. Then we show, for the case of hypersurfaces of the unit sphere, how much one can relax the condition of M being contained in an open hemisphere (necessary to apply the theorem of Barbosa and Colares), obtaining a result (theorem 4.11) that also works as a sort of analogue to the gap theorem of Alencar and do Carmo for general $1 \leq r \leq n$. We finish our discussion with some remarks on noncompact complete hypersurfaces of the Euclidean space $\mathbf{R}^{n+1}$, giving, in theorem 4.12, a characterization of r-cilinders $\mathbf{S}^r \times \mathbf{R}^{n-r}$.

This paper is organized in the following manner: in section 2 we establish some notation and recall several results needed for further developments. Then, in section 3, we obtain the formula for $L_r(S_r)$ as a corollary of the more general computation of $L_q(S_r)$. Finally, in section 4, we state and prove the applications referred to in the above paragraph.

2. Preliminaries

Unless stated otherwise, we mean by M^n, or simply M, an n-dimensional orientable Riemannian manifold, with Riemannian metric $g = \langle\, , \rangle$, Levi-Civita connection ∇ and curvature tensor R; $\mathscr{D}(M)$ denotes the commutative ring of smooth real functions on M.

2.1. Tensor fields. Let $\phi = \langle T\cdot, \cdot\rangle$ denote an arbitrary 2-tensor on M, and $\nabla\phi$ and $\nabla^2\phi = \nabla(\nabla\phi)$ be its first and second covariant differentials. For each $V \in \mathscr{X}(M)$, the recipe $(\nabla_V\phi)(X, Y) = (\nabla\phi)(X, Y, V)$ defines another 2-tensor on M, called the covariant derivative of ϕ in the direction of V. If $\nabla_V T$ denotes the linear operator associated to $\nabla_V\phi$, one has

$$(\nabla_V T)(X) = \nabla_V(TX) - T(\nabla_V X).$$

Let $\{e_i\}$ be a moving frame on an open neighborhood $U \subset M$, with coframe $\{\omega_i\}$ and connection 1-forms ω_{ij}. Letting ϕ_{ij}, ϕ_{ijk} and ϕ_{ijkl} denote the components of ϕ, $\nabla\phi$ e $\nabla^2\phi$ with respect to $\{e_i\}$, the following relations take place:

$$\sum_k \phi_{ijk}\omega_k = d\phi_{ij} - \sum_k \phi_{kj}\omega_{ik} - \sum_k \phi_{ik}\omega_{jk}; \tag{1}$$

$$\sum_l \phi_{ijkl}\omega_l = d\phi_{ijk} - \sum_l \phi_{ljk}\omega_{il} - \sum_l \phi_{ilk}\omega_{jl} - \sum_l \phi_{ijl}\omega_{kl}. \tag{2}$$

The proof of the following lemma can be found in [5].

Lemma 2.1. *Let ϕ be a 2-tensor on M. With respect to an arbitrary moving frame $\{e_k\}$ on M, and letting $R_{irkl} = R(e_i, e_r, e_k, e_l)$, one has*

$$\phi_{ijkl} - \phi_{ijlk} = -\sum_r \phi_{rj}R_{irkl} - \sum_r \phi_{ir}R_{rjlk}.$$

The following remarks on components of tensors with respect to a given moving frame will be used in the next section.

Remark 2.2. A moving frame $\{e_k\}$ on (an open neighborhood of) M is called geodesic at p when $(\nabla_{e_k}e_i)(p) = 0$ for all $1 \le i, k \le n$, which is in turn equivalent to $\omega_{ij}(p) = 0$ for all $1 \le i, j \le n$. The usual way to build frames on M, geodesic at $p \in M$, is by fixing a normal neighborhood of p and parallel transporting the elements of an arbitrary orthonormal basis of T_pM along the geodesic rays issuing from p. Whenever we speak of a frame on M, geodesic at some point $p \in M$, we will always assume that it has been built this way.

Remark 2.3. Note also that, for fixed $1 \le k \le n$, the above recipe gives $(\nabla_{e_k} e_i)(q) = 0$, for every $1 \le i \le n$ and every point q along the geodesic ray issuing from p with velocity vector e_k. Therefore, $\omega_{ij}(q)(e_k) = 0$ for all such i, j and q, and setting $\phi_{ij;k} = e_k(\phi_{ij})$ and $\phi_{ij;kk} = e_k(e_k(\phi_{ij}))$ one has, along the geodesic ray issuing from p with velocity vector e_k,

$$\phi_{ijk} = \phi_{ij;k} \quad \text{and} \quad \phi_{ijkk} = \phi_{ij;kk}. \tag{3}$$

The first part of (3) follows from (1), while the second one follows from substituting the first into (2).

Remark 2.4. A 2-tensor ϕ on M is Codazzi when $\phi_{ijk} = \phi_{ikj}$ for all $1 \le i, j, k \le n$, and with respect to any moving frame $\{e_k\}$ on M. If this is the case, changing indices j and k in (2) gives

$$\phi_{ijkl} = \phi_{ikjl}, \tag{4}$$

for all $1 \le i, j, k, l \le n$.

A 2-tensor ϕ on M is symmetric if $\phi(X, Y) = \phi(Y, X)$ for all $X, Y \in \mathscr{X}(M)$, or equivalently, when its associated linear operator T is self-adjoint. If $X \in \mathscr{X}(M)$ then $\nabla_X \phi$ is symmetric whenever ϕ is symmetric, so that $\nabla_X T$ is self-adjoint whenever T is self-adjoint. With respect to an arbitrary moving frame $\{e_k\}$ on M, the symmetry of ϕ is equivalent to $\phi_{ij} = \phi_{ji}$, for all $1 \le i, j \le n$. We define the squared norm of a symmetric 2-tensor ϕ on M by setting

$$|\phi|^2 = \operatorname{tr}(T^2) = \sum_{i,j} \phi_{ij}^2,$$

where tr denotes the *trace* of its associated linear operator T.

2.2. Isometric immersions. Let $x : M^n \to \overline{M}^{n+1}$ denote an isometric immersion from M^n into an $(n+1)$-dimensional, oriented Riemannian manifold $\overline{M}^{n+1}$. Also, suppose M oriented by the choice of a unit normal vetor field N, and denote by A the corresponding second fundamental form. When $\overline{M}^{n+1}$ has constant sectional curvature c, we recall Gauss' and Codazzi's equations: for $W, X, Y, Z \in \mathscr{X}(M)$, one has

$$\begin{aligned} \langle R(W,X)Y,Z\rangle = {} & c[\langle W,Y\rangle\langle X,Z\rangle - \langle W,Z\rangle\langle X,Y\rangle] \\ & + [\langle AW,Y\rangle\langle AX,Z\rangle - \langle AW,Z\rangle\langle AX,Y\rangle] \end{aligned} \tag{5}$$

and

$$(\nabla_X A)Y = (\nabla_Y A)X. \tag{6}$$

Note that in this case Codazzi's equation (6) is exactly what it means for the second fundamental form A to be a Codazzi tensor.

Associated to the second fundamental form A of a general isometric immersion $x : M^n \to \overline{M}^{n+1}$ one has n invariants S_r, $1 \le r \le n$, given by the equality

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in M$ and $\{e_k\}$ is a basis of T_pM formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbf{R}[X_1, \dots, X_n]$ is the r-th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$. In particular,

$$|A|^2 + 2S_2 = S_1^2.$$

The following lemma appears, in a slightly different form, in [2].

Lemma 2.5. *Let $x : M^n \to \overline{M}^{n+1}$ be an isometric immersion. If S_2 is constant on M, then*

$$S_1^2(|\nabla A|^2 - |\nabla S_1|^2) \ge 2S_2|\nabla A|^2. \tag{7}$$

In particular, if $S_2 \ge 0$ then $|\nabla A|^2 - |\nabla S_1|^2 \ge 0$.

Note also that if R denotes the scalar curvature of M, and $\overline{M}$ has constant sectional curvature c, it follows from Gauss' equation that

$$2S_2 = n(n-1)(R - c), \tag{8}$$

so that S_2 is constant on M if and only if R is constant on M.

2.3. Newton transformations. For $0 \le r \le n$, one defines the r-th Newton operator P_r on M by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recursion formulae

$$P_r = S_r I - AP_{r-1}.$$

A trivial induction shows that

$$P_r = S_r I - S_{r-1}A + S_{r-2}A^2 - \dots + (-1)^r A^r,$$

from where Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial on A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM diagonalizing A at $p \in M$ also diagonalize all of the

P_r at p. Hence, denoting by A_i the restriction of A to $\langle e_i\rangle^\perp \subset T_pM$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1}(-1)^k S_k(A_i)t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1\le j_1<\cdots<j_k\le n\\ j_1,\dots,j_k\neq i}} \lambda_{j_1}\cdots\lambda_{j_k}.$$

With the above notations, one can easily prove that $P_re_i = S_r(A_i)e_i$, and also that

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$.
(b) $\mathrm{tr}(P_r) = \sum_{i=1}^n S_r(A_i) = (n-r)S_r$.
(c) $\mathrm{tr}(AP_r) = \sum_{i=1}^n \lambda_i S_r(A_i) = (r+1)S_{r+1}$.
(d) $\mathrm{tr}(A^2P_r) = \sum_{i=1}^n \lambda_i^2 S_r(A_i) = S_1S_{r+1} - (r+2)S_{r+2}$.

