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NON-UNIQUENESS OF OBSTACLE PROBLEM ON FINITE RIEMANN SURFACE

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Abstract

In [1], R. Fehlmann and F. P. Gardiner studied an extremal problem for a finite Riemann surface to establish a slit mapping theorem. In this article, we give a condition for non-uniqueness of such slit mappings, by using a deformation of a Riemann surface.

1. Introduction

Let S be an analytically finite Riemann surface, namely, a compact Riemann surface minus finitely many points. Though all the results in the present note are generalized for topologically finite Riemann surfaces in an appropriate way (see [5]), for simplicity, we restrict ourselves to this case.

Let $A(S)$ be the set of integrable holomorphic quadratic differentials φ on S. For $\varphi \in A(S)$ set $\|\varphi\| = \iint_S |\varphi|\, dxdy$, $z = x + iy$. Let $\mathfrak{S}(S)$ be the family of simple closed curves on S, which are homotopic neither to a point of S nor to a puncture of S. Let $\mathfrak{S}[S]$ be the set of free homotopy classes of elements of $\mathfrak{S}(S)$. For $\varphi \in A(S)$ and $\gamma \in \mathfrak{S}(S)$, we denote the height of γ with respect to φ by

$$\mathrm{height}_\varphi(\gamma) = \int_\gamma |\mathrm{Im}(\sqrt{\varphi(z)}\, dz)|$$

and the height of the homotopy class $[\gamma]$ by

$$\mathrm{height}_\varphi[\gamma] = \inf_\beta \mathrm{height}_\varphi(\beta),$$

where the infimum is taken over all closed curves $\beta \in \mathfrak{S}(S)$ freely homotopic to γ in S.

Definition 1.1. We say that E is an *obstacle* in S if E is a compact subset of S, if $S \backslash E$ is connected and if E is contained in a topological disk in S.

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Remark 1.2. In [1] Fehlmann and Gardiner called E an obstacle if E is a compact subset in S consisting of finitely many components, each of which is simply connected, and if the natural embedding $S\backslash E \to S$ induces a surjective homomorphism $\pi_1(S\backslash E) \to \pi_1(S)$. An obstacle in the sense of Definition 1.1 satisfies these conditions (see Lemma 2.3 in [5]). A compact set consisting of finitely many simply connected components may not be an obstacle in the sense of Fehlmann and Gardiner. The next example was learned from Professor Masahiko Taniguchi. Let $E_0' = \{x + i\sin(\pi/x) \mid x \in (-1,0) \cup (0,1]\} \cup \{iy \mid -1 \le y \le 1\}$ and set $E_0 = \{e^{\pi i z} \mid z \in E_0'\}$. Then we can see that the compact set E_0 is connected and simply connected and E_0 separates 0 from ∞ in $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$. Let γ be a non-trivial simple closed curve on an analytically finite Riemann surface S. Then there is a topological embedding $g : \mathbf{C}^* \to S$ such that the image of the unite circle S^1 under g is freely homotopic to γ in S. The set $E = g(E_0)$ is connected and simply connected. Since $S\backslash E$ is homeomorphic to $S\backslash\gamma$ the homomorphism $\pi_1(S\backslash E) \to \pi_1(S)$ is not surjective.

For an obstacle E of S, let $\mathfrak{F}(S,E)$ be the family of pairs (f, S_f), where f is a conformal map of $S\backslash E$ into another Riemann surface S_f of the same analytic type as S such that f maps each puncture of S to a puncture of S_f. Then $(f, S_f) \in \mathfrak{F}(S,E)$ induces an isomorphism ι_f of the fundamental group $\pi_1(S)$ of S onto $\pi_1(S_f)$ (cf. [5, Lemma 2.5]). We denote by $[S_f, \iota_f]$ the Teichmüller (equivalence) class of (S_f, ι_f) in $T(S)$. Here, pairs (R_j, ι_j), $j = 1,2$, of Riemann surfaces R_j and orientation-preserving isomorphisms $\iota_j : \pi_1(S) \to \pi_1(R_j)$ are said to be Teichmüller equivalent if there exists a conformal map $h : R_1 \to R_2$ such that $\iota_2 = h_* \circ \iota_1$. We refer to [4] for basic facts about Teichmüller spaces. We remark that, for every $(f, S_f) \in \mathfrak{F}(S,E)$ the set $f(E) := S_f\backslash f(S\backslash E)$ is an obstacle of S_f.