Concerning general bases of T_pM, the following lemma is due to R. Reilly ([14]). For the sake of completeness, and also to establish some notation, we include a short proof of it.

LEMMA 2.6. *If (h_{ij}) denotes the matrix of A with respect to a certain orthonormal basis $\beta = \{e_k\}$ of T_pM, then the matrix (h^r_{ij}) of P_r with respect to the same basis is given by*

$$h^r_{ij} = \frac{1}{r!}\sum_{i_k,j_k=1}^n \epsilon^{j_1\cdots j_r j}_{i_1\cdots i_r i} h_{j_1i_1}\cdots h_{j_ri_r}, \tag{9}$$

where

$$\epsilon^{j_1\cdots j_r}_{i_1\cdots i_r} = \begin{cases} sgn(\sigma), & \text{if the } i_k \text{ are pairwise distinct and}\\ & \sigma = (j_k) \text{ is a permutation of them;}\\ 0, & \text{else.}\end{cases}$$

Proof. Recall that $P_r = \sum_{j=0}^n(-1)^jS_{r-j}A^j$, with the coefficients S_{r-j} not depending on the chosen basis of T_pM. Therefore, it suffices to verify the above formula when β diagonalizes A at p, with, say, $Ae_k = \lambda_ke_k$ for $1\le k\le n$. In this case, the right hand side of (9) sucessively equals

$$\begin{aligned}\frac{1}{r!}&\sum_{i_k,j_k=1}^n \epsilon^{j_1\cdots j_r j}_{i_1\cdots i_r i}\delta_{j_1i_1}\cdots\delta_{j_ri_r}\lambda_{j_1}\cdots\lambda_{j_r}\\ &= \frac{1}{r!}\sum_{i_k\neq i}\epsilon^{i_1\cdots i_r j}_{i_1\cdots i_r i}\lambda_{i_1}\cdots\lambda_{i_r} = \delta_{ij}\sum_{\substack{i_1<\cdots<i_r\\ i_k\neq i}}\lambda_{i_1}\cdots\lambda_{i_r}\\ &= \delta_{ij}S_r(A_i) = \langle P_re_i, e_j\rangle = h^r_{ij}.\end{aligned}$$

□

We use the above lemma to compute first derivatives of h^r_{ij}:

LEMMA 2.7. *Let $\{e_k\}$ be a moving frame on a neighborhood of $p \in M$, diagonalizing the second fundamental form A at p, with $Ae_k = \lambda_k e_k$ for $1 \leq k \leq n$. Then, for $1 \leq i, j, k \leq n$, $i \neq j$, one has at p*

$$e_k(h^r_{ii}) = \sum_{l \neq i} S_{r-1}(A_{il}) h_{ll;k} \tag{10}$$

and

$$e_k(h^r_{ij}) = -S_{r-1}(A_{ij}) h_{ij;k}, \tag{11}$$

where A_{ij} denotes the restriction of A to $\{e_i, e_j\}^\perp \subset T_pM$.

Proof. Forgetting for the moment the restriction of being $i \neq j$, it follows from (9) that

$$\begin{aligned} e_k(h^r_{ij}) = &\frac{1}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon^{j_1 \cdots j_r j}_{i_1 \cdots i_r i} h_{j_1 i_1; k} h_{j_2 i_2} \cdots h_{j_r i_r} + \cdots \\ &+ \frac{1}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon^{j_1 \cdots j_r j}_{i_1 \cdots i_r i} h_{j_1 i_1} \cdots h_{j_{r-1} i_{r-1}} h_{j_r i_r; k}. \end{aligned} \tag{12}$$

At p, the first summand on the right hand side equals

$$\frac{1}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon^{j_1 \cdots j_r j}_{i_1 \cdots i_r i} \delta_{i_2 j_2} \cdots \delta_{i_r j_r} h_{j_1 i_1; k} \lambda_{i_2} \cdots \lambda_{i_r} = \frac{1}{r!} \sum_{i_k, j_1 = 1}^{n} \epsilon^{j_1 i_2 \cdots i_r j}_{i_1 i_2 \cdots i_r i} h_{j_1 i_1; k} \lambda_{i_2} \cdots \lambda_{i_r}. \tag{13}$$

Now, consider two separate cases: for $i = j$,

$$\begin{aligned} (13) &= \frac{1}{r!} \sum_{i_k, j_1 = 1}^{n} \epsilon^{j_1 i_2 \cdots i_r i}_{i_1 i_2 \cdots i_r i} h_{j_1 i_1; k} \lambda_{i_2} \cdots \lambda_{i_r} \\ &= \frac{1}{r!} \sum_{1 \leq i_k \leq n} \epsilon^{i_1 i_2 \cdots i_r i}_{i_1 i_2 \cdots i_r i} h_{i_1 i_1; k} \lambda_{i_2} \cdots \lambda_{i_r} \\ &= \frac{1}{r} \sum_{l \neq i} \sum_{\substack{i_2 < \cdots < i_r \\ i_k \neq i, l}} h_{ll; k} \lambda_{i_2} \cdots \lambda_{i_r} = \frac{1}{r} \sum_{l \neq i} S_{r-1}(A_{il}) h_{ll; k}. \end{aligned}$$

Since the same is true for all other summands in (12), one gets (10). For $i \neq j$, it follows from the definition of $\epsilon^{j_1 i_2 \cdots i_r j}_{i_1 i_2 \cdots i_r i}$ that

$$(13) = \frac{1}{r!}\sum_{i_k \neq i,j} \epsilon^{ii_2\cdots i_r j}_{ji_2\cdots i_r i} h_{ij;k}\lambda_{i_2}\cdots\lambda_{i_r}$$

$$= -\frac{1}{r}\sum_{\substack{i_k \neq i,j \\ i_2<\cdots<i_r}} h_{ij;k}\lambda_{i_2}\cdots\lambda_{i_r} = -\frac{1}{r}S_{r-1}(A_{ij})h_{ij;k},$$

so that (11) now follows from (12). □

In the sequel, we will need the following proposition. Item (a) is essentially the content of lemma 1.1 and equation (1.3) in [10], while item (b) is quoted as proposition 1.5 in [11].

Proposition 2.8. *Let* $x : M^n \to \overline{M}^{n+1}$ *be an isometric immersion, and* $1 \leq r < n$, $p \in M$.
(a) *If* $S_r(p) = 0$, *then* P_{r-1} *is semi-definite at* p.
(b) *If* $S_r(p) = 0$ *and* $S_{r+1}(p) \neq 0$, *then* P_{r-1} *is definite at* p.

Finally, for $0 \leq r \leq n$, let $L_r : \mathscr{D}(M) \to \mathscr{D}(M)$ be the second order differential operator given by

$$L_r(f) = \mathrm{tr}(P_r \, Hess\, f).$$

When $\overline{M}^{n+1}$ has constant seccional curvature, it was proved by H. Rosenberg in [13] that

$$L_r(f) = div(P_r \nabla f),$$

where *div* stands for the divergence of a vector field on M. Thus, for $f, g \in \mathscr{D}(M)$, it follows from the properties of the divergence of vector fields that

$$L_r(fg) = fL_r(g) + gL_r(f) + 2\langle P_r\nabla f, \nabla g\rangle.$$

3. The formula for $L_r(S_r)$

As in the previous section, $x : M^n \to \overline{M}_c^{n+1}$ denotes an isometric immersion between oriented Riemannian manifolds, and A denotes the corresponding second fundamental form.

Proposition 3.1. *Let* $x : M^n \to \overline{M}_c^{n+1}$ *be an isometric immersion, and* $0 \leq q < n$, $0 < r < n$. *If* $\{e_k\}$ *is any orthonormal frame on* M, *then*

$$(14) \qquad \begin{aligned} L_q(S_r) = {} & L_{r-1}(S_{q+1}) \\ & + \sum_k \mathrm{tr}\{[P_q(\nabla_{e_k}P_{r-1}) - P_{r-1}(\nabla_{e_k}P_q)](\nabla_{e_k}A)\} \\ & + c[\mathrm{tr}(AP_{r-1})\,\mathrm{tr}(P_q) - \mathrm{tr}(P_{r-1})\,\mathrm{tr}(AP_q)] \\ & + \mathrm{tr}(A^2P_{r-1})\,\mathrm{tr}(AP_q) - \mathrm{tr}(AP_{r-1})\,\mathrm{tr}(A^2P_q). \end{aligned}$$

Proof. First of all, observe that the validity of (14) does not depend on the particular chosen frame $\{e_k\}$. So, let $p \in M$ and $\{e_k\}$ be a moving frame on a neighborhood $U \subset M$ of p, diagonalizing A at p, with $Ae_k = \lambda_k e_k$ for $1 \le k \le n$. Denote respectively by h_{ij} and h_{ij}^r the components of A and P_r with respect to such a frame. It follows from equation (9) that

$$
\begin{aligned}
h_{ii}^r &= \frac{1}{r!} \sum_{i_k, j_k = 1}^{n} \epsilon_{i_1 \cdots i_r i}^{j_1 \cdots j_r i} h_{j_1 i_1} \cdots h_{j_r i_r} \\
&= \frac{1}{r!} \sum_{i_k \neq i, \sigma = (j_k)} sgn(\sigma) h_{j_1 i_1} \cdots h_{j_r i_r} \\
&= \sum_{\substack{i_1 < \cdots < i_r \\ i_k \neq i}} \sum_{\sigma = (j_k)} sgn(\sigma) h_{j_1 i_1} \cdots h_{j_r i_r} \\
&= \sum_{\substack{i_1 < \cdots < i_r \\ i_k \neq i}} A(c_{i_1}, \ldots, c_{i_r}),
\end{aligned} \tag{15}
$$

where by $A(c_{i_1}, \ldots, c_{i_r})$ we mean the $r \times r$ determinant minor of A, obtained by choosing lines and columns of A with indices $i_1 < \cdots < i_r$. Hence,

$$
\begin{aligned}
S_r = \frac{1}{n-r} \operatorname{tr}(P_r) &= \frac{1}{n-r} \sum_i \sum_{\substack{i_1 < \cdots < i_r \\ i_k \neq i}} A(c_{i_1}, \ldots, c_{i_r}) \\
&= \sum_{i_1 < \cdots < i_r} A(c_{i_1}, \ldots, c_{i_r})
\end{aligned}
$$

for once one has chosen $1 \le i_1 < \cdots < i_r \le n$, there will be left $n - r$ possible choices for i in $\{1, \ldots, n\}$. Since determinants are multilinear functions of their columns, one gets

$$
e_k(S_r) = \sum_{i_1 < \cdots < i_r} [A(c_{i_1;k}, c_{i_2}, \ldots, c_{i_r}) + \cdots + A(c_{i_1}, \ldots, c_{i_{r-1}}, c_{i_r;k})] \tag{16}
$$

on U. At p, one has

$$
A(c_{i_1;k}, c_{i_2}, \ldots, c_{i_r}) = \begin{vmatrix} h_{i_1 i_1;k} & 0 & \cdots & 0 \\ h_{i_2 i_1;k} & \lambda_{i_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_r i_1;k} & 0 & \cdots & \lambda_{i_r} \end{vmatrix} = h_{i_1 i_1;k} \lambda_{i_2} \cdots \lambda_{i_r},
$$

and analogously for the remaining summands, so that

$$
\begin{aligned}
e_k(S_r) &= \sum_{i_1 < \cdots < i_r} (h_{i_1 i_1;k} \lambda_{i_2} \cdots \lambda_{i_r} + \cdots + \lambda_{i_1} \cdots \lambda_{i_{r-1}} h_{i_r i_r;k}) \\
&= \sum_{i=1}^{n} h_{ii;k} S_{r-1}(A_i).
\end{aligned} \tag{17}
$$

The last equality follows from the fact that, for fixed $1 \le i \le n$, $h_{ii;k}$ appears in the above sum together with all products $\lambda_{j_1} \cdots \lambda_{j_{r-1}}$, with $j_1, \dots, j_{r-1} \ne i$ (note that the above formula for $e_k(S_r)$ could have been obtained directly from (10). This alternative approach was chosen to ease, in what comes next, the computation of second derivatives).

To compute second derivatives, suppose further $\{e_k\}$ geodesic at p. It follows from (16) that

$$e_k(e_k(S_r)) = \sum_{i_1 < \cdots < i_r} [A(c_{i_1;kk}, c_{i_2}, \dots, c_{i_r}) + \cdots + A(c_{i_1}, \dots, c_{i_{r-1}}, c_{i_r;kk})] + \sum_{s \ne t} \sum_{i_1 < \cdots < i_r} A(c_{i_1}, \dots, c_{i_s;k}, \dots, c_{i_t;k}, \dots, c_{i_r}),$$

and one gets at p

$$e_k(e_k(S_r)) = \sum_{i_1 < \cdots < i_r} (h_{i_1 i_1;kk} \lambda_{i_2} \cdots \lambda_{i_r} + \cdots + \lambda_{i_1} \cdots \lambda_{i_{r-1}} h_{i_r i_r;kk}) + \sum_{\substack{i_1 < \cdots < i_r \\ s \ne t}} (h_{i_s i_s;k} h_{i_t i_t;k} - h_{i_s i_t;k} h_{i_t i_s;k}) \lambda_{i_1} \cdots \hat{\lambda}_{i_s} \cdots \hat{\lambda}_{i_t} \cdots \lambda_{i_r}.$$

Grouping equal occurrences of $(r-2)$-tuples $i_1 < \cdots < i_{r-2}$ in the last expression above, $e_k(e_k(S_r))$ equals

$$\sum_i \sum_{\substack{i_1 < \cdots < i_{r-1} \\ i_k \ne i}} h_{ii;kk} \lambda_{i_1} \cdots \lambda_{i_{r-1}} + \sum_{i \ne j} \sum_{\substack{i_1 < \cdots < i_{r-2} \\ i_k \ne i,j}} [h_{ii;k} h_{jj;k} - h_{ij;k}^2] \lambda_{i_1} \cdots \lambda_{i_{r-2}},$$

and finally

$$e_k(e_k(S_r)) = \sum_i S_{r-1}(A_i) h_{ii;kk} + \sum_{i \ne j} S_{r-2}(A_{ij}) [h_{ii;k} h_{jj;k} - h_{ij;k}^2].$$

Therefore, we get at p

$$\begin{aligned}
(18) \qquad L_q(S_r) &= \operatorname{tr}(P_q \, Hess(S_r)) = \sum_{k=1}^n S_q(A_k) e_k(e_k(S_r)) \\
&= \sum_{i,k} S_q(A_k) S_{r-1}(A_i) h_{ii;kk} + \sum_{\substack{i,j,k \\ i \ne j}} S_q(A_k) S_{r-2}(A_{ij}) [h_{ii;k} h_{jj;k} - h_{ij;k}^2] \\
&= \sum_i S_{r-1}(A_i) L_q(h_{ii}) + \sum_{\substack{i,j,k \\ i \ne j}} S_q(A_k) S_{r-2}(A_{ij}) h_{ii;k} h_{jj;k} \\
&\quad - \sum_{\substack{i,j,k \\ i \ne j}} S_q(A_k) S_{r-2}(A_{ij}) h_{ij;k}^2.
\end{aligned}$$

Lemma 2.1, as well as the remarks on commutation of indices in geodesic frames made right after it, allow one to conclude that, at p,

$$
\begin{aligned}
(19)\qquad & \sum_i S_{r-1}(A_i)L_q(h_{ii}) \\
&= \sum_{i,k} S_{r-1}(A_i)S_q(A_k)h_{iikk} = \sum_{i,k} S_{r-1}(A_i)S_q(A_k)h_{ikik} \\
&= \sum_{i,k} S_{r-1}(A_i)S_q(A_k)(h_{ikik} - h_{ikki} + h_{ikki} - h_{kkii} + h_{kkii}) \\
&= \sum_{i,k} S_{r-1}(A_i)S_q(A_k)(h_{ikik} - h_{ikki}) + \sum_{i,k} S_{r-1}(A_i)S_q(A_k)h_{kkii} \\
&= -\sum_{i,j,k} S_{r-1}(A_i)S_q(A_k)(h_{jk}R_{ijik} + h_{ij}R_{jkki}) + \sum_{i,k} S_{r-1}(A_i)S_q(A_k)h_{kkii} \\
&= -\sum_{i,k} S_{r-1}(A_i)S_q(A_k)\lambda_k R_{ikik} - \sum_{i,k} S_{r-1}(A_i)S_q(A_k)\lambda_i R_{ikki} \\
&\quad + \sum_k S_q(A_k)L_{r-1}(h_{kk}).
\end{aligned}
$$

Now, write $r-1$ in place of q and $q+1$ in place of r in relation (18) to get

$$
\begin{aligned}
(20)\qquad L_{r-1}(S_{q+1}) = \sum_i S_q(A_i)L_{r-1}(h_{ii}) &+ \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k} \\
&- \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ij;k}^2.
\end{aligned}
$$

Substituting the result of (19) into (20), we arrive at

$$
\begin{aligned}
(21)\qquad L_{r-1}(S_{q+1}) = \sum_i S_{r-1}(A_i)L_q(h_{ii}) &+ \sum_{i,k} S_{r-1}(A_i)S_q(A_k)\lambda_k R_{ikik} \\
&+ \sum_{i,k} S_{r-1}(A_i)S_q(A_k)\lambda_i R_{ikki} + \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k} \\
&- \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ij;k}^2.
\end{aligned}
$$

Finally, subtracting (21) from (18) gives

$$
\begin{aligned}
(22)\qquad L_q(S_r) = L_{r-1}(S_{q+1}) &- \sum_{i,k} S_{r-1}(A_i)S_q(A_k)\lambda_k R_{ikik} \\
&- \sum_{i,k} S_{r-1}(A_i)S_q(A_k)\lambda_i R_{ikki} + \sum_{\substack{i,j,k \\ i\neq j}} S_q(A_k)S_{r-2}(A_{ij})h_{ii;k}h_{jj;k}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{i,j,k \\ i\neq j}} S_q(A_k) S_{r-2}(A_{ij}) h_{ij;k}^2 - \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k) S_{q-1}(A_{ij}) h_{ii;k} h_{jj;k} \\
&+ \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k) S_{q-1}(A_{ij}) h_{ij;k}^2.
\end{aligned}
$$

In order to better examine the summands at the right hand side of (22), let

$$
I = \sum_{i,k} S_{r-1}(A_i) S_q(A_k) \lambda_k R_{ikik}, \quad II = \sum_{i,k} S_{r-1}(A_i) S_q(A_k) \lambda_i R_{ikki}.
$$

Using Gauss' equation, one gets

$$
\begin{aligned}
I &= \sum_{i,k} \langle R(P_{r-1}e_i, P_q e_k) e_i, A e_k \rangle \\
&= c \sum_{i,k} [\langle P_{r-1}e_i, e_i \rangle \langle P_q e_k, A e_k \rangle - \langle P_{r-1} e_i, A e_k \rangle \langle P_q e_k, e_i \rangle] \\
&\quad + \sum_{i,k} [\langle A P_{r-1} e_i, e_i \rangle \langle A P_q e_k, A e_k \rangle - \langle A P_{r-1} e_i, A e_k \rangle \langle A P_q e_k, e_i \rangle] \\
&= c \left[\operatorname{tr}(P_{r-1}) \operatorname{tr}(A P_q) - \sum_k \langle A P_{r-1} e_k, P_q e_k \rangle \right] \\
&\quad + \operatorname{tr}(A P_{r-1}) \operatorname{tr}(A^2 P_q) - \sum_k \langle A^2 P_{r-1} e_k, A P_q e_k \rangle \\
&= c[\operatorname{tr}(P_{r-1}) \operatorname{tr}(A P_q) - \operatorname{tr}(A P_{r-1} P_q)] + \operatorname{tr}(A P_{r-1}) \operatorname{tr}(A^2 P_q) - \operatorname{tr}(A^3 P_{r-1} P_q)
\end{aligned}
$$

and

$$
\begin{aligned}
II &= \sum_{i,k} \langle R(A e_i, P_q e_k) e_k, P_{r-1} e_i \rangle \\
&= c \sum_{i,k} [\langle A e_i, e_k \rangle \langle P_q e_k, P_{r-1} e_i \rangle - \langle A e_i, P_{r-1} e_i \rangle \langle P_q e_k, e_k \rangle] \\
&\quad + \sum_{i,k} [\langle A^2 e_i, e_k \rangle \langle A P_q e_k, P_{r-1} e_i \rangle - \langle A^2 e_i, P_{r-1} e_i \rangle \langle A P_q e_k, e_k \rangle] \\
&= c \left[\sum_k \langle A e_k, P_{r-1} P_q e_k \rangle - \operatorname{tr}(A P_{r-1}) \operatorname{tr}(P_q) \right] \\
&\quad + \sum_k \langle A^2 e_k, A P_{r-1} P_q e_k \rangle - \operatorname{tr}(A^2 P_{r-1}) \operatorname{tr}(A P_q) \\
&= c[\operatorname{tr}(A P_{r-1} P_q) - \operatorname{tr}(A P_{r-1}) \operatorname{tr}(P_q)] + \operatorname{tr}(A^3 P_{r-1} P_q) - \operatorname{tr}(A^2 P_{r-1}) \operatorname{tr}(A P_q).
\end{aligned}
$$

On the other hand, letting

$$III = \sum_{\substack{i,j,k \\ i\neq j}} S_q(A_k)S_{r-2}(A_{ij})h_{ii;k}h_{jj;k} - \sum_{\substack{i,j,k \\ i\neq j}} S_q(A_k)S_{r-2}(A_{ij})h^2_{ij;k}$$

and

$$IV = \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k} - \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h^2_{ij;k},$$

it follows from lemma 2.7 that, at p,

$$\begin{aligned}\sum_{\substack{i,j,k \\ i\neq j}} S_q(A_k)S_{r-2}(A_{ij})h_{ii;k}h_{jj;k} &= \sum_{i,k} S_q(A_k)h_{ii;k}\sum_{j\neq i} S_{r-2}(A_{ij})h_{jj;k} \\ &= \sum_{i,k} S_q(A_k)h_{ii;k}e_k(h^{r-1}_{ii})\end{aligned}$$

and

$$-\sum_{\substack{i,j,k \\ i\neq j}} S_q(A_k)S_{r-2}(A_{ij})h^2_{ij;k} = \sum_{\substack{i,j,k \\ i\neq j}} S_q(A_k)h_{ij;k}e_k(h^{r-1}_{ij}).$$

Adding these two relations, one gets

$$III = \sum_{i,j,k} S_q(A_k)e_k(h^{r-1}_{ij})h_{ij;k} = \sum_k \operatorname{tr}[P_q(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)].$$

Again from lemma 2.7, one has at p

$$\begin{aligned}\sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h_{ii;k}h_{jj;k} &= \sum_{i,k} S_{r-1}(A_k)h_{ii;k}\sum_{j\neq i} S_{q-1}(A_{ij})h_{jj;k} \\ &= \sum_{i,k} S_{r-1}(A_k)h_{ii;k}e_k(h^{q}_{ii})\end{aligned}$$

and

$$-\sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)S_{q-1}(A_{ij})h^2_{ij;k} = \sum_{\substack{i,j,k \\ i\neq j}} S_{r-1}(A_k)h_{ij;k}e_k(h^{q}_{ij}),$$

so that

$$IV = \sum_{i,j,k} S_{r-1}(A_k)e_k(h^{q}_{ij})h_{ij;k} = \sum_k \operatorname{tr}[P_{r-1}(\nabla_{e_k}P_q)(\nabla_{e_k}A)].$$

It now suffices to substitute the expressions for I, II, III and IV into (22). $\square$

As a byproduct of the computations in the proof of the previous proposition, we get a proof for the following well-known

LEMMA 3.2. *Let $x : M^n \to \overline{M}_c^{n+1}$ be an isometric immersion, and $V \in \mathcal{X}(M)$. Then*

$$\operatorname{tr}(P_{r-1}(\nabla_V A)) = V(S_r). \tag{23}$$

Proof. Let $p \in M$, and $\{e_k\}$ be a moving frame on a neighborhood of $p \in M$, geodesic at p and such that, at p, $Ae_k = \lambda_k e_k$ for $1 \le k \le n$. Since both sides of (23) are linear on V, it suffices to prove that $\operatorname{tr}(P_{r-1}(\nabla_{e_k} A)) = e_k(S_r)$.

$$\begin{aligned}\operatorname{tr}(P_{r-1}(\nabla_{e_k} A)) &= \sum_{i=1}^{n} \langle P_{r-1}(\nabla_{e_k} A)e_i, e_i\rangle = \sum_{i=1}^{n} S_{r-1}(A_i)\langle(\nabla_{e_k} A)e_i, e_i\rangle \\ &= \sum_{i=1}^{n} S_{r-1}(A_i)h_{iik}.\end{aligned}$$

Since the frame is geodesic at p, we have $h_{iik} = h_{ii;k}$ at p, and (17) gives the desired result. $\square$

COROLLARY 3.3. *Let $x : M^n \to \overline{M}_c^{n+1}$ be an isometric immersion, and $0 < r \le n$. Then*

$$\begin{aligned}L_r(S_r) &= L_{r-1}(S_{r+1}) + S_r[\Delta S_r - L_{r-1}(S_1)] \\ &\quad + \sum_k |P_{r-1}\nabla_{e_k} A|^2 - |\nabla S_r|^2 \\ &\quad + \operatorname{tr}(AP_{r-1})\{S_r(|A|^2 - cn) - [\operatorname{tr}(A^2P_r) - c\operatorname{tr}(P_r)]\} \\ &\quad - \operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(A^2P_{r-1}) - c\operatorname{tr}(P_{r-1})],\end{aligned} \tag{24}$$

where $\{e_k\}$ is any orthonormal frame on M, or still

$$\begin{aligned}L_r(S_r) &= L_{r-1}(S_{r+1}) + S_r[\Delta S_r - L_{r-1}(S_1)] + \sum_k |P_{r-1}\nabla_{e_k} A|^2 - |\nabla S_r|^2 \\ &\quad + \frac{1}{2}\sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}),\end{aligned} \tag{25}$$

at $p \in M$, where $\{e_k\}$ is an orthonormal frame on M, diagonalizing A at p, with $Ae_k = \lambda_k e_k$ at p, and σ_{ij} denotes the 2-dimensional subspace of T_pM generated by e_i and e_j.

Proof. It follows from proposition 3.1 that

$$
\begin{aligned}
(26)\qquad L_r(S_r) = L_{r-1}(S_{r+1}) &+ \sum_k \operatorname{tr}\{[P_r(\nabla_{e_k}P_{r-1}) - P_{r-1}(\nabla_{e_k}P_r)](\nabla_{e_k}A)\}\\
&+ c[\operatorname{tr}(AP_{r-1})\operatorname{tr}(P_r) - \operatorname{tr}(P_{r-1})\operatorname{tr}(AP_r)]\\
&+ \operatorname{tr}(A^2P_{r-1})\operatorname{tr}(AP_r) - \operatorname{tr}(AP_{r-1})\operatorname{tr}(A^2P_r),
\end{aligned}
$$

where $\{e_k\}$ is any orthonormal frame on M. Making

$$T_k = [P_r(\nabla_{e_k}P_{r-1}) - P_{r-1}(\nabla_{e_k}P_r)](\nabla_{e_k}A),$$

we get

$$
\begin{aligned}
T_k &= [(S_rI - AP_{r-1})(\nabla_{e_k}P_{r-1}) - P_{r-1}(\nabla_{e_k}(S_rI - AP_{r-1})](\nabla_{e_k}A)\\
&= S_r(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A) - AP_{r-1}(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)\\
&\quad - P_{r-1}[e_k(S_r)I - (\nabla_{e_k}A)P_{r-1} - A(\nabla_{e_k}P_{r-1})](\nabla_{e_k}A)\\
&= S_r(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A) - e_k(S_r)P_{r-1}(\nabla_{e_k}A) + (P_{r-1}\nabla_{e_k}A)^2,
\end{aligned}
$$

so that

$$
\begin{aligned}
(27)\qquad \sum_k \operatorname{tr}(T_k) = S_r &\sum_k \operatorname{tr}[(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)]\\
&- \sum_k \operatorname{tr}[e_k(S_r)P_{r-1}(\nabla_{e_k}A)] + \sum_k |P_{r-1}\nabla_{e_k}A|^2.
\end{aligned}
$$

Now, lemma 3.2 gives

$$(28)\qquad \sum_k \operatorname{tr}[e_k(S_r)P_{r-1}(\nabla_{e_k}A)] = \operatorname{tr}[P_{r-1}(\nabla_{\nabla S_r}A)] = |\nabla S_r|^2.$$

On the other hand, making $q = 0$ in proposition 3.1 one gets

$$
\begin{aligned}
\Delta S_r &= L_{r-1}(S_1) + \sum_k \operatorname{tr}\{(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)\}\\
&= +c[\operatorname{tr}(AP_{r-1})n - \operatorname{tr}(P_{r-1})S_1] + \operatorname{tr}(A^2P_{r-1})S_1 - \operatorname{tr}(AP_{r-1})|A|^2,
\end{aligned}
$$

so that

$$
\begin{aligned}
(29)\qquad \sum_k \operatorname{tr}\{(\nabla_{e_k}P_{r-1})(\nabla_{e_k}A)\} &= \Delta S_r - L_{r-1}(S_1) - c[\operatorname{tr}(AP_{r-1})n - \operatorname{tr}(P_{r-1})S_1]\\
&\quad - \operatorname{tr}(A^2P_{r-1})S_1 + \operatorname{tr}(AP_{r-1})|A|^2,
\end{aligned}
$$

Substituting (28) and (29) into (27), and then into (26), we finally arrive at

$$
\begin{aligned}
L_r(S_r) = L_{r-1}(S_{r+1}) &+ S_r[\Delta S_r - L_{r-1}(S_1)]\\
&+ \sum_k |P_{r-1}\nabla_{e_k}A|^2 - |\nabla S_r|^2 - cS_r[\operatorname{tr}(AP_{r-1})n - \operatorname{tr}(P_{r-1})S_1]
\end{aligned}
$$

$$
\begin{aligned}
&- S_r \operatorname{tr}(A^2P_{r-1})S_1 + S_r \operatorname{tr}(AP_{r-1})|A|^2\\
&+ c[\operatorname{tr}(AP_{r-1}) \operatorname{tr}(P_r) - \operatorname{tr}(P_{r-1}) \operatorname{tr}(AP_r)]\\
&+ \operatorname{tr}(A^2P_{r-1}) \operatorname{tr}(AP_r) - \operatorname{tr}(AP_{r-1}) \operatorname{tr}(A^2P_r),
\end{aligned}
$$

from where (24) easily follows. In order to get (25), let

$$
\begin{aligned}
T = {}& \operatorname{tr}(AP_{r-1})\{S_r(|A|^2 - cn) - [\operatorname{tr}(A^2P_r) - c \operatorname{tr}(P_r)]\}\\
&- \operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(A^2P_{r-1}) - c \operatorname{tr}(P_{r-1})]
\end{aligned}
$$

and take a basis $\{e_k\}$ of T_pM as in the statement of the corollary. Then

$$
\begin{aligned}
T &= \sum_i \lambda_i S_{r-1}(A_i)S_r(|A|^2 - cn) + \sum_{i,j} \lambda_i S_{r-1}(A_i)S_r(A_j)(c - \lambda_j^2)\\
&\quad + \sum_{i,j} \lambda_i^2 S_{r-1}(A_i)S_{r-1}(A_j)(c - \lambda_j^2)\\
&= \sum_i \lambda_i S_{r-1}(A_i) \cdot S_r(|A|^2 - cn)\\
&\quad + \sum_i \lambda_i S_{r-1}(A_i) \cdot \sum_j (c - \lambda_j^2)[S_r(A_j) + \lambda_i S_{r-1}(A_j)].
\end{aligned}
$$

Observing that

$$
\begin{aligned}
&S_r(|A|^2 - cn) + \sum_j (c - \lambda_j^2)[S_r(A_j) + \lambda_i S_{r-1}(A_j)]\\
&\qquad = S_r(|A|^2 - cn) + \sum_j (c - \lambda_j^2)[S_r + (\lambda_i - \lambda_j)S_{r-1}(A_j)]\\
&\qquad = \sum_j (c - \lambda_j^2)(\lambda_i - \lambda_j)S_{r-1}(A_j),
\end{aligned}
$$

we get

$$
T = \sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)\lambda_i(\lambda_i - \lambda_j)(c - \lambda_j^2).
$$

Doing the same computation as the one above, this time changing i by j from the very beginning, we arrive at

$$
T = \sum_{i,j} S_{r-1}(A_j)S_{r-1}(A_i)\lambda_j(\lambda_j - \lambda_i)(c - \lambda_i^2),
$$

so that

$$\begin{aligned} 2T &= \sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)[\lambda_i(c - \lambda_j^2) - \lambda_j(c - \lambda_i^2)] \\ &= \sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)^2(c + \lambda_i\lambda_j), \\ &= \sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}), \end{aligned}$$

where Gauss' equation was used in the last equality. □

Corollary 3.4. *Let* $x : M^n \to \overline{M}_c^{n+1}$ *be an isometric immersion. Then*

$$\begin{aligned} L_1(S_1) = \Delta S_2 + \{|\nabla A|^2 - |\nabla S_1|^2\} + \operatorname{tr}(AP_1)(|A|^2 - cn) \\ - S_1[\operatorname{tr}(A^2P_1) - c\operatorname{tr}(P_1)]. \end{aligned} \tag{30}$$

4. Applications

From now on, all manifolds are supposed to be connected. Let $x_1 : \mathbf{S}_{r_1}^{n_1} \to \mathbf{R}^{n_1+1}$ and $x_2 : \mathbf{S}_{r_2}^{n_2} \to \mathbf{R}^{n_2+1}$ be the standard immersions, with second (vector) fundamental forms α_1 and α_2. If $r_1^2 + r_2^2 = 1$ and $n = n_1 + n_2$, the product immersion $x = (x_1, x_2)$ satisfies $x(\mathbf{S}_{r_1}^{n_1} \times \mathbf{S}_{r_2}^{n_2}) \subset \mathbf{S}_1^{n+1}$. The Clifford torus $\mathbf{S}_{r_1}^{n_1} \times \mathbf{S}_{r_2}^{n_2}$ is the induced immersion $\mathbf{S}_{r_1}^{n_1} \times \mathbf{S}_{r_2}^{n_2} \hookrightarrow \mathbf{S}_1^{n+1}$.

Orient the Clifford torus via $N = \left(-\frac{r_2}{r_1}x_1, \frac{r_1}{r_2}x_2\right)$, and let $(p,q) \in \mathbf{S}_{r_1}^{n_1} \times \mathbf{S}_{r_2}^{n_2}$, $\{e_1, \dots, e_{n_1}\}$ be any orthonormal basis of $T_p\mathbf{S}_{r_1}^{n_1}$ and $\{e_{n_1+1}, \dots, e_{n_1+n_2}\}$ be any orthonormal basis of $T_q\mathbf{S}_{r_2}^{n_2}$. Making $E_i = (e_i, 0)$ for $1 \le i \le n_1$, and $E_i = (0, e_i)$ for $n_1 + 1 \le i \le n_1 + n_2$, we get an orthonormal basis for $T_{(p,q)}(\mathbf{S}_{r_1}^{n_1} \times \mathbf{S}_{r_2}^{n_2})$. Following [3], the matrix of the second fundamental form $A = A_N$ of the Clifford torus with respect to such a basis is given by

$$A = \begin{bmatrix} \frac{r_2}{r_1} I_{n_1} & 0 \\ 0 & -\frac{r_1}{r_2} I_{n_2} \end{bmatrix}. \tag{31}$$

Therefore, the fundamental principle of counting allows one to immediately read from this matrix the following expression for the elementary symmetric function S_r on the Clifford torus $\mathbf{S}_{r_1}^{n_1} \times \mathbf{S}_{r_2}^{n_2}$ in the chosen orientation:

$$S_r = \sum_{0 \le k \le r} (-1)^{r-k} \binom{n_1}{k} \binom{n_2}{r-k} \left(\frac{r_2}{r_1}\right)^k \left(\frac{r_1}{r_2}\right)^{r-k}, \tag{32}$$

with the convention that $\binom{m}{j} = 0$ whenever $j > m$. Yet another useful relation is true, as asserted by the following

LEMMA 4.1. *Let r_1, r_2 be positive real numbers such that $r_1^2 + r_2^2 = 1$, and n_1, n_2 be positive integers, $n = n_1 + n_2$. Concerning the Clifford torus $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2} \hookrightarrow \mathbf{S}^{n+1}$ one has, for $1 \le r \le n$,*

$$\begin{aligned}(33)\qquad &\operatorname{tr}(AP_{r-1})\{S_r(|A|^2 - n) - [\operatorname{tr}(A^2P_r) - \operatorname{tr}(P_r)]\}\\ &\quad - \operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(A^2P_{r-1}) - \operatorname{tr}(P_{r-1})] = 0.\end{aligned}$$

Proof. Letting A denote the second fundamental form of the standard immersion $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2} \hookrightarrow \mathbf{S}^{n+1}$, with respect to $N = \left(-\frac{r_2}{r_1}x_1, \frac{r_1}{r_2}x_2\right)$, it follows from (31) that

$$A^2 - I = \left(\frac{r_2}{r_1} - \frac{r_1}{r_2}\right)A.$$

Therefore, letting $\gamma = \frac{r_2}{r_1} - \frac{r_1}{r_2}$, one has for $0 \le r \le n$ that

$$A^2P_r - P_r = (A^2 - I)P_r = \gamma \cdot AP_r,$$

and thus

$$\begin{aligned}&\operatorname{tr}(AP_{r-1})\{S_r(|A|^2 - n) - [\operatorname{tr}(A^2P_r) - \operatorname{tr}(P_r)]\}\\ &\qquad - \operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(A^2P_{r-1}) - \operatorname{tr}(P_{r-1})]\\ &\quad = \operatorname{tr}(AP_{r-1})\{S_r \operatorname{tr}(A^2P_0 - P_0) - \operatorname{tr}(A^2P_r - P_r)\}\\ &\qquad - \operatorname{tr}(A^2P_{r-1}) \operatorname{tr}(A^2P_{r-1} - P_{r-1})\\ &\quad = \operatorname{tr}(AP_{r-1})[S_r\gamma \cdot \operatorname{tr}(AP_0) - \gamma \cdot \operatorname{tr}(AP_r)] - \operatorname{tr}(A^2P_{r-1})\gamma \cdot \operatorname{tr}(AP_{r-1})\\ &\quad = \gamma \cdot \operatorname{tr}(AP_{r-1})[S_1S_r - (r+1)S_{r+1}] - \gamma \cdot \operatorname{tr}(A^2P_{r-1}) \operatorname{tr}(AP_{r-1})\\ &\quad = \gamma \cdot \operatorname{tr}(AP_{r-1}) \operatorname{tr}(A^2P_{r-1}) - \gamma \cdot \operatorname{tr}(A^2P_{r-1}) \operatorname{tr}(AP_{r-1}) = 0. \qquad \square\end{aligned}$$

We now state a slightly modified version of remark 2.1 of [3]:

LEMMA 4.2. *Let $x : M^n \to \overline{M}^{n+1}_c$ be an isometric immersion. Assume that the mean curvature H of M does not change sign, and choose the orientation in such a way that $H \ge 0$.*[1] *If the scalar curvature R of M satisfies $R \ge c$, then $P_1 \ge 0$. If $R > c$, then $P_1 > 0$.*

Proof. It follows from equation (8) that $R \ge c$ if and only if $S_2 \ge 0$. Denoting by $\lambda_1, \dots, \lambda_n$ the eigenvalues of the second fundamental form A of x, one has

[1] If M has scalar curvature $R > c$, then $S_2 > 0$. It then follows from $S_1^2 = 2S_2 + |A|^2$ that $H \ne 0$. Therefore, H does not change sign on M.

$$S_1^2 = |A|^2 + 2S_2 \geq |A|^2 \geq \lambda_i^2, \tag{34}$$

so that $-S_1 \leq \lambda_i \leq S_1$. Hence, $S_1(A_i) = S_1 - \lambda_i \geq 0$, and $P_1 \geq 0$. If, at some $p \in M$, one has $S_1(A_i) = 0$, it follows that $S_1 = \lambda_i$, and (34) gives $S_2 = 0$ and $\lambda_j = 0$ for all $j \neq i$. Therefore, $S_1 \geq 0$, $S_2 > 0 \Rightarrow P_1 > 0$. $\square$

Our first result generalizes theorem 3.1 of [3].

THEOREM 4.3. *Let $x : M^n \to \mathbf{S}^{n+1}$ be a closed orientable hypersurface of the sphere, with scalar curvature $R \geq 1$. Assume that the mean curvature H of M does not change sign, and orient M in such a way that $H \geq 0$. If H or R is constant on M, and*

$$H[\mathrm{tr}(P_1) - \mathrm{tr}(A^2P_1)] + (n-1)(R-1)(|A|^2 - n) \geq 0$$

on M, then

(a) *$H[\mathrm{tr}(P_1) - \mathrm{tr}(A^2P_1)] + (n-1)(R-1)(|A|^2 - n) = 0$ on M.*

(b) *M is either totally geodesic or a Clifford torus $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, with $r_1^2 + r_2^2 = 1$, $n_1, n_2 > 0$ and $\beta = \left(\frac{r_2}{r_1}\right)^2 \geq 1$ satisfying*

$$n_1(n_1 - 1)\beta^2 - 2n_1n_2\beta + n_2(n_2 - 1) = n(n-1)(R-1)\beta. \tag{35}$$

Proof. It follows from (30) that

$$L_1(S_1) = \Delta S_2 + |\nabla A|^2 - |\nabla S_1|^2 + 2S_2(|A|^2 - n) + S_1[\mathrm{tr}(P_1) - \mathrm{tr}(A^2P_1)],$$

and upon integration over M we get

$$0 = \int_M \{|\nabla A|^2 - |\nabla S_1|^2 + 2S_2(|A|^2 - n) + S_1[\mathrm{tr}(P_1) - \mathrm{tr}(A^2P_1)]\}\, dM.$$

Now, since $2S_2 = n(n-1)(R-1) \geq 0$, lemma 2.5 gives for the case of constant R that $|\nabla A|^2 - |\nabla S_1|^2 \geq 0$. Since this inequality is obviously true when H is constant, we thus get

$$|\nabla A|^2 - |\nabla S_1|^2 = 0 \tag{36}$$

and

$$H[\mathrm{tr}(P_1) - \mathrm{tr}(A^2P_1)] + (n-1)(R-1)(|A|^2 - n) = 0 \tag{37}$$

on M. Returning to the expression for $L_1(S_1)$, it follows that $L_1(S_1) = \Delta S_2$. Therefore, whether H or R is constant, we get $\Delta S_2 = 0$, and Hopf's strong maximum principle assures that S_2 is constant. This in turn gives us $L_1(S_1) = 0$ in both cases, and by using $|A|^2 = S_1^2 - 2S_2$ we arrive at

$$\frac{1}{2}L_1|A|^2 = S_1L_1(S_1) + \langle P_1\nabla S_1, \nabla S_1\rangle - L_1(S_2) = \langle P_1\nabla S_1, \nabla S_1\rangle.$$

Integrating again over M gives $\int_M \langle P_1\nabla S_1, \nabla S_1\rangle\, dM = 0$, and the previous lemma gives $\langle P_1\nabla S_1, \nabla S_1\rangle = 0$ on M.

If $S_2 = 0$ (or, equivalently, $R = 1$) on M, it was proved in [3] that M is either totally geodesic or a Clifford torus. Otherwise, $S_2 > 0$ on M, and lemma 4.2 gives $P_1 > 0$ on M. Thus, $\nabla S_1 = 0$ on M, and it follows from (36) that $\nabla A = 0$. By theorem 4 of [12], M^n is a Clifford torus $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, with $r_1^2 + r_2^2 = 1$. Finally, lemma 4.1 assures that

$$nH[\mathrm{tr}(P_1) - \mathrm{tr}(A^2P_1)] + n(n-1)(R-1)(|A|^2 - n) = 0$$

on all of these tori, so that the only algebraic condition to be satisfied by M is

$$2S_2 = n(n-1)(R-1).$$

It now suffices to refer to expression (32) for S_2 on Clifford tori. $\square$

Example 4.4. We construct some families of nontrivial Clifford tori having constant prescribed scalar curvature $1 \leq R < 2$.

When $R \geq 2$, every Clifford torus satisfying (35) also satisfies $\beta > 1$. In fact, otherwise one would have from (32) that

$$n(n-1)(R-1) = n_1(n_1-1) - 2n_1n_2 + n_2(n_2-1) = (n_1-n_2)^2 - n,$$

which would give in turn

$$(n-2)^2 \geq (n_1-n_2)^2 = n[(n-1)(R-1)+1] \geq n^2,$$

a contradiction. For $1 \leq R < 2$ there are several families of nontrivial tori (i.e., non-minimal ones) with $\beta = 1$ and satisfying (35). For $R = 1$, for instance, relation (35) reduces to $(n_1 - n_2)^2 = n_1 + n_2$. Any solution $n_1 = a_1$, $n_2 = a_2$ ($a_1 < a_2$) for this equation generates a whole family $n_1 = a_k$, $n_2 = a_{k+1}$ of solutions, where the sequence $(a_k)_{k \geq 1}$ satisfies, for $k \geq 1$, the recurrence relation $a_{k+2} = 2a_{k+1} - a_k + 1$. Since $a_1 = 1$, $a_2 = 3$ is one solution, we get in this case the family

$$\mathbf{S}^{\binom{k}{2}}_{1/\sqrt{2}} \times \mathbf{S}^{\binom{k+1}{2}}_{1/\sqrt{2}} \hookrightarrow \mathbf{S}^{\binom{k+1}{3}+1}_{1}.$$

For $R = 3/2$, (35) reduces to $(n_1+n_2)(n_1+n_2-1) = 8n_1n_2$. Once again, any solution $n_1 = a_1$, $n_2 = a_2$ for this equation generates a whole family $n_1 = a_k$, $n_2 = a_{k+1}$ of solutions, where the sequence $(a_k)_{k \geq 1}$ satisfies, for $k \geq 1$, the recurrence relation $a_{k+2} = 6a_{k+1} - a_k + 1$. Thus, the solution $n_1 = 1$, $n_2 = 7$ generates a family of Clifford tori having scalar curvature $3/2$, the first member of which is $\mathbf{S}^1_{1/\sqrt{2}} \times \mathbf{S}^7_{1/\sqrt{2}} \hookrightarrow \mathbf{S}^9_1$.

From now on we state and prove our main results, the first of which being a gap theorem that generalizes theorem 5.3.2 of [15] for $(r-1)$-minimal hypersurfaces of the sphere:

THEOREM 4.5. *If $x: M^n \to \mathbf{S}^{n+1}$ is a closed oriented hypersurface of the sphere for which $S_r = 0$, then*

$$\int_M \operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(P_{r-1}) - \operatorname{tr}(A^2P_{r-1})]\, dM \le 0. \tag{38}$$

Moreover, if $S_{r+1} \neq 0$ *on* M, *then* $\operatorname{tr}(A^2P_{r-1})\operatorname{tr}(P_{r-1}) \ge \operatorname{tr}(A^2P_{r-1})^2$ *on* M *if and only if* M^n *is a Clifford torus* $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, *with* $r_1^2 + r_2^2 = 1$, $n_1 + n_2 = n$ *and* $S_r = 0$.

Proof. It follows from corollary 3.3 that

$$0 = L_{r-1}(S_{r+1}) + \sum_k |P_{r-1}\nabla_{e_k}A|^2 + \operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(P_{r-1}) - \operatorname{tr}(A^2P_{r-1})]. \tag{39}$$

Integrating over M, we get

$$0 = \int_M \sum_k |P_{r-1}\nabla_{e_k}A|^2\, dM + \int_M \operatorname{tr}(A^2P_{r-1})[\operatorname{tr}(P_{r-1}) - \operatorname{tr}(A^2P_{r-1})]\, dM,$$

and hence the first part of the theorem.

To the second part, note firstly that, by lemma 4.1, all Clifford tori $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$ with $S_r = 0$ do satisfy $\operatorname{tr}(A^2P_{r-1}) = \operatorname{tr}(P_{r-1})$. Conversely, suppose that $\operatorname{tr}(A^2P_{r-1})\operatorname{tr}(P_{r-1}) \ge \operatorname{tr}(A^2P_{r-1})^2$ and $S_{r+1} \neq 0$ on M. Then

$$\operatorname{tr}(A^2P_{r-1})\operatorname{tr}(P_{r-1}) = \operatorname{tr}(A^2P_{r-1})^2$$

and

$$\sum_k |P_{r-1}\nabla_{e_k}A|^2 = 0$$

on M, from where it follows that $P_{r-1}\nabla_{e_k}A = 0$ for all $1 \le k \le n$. Since $S_r = 0$ and $S_{r+1} \neq 0$, item (b) of proposition 2.8 assures that P_{r-1} is invertible, so that $\nabla_{e_k}A = 0$ for all $1 \le k \le n$, or still $\nabla A = 0$. Hence, by theorem 4 of [12], M is an open submanifold of a Clifford torus $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, with $r_1^2 + r_2^2 = 1$ and $n_1 + n_2 = n$. Since M is also closed and connected, it follows that $M = \mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$. □

The nonnegativeness of the sectional curvature K_M of M suffices to guarantee that $\operatorname{tr}(A^2P_{r-1})\operatorname{tr}(P_{r-1}) \ge \operatorname{tr}(A^2P_{r-1})^2$, as asserted by the following

COROLLARY 4.6. *Let* $x : M^n \to \mathbf{S}^{n+1}$ *be a closed, oriented hypersurface of the sphere, with* $S_r = 0$ *for some* $1 \le r < n$. *If* $S_{r+1} \neq 0$ *on* M^n, *and* M^n *has sectional curvature* $K_M \ge 0$, *then* M^n *is a Clifford torus* $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, *with* $r_1^2 + r_2^2 = 1$, $n_1 + n_2 = n$ *and* $S_r = 0$.

Proof. The proof of corollary 3.3, together with $S_r = 0$, guarantee that, for $p \in M$,

$$\mathrm{tr}(A^2P_{r-1})[\mathrm{tr}(P_{r-1}) - \mathrm{tr}(A^2P_{r-1})] = \frac{1}{2}\sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}),$$

where $\{e_k\}$ is a moving frame on M^n, diagonalizing A at p, with $Ae_k = \lambda_k e_k$ at p, and σ_{ij} denotes the 2-dimensional subspace of T_pM generated by e_i and e_j.

Now, item (b) of proposition 2.8 guarantees that P_{r-1} is definite, so that $S_{r-1}(A_i)S_{r-1}(A_j) > 0$ for all $1 \le i, j \le n$. Therefore, since $K_M \ge 0$, we get

$$\sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}) \ge 0$$

on M^n, or still $\mathrm{tr}(A^2P_{r-1})[\mathrm{tr}(P_{r-1}) - \mathrm{tr}(A^2P_{r-1})] \ge 0$ on M^n. □

Remark 4.7. In both results above, condition $S_{r+1} \neq 0$ eliminates the possibility of M^n being totally geodesic. Moreover, it follows from (31) that, for all Clifford tori $M = \mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, one has

$$K_M(\sigma_{ij}) = 1 + \lambda_i\lambda_j = \begin{cases} 1 + \dfrac{r_2^2}{r_1^2}, & \text{if } 1 \le i, j \le n_1 \\ 0, & \text{if } 1 \le i \le n_1 < j \le n \\ 1 + \dfrac{r_1^2}{r_2^2}, & \text{if } n_1 < i, j \le n \end{cases}.$$

Hence, $K_M \ge 0$.

For what comes next, we stress that the ellipticity of the operator L_r is equivalent to the positive definiteness of P_r. In the proof of the next result we use proposition 3.2 of [4], stated below.

PROPOSITION 4.8. *Let M^n be a closed hypersurface of $\mathbf{R}^{n+1}$, $\mathbf{H}^{n+1}$ or of an open hemisphere of $\mathbf{S}^{n+1}$, such that $S_r > 0$ on M^n for some $2 \le r \le n$. Then, for $1 \le j \le r-1$, one has $H_j > 0$ and L_j elliptic on M^n.*

THEOREM 4.9. *Let $x : M^n \to \overline{M}^{n+1}_c$ be a closed orientable hypersurface of $\overline{M}^{n+1}_c$, where $\overline{M}^{n+1}_c$ denotes $\mathbf{H}^{n+1}$, $\mathbf{R}^{n+1}$ or an open hemisphere of $\mathbf{S}^{n+1}$, according to whether $c = -1, 0$ or 1. If $S_r \neq 0$ is constant on M^n for some $2 \le r < n$, and M^n has sectional curvature $K_M \ge 0$, then M is a geodesic hypersphere and x is an embedding.*

Proof. First of all, it follows from the hypotheses on M^n and $\overline{M}^{n+1}$ the existence of a point $p_0 \in M$ where all principal curvatures of x have the same sign. Orienting M^n in such a way that these curvatures are all positive at p_0, we get $S_r(p_0) > 0$, and thus $S_r > 0$ on M^n. On the other hand, since S_r is constant on M^n, equation (25) gives, at $p \in M$,

$$0 = L_{r-1}(S_{r+1} - S_1 S_r) + \sum_k |P_{r-1}\nabla_{e_k} A|^2 + \frac{1}{2}\sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}), \tag{40}$$

where $\{e_k\}$ is a moving frame on a neighborhood of p in M, diagonalizing A at p, with $Ae_k = \lambda_k e_k$, and σ_{ij} denotes the 2-dimensional subspace of T_pM generated by e_i and e_j.

Now, proposition 4.8 assures the ellipticity of L_{r-1}, so that P_{r-1} is positive definite, and $S_{r-1}(A_i) > 0$ for all $1 \le i \le n$. Therefore,

$$\sum_{i,j} S_{r-1}(A_i)S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}) \ge 0,$$

and equation (40) gives

$$L_{r-1}(S_{r+1} - S_1 S_r) + \sum_k |P_{r-1}\nabla_{e_k} A|^2 \le 0.$$

from where $L_{r-1}(S_{r+1} - S_1 S_r) \le 0$. Since M is closed and L_{r-1} is elliptic, it follows from Hopf's strong maximum principle ([8]) that $S_{r+1} - S_1 S_r$ is constant on M, so that $\sum_k |P_{r-1}\nabla_{e_k} A|^2 = 0$. Using again the fact that P_{r-1} is positive definite, we get $\nabla_{e_k} A = 0$ for $1 \le k \le n$, or still $\nabla A = 0$ on M. Let's now consider three separate cases:

For $c = 0$, theorem 4 of [12] assures that, up to isometries of $\mathbf{R}^{n+1}$, M is an open subset of $\mathbf{S}^{n_1}_{r_1} \times \mathbf{R}^{n_2}$, where $n_1 + n_2 = n$ and $n_1, n_2 \ge 0$. Since M is closed (i.e., compact whithout boundary), we have $M^n = \mathbf{S}^n_{r_1}$. The reasoning for $c = -1$ is essentially the same. Suppose now that M^n is contained in an open hemisphere of $\mathbf{S}^{n+1}$. It follows again from theorem 4 of [12] that, up to isometries of $\mathbf{S}^{n+1}$, M is a Clifford torus $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, where $n_1 + n_2 = n$ and $n_1, n_2 \ge 0$. However, were $n_1, n_2 > 0$, M would not be contained in an open hemisphere of $\mathbf{S}^{n+1}$. Therefore, $\min\{n_1, n_2\} = 0$, and the closedness of M gives that it is isometric to $\mathbf{S}^n_{r_1}$, for some $0 < r_1 < 1$.

Finally, since in all of the above cases $x : M \to x(M)$ is a covering of the simply connected space $x(M)$, it follows that x is an embedding. □

The hypothesis of M^n being contained in an open hemisphere of $\mathbf{S}^{n+1}$ in the above theorem is somewhat restrictive, but can be relaxed once one imposes conditions on M^n sufficient to guarantee the ellipticity of L_r, for some $1 \le r < n$. The following lemma gives one such set of conditions:

LEMMA 4.10. *Let M^n be an orientable Riemannian manifold, of Ricci curvature* $\mathrm{Ric} \ge c$, *and $x : M^n \to \overline{M}^{n+1}_c$ be an isometric immersion. Suppose that the mean curvature H of M^n does not change sign, and orient M in such a way that $H \ge 0$. If $S_r(p) \ne 0$ for some $2 \le r \le n$, then L_{r-1} is elliptic at p.*

Proof. Fix $p \in M$ and choose a basis $\{e_k\}$ of T_pM, diagonalizing A at p, with $Ae_k = \lambda_k e_k$ for $1 \leq k \leq n$. Gauss' equation gives

$$\mathrm{Ric}_p(e_k) = \frac{1}{n-1}\sum_{i \neq k}(c + \lambda_k \lambda_i) = c + \frac{1}{n-1}\lambda_k(S_1 - \lambda_k).$$

Then, $\mathrm{Ric}_p(e_k) \geq c$ and $S_1(p) \geq 0$ give $0 \leq \lambda_k \leq S_1(p)$ for $1 \leq k \leq n$. It follows that all summands of $S_r(p)$ are nonnegative, so that $S_r(p) \geq 0$. If $S_r(p) \neq 0$ then $S_r(p) > 0$, and at least r of the λ_k are positive, so that, at p, at least one of the summands in $S_{r-1}(A_i)$ is positive, for each $1 \leq i \leq n$. Hence, P_{r-1} is positive definite at p. □

Theorem 4.11. *Let $x : M^n \to \mathbf{S}^{n+1}$ be a complete orientable hypersurface of the unit sphere, with Ricci curvature* Ric ≥ 1. *Assume that the mean curvature H of M does not change sign, and orient M in such a way that $H \geq 0$. If, for some $2 \leq r \leq n$, $S_r \neq 0$ is constant on M, then*

$$\begin{aligned}(41)\qquad &\mathrm{tr}(AP_{r-1})\{S_r(|A|^2 - n) + [\mathrm{tr}(P_r) - \mathrm{tr}(A^2P_r)]\}\\ &\quad + \mathrm{tr}(A^2P_{r-1})[\mathrm{tr}(P_{r-1}) - \mathrm{tr}(A^2P_{r-1})] \geq 0,\end{aligned}$$

on M if and only if M is a Clifford torus $\mathbf{S}^{n_1}_{r_1} \times \mathbf{S}^{n_2}_{r_2}$, with $n_1 + n_2 = n$, and $r_1^2 + r_2^2 = 1$.

Proof. When M is a Clifford torus, it follows from lemma 4.1 that (41) becomes an equality. Conversely, suppose that (41) is valid on M. Since S_r is constant on M, one has once more

$$L_{r-1}(S_{r+1} - S_1S_r) + \sum_k |P_{r-1}\nabla_{e_k}A|^2 \leq 0.$$

The condition on the Ricci curvature of M assures, via Bonnet-Myers thorem, that M is closed. On the other hand, since $S_r \neq 0$ on M, the preceeding lemma assures the ellipticity of L_{r-1}, and from this point on the reasoning is identical to that of the previous result. □

Concerning the non-compact case, we have the following result:

Theorem 4.12. *Let $x : M^n \to \mathbf{R}^{n+1}$ be a complete, non-compact, oriented hypersurface of the Euclidean space, with sectional curvature $K_M \geq 0$. If, for some $2 \leq r < n$, $S_r \neq 0$ is constant on M, and $S_1S_r - S_{r+1}$ attains a global maximum on M, then M is isometric to $\mathbf{S}^{n_1}_{r_1} \times \mathbf{R}^{n_2}$ for some $r_1 > 0$, where $n_1 + n_2 = n$ and $r \leq n_1 < n$. In particular, if $S_{r+1} = 0$ on M and H attains a global maximum on M, then M is isometric to $\mathbf{S}^r_{r_1} \times \mathbf{R}^{n-r}$, for some $r_1 > 0$.*

Proof. Since $K_M \geq 0$, it follows that M has Ricci curvature $\mathrm{Ric} \geq 0$. Moreover, letting $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the second fundamental form A of M, it also follows from $K_M \geq 0$ that, at each point of M, one has either $\lambda_1, \dots, \lambda_n \geq 0$ or $\lambda_1, \dots, \lambda_n \leq 0$. Therefore, S_1 does not change sign on M, for otherwise there would exist $p \in M$ such that $S_1(p) = 0$, and hence $\lambda_1 = \cdots = \lambda_n = 0$ at p; this would contradict $S_r(p) \neq 0$. The ellipticity of L_{r-1} now follows from lemma 4.10.

Using equation (25), we get at $p \in M$ and for an appropriate frame $\{e_k\}$ that

$$L_{r-1}(S_1 S_r - S_{r+1}) = \sum_k |P_{r-1} \nabla_{e_k} A|^2 + \frac{1}{2} \sum_{i,j} S_{r-1}(A_i) S_{r-1}(A_j)(\lambda_i - \lambda_j)^2 K_M(\sigma_{ij}).$$

Since both terms on the right hand side are nonnegative, it follows that $L_{r-1}(S_1 S_r - S_{r+1}) \geq 0$. The hypothesis of $S_1 S_r - S_{r+1}$ attaining a global maximum on M guarantees, via Hopf's strong maximum principle, that $S_1 S_r - S_{r+1}$ is constant on M. It then follows from the above relation that

$$\sum_k |P_{r-1} \nabla_{e_k} A|^2 = 0$$

on M, and thus $\nabla A = 0$ on M, for P_{r-1} is positive definite.

Finally, applying once more theorem 4 of [12] we get M^n isometric to $\mathbf{S}^{n_1}_{r_1} \times \mathbf{R}^{n_2}$, where $n_1, n_2 \geq 0$ and $n_1 + n_2 = n$. However, since M is non-compact and $S_r \neq 0$ over it, it must happen that $r \leq n_1 < n$. To finish, note that $S_{r+1} = 0$ gives $n_1 = r$ as the only possible option. □

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