The heights mapping theorem (cf. [3]) states that, for every $(f, S_f) \in \mathfrak{F}(S,E)$ and $\varphi \in A(S)\backslash\{0\}$, there exists the unique holomorphic quadratic differential $\varphi_f \in A(S_f)\backslash\{0\}$ such that

$$\mathrm{height}_\varphi[\gamma] = \mathrm{height}_{\varphi_f}(\iota_f[\gamma]) \quad \text{for every } [\gamma] \in \mathfrak{S}[S].$$

Definition 1.3. A compact subset E of S is said to be a *horizontal slit* for $\varphi \in A(S)\backslash\{0\}$ if each connected component of E is either a horizontal arc of φ or a finite union of horizontal arcs and critical points of φ.

Let E be an obstacle of S and $\varphi \in A(S)\backslash\{0\}$. Fehlmann and Gardiner [1] posed an *obstacle problem for* (S, E, φ) which asks the existence of $(f, S_f) \in \mathfrak{F}(S,E)$ maximizing the quantity

$$M_f = \|\varphi_f\|_{L^1(S_f)} = \iint_{S_f} |\varphi_f|$$

in $\mathfrak{F}(S,E)$, and showed the following result.

Theorem 1.4 (Fehlmann-Gardiner). *Suppose that S is an analytically finite Riemann surface, and that $\varphi \in A(S)\backslash\{0\}$. Let E be an obstacle of S with finitely* *many components. Then there exists an element* $(g, S_g) \in \mathfrak{F}(S, E)$ *such that* M_g *attains the supremum*:

$$M_g = \sup_{(f, S_f) \in \mathfrak{F}(S, E)} M_f.$$

Moreover, $g(E)$ *is a horizontal slit for* φ_g. *Furthermore if* $(f, S_f) \in \mathfrak{F}(S, E)$ *is also extremal for* (S, E, φ), *then* $f^*\varphi_f = g^*\varphi_g$ *on* $S \backslash E$.

The point $(g, S_g) \in \mathfrak{F}(S, E)$ in Theorem 1.4 is called *extremal* for (S, E, φ), and the associated differential φ_g is called the *extremal differential*.

Fehlmann and Gardiner asserted in [1] moreover that if $(f, S_f) \in \mathfrak{F}(S, E)$ is also extremal for (S, E, φ), then $g \circ f^{-1}$ extends to a conformal map of S_f onto S_g. This is not necessarily valid. We show it in the following theorem. To state the result, we introduce a technical concept.

Definition 1.5. Let S be an analytically finite Riemann surface and m be an integer with $m \geq 2$. Suppose that an obstacle E of S is a horizontal slit for $\varphi \in A(S) \backslash \{0\}$. We will call $p_0 \in E$ a *refolding point of order* m *for* (S, E, φ) if p_0 is a zero of φ of order m and if E contains two horizontal arcs ℓ_1 and ℓ_2 with common end point p_0 such that the angle formed by them at p_0 is greater than $2\pi/(m+2)$.

Theorem 1.6. *Let* E *be an obstacle of an analytically finite Riemann surface* S *and* $\varphi \in A(S) \backslash \{0\}$. *Suppose that* $(g, S_g) \in \mathfrak{F}(S, E)$ *is extremal for* (S, E, φ) *and that* $g(E)$ *has a refolding point* p_0 *of order* $m \geq 3$ *for* $(S_g, g(E), \varphi_g)$. *Then, there exists another extremal element* $(f, S_f) \in \mathfrak{F}(S, E)$ *for* (S, E, φ) *such that* S_f *is not conformally equivalent to* S_g.

Remark 1.7. In the proof, by parametrizing the arcs κ_j, $j = 1, 2$, by $t \in [0, 1]$ so that the φ_g-length of $\kappa_j([0, t])$ is t times that of $\kappa_j([0, 1])$, we can actually construct a family of extremal elements $(f_t, S_{f_t}) \in \mathfrak{F}(S, E)$, $0 \leq t \leq 1$, for the same obstacle problem for (S, E, φ) satisfying

(i) $(f_0, S_{f_0}) = (g, S_g)$,
(ii) the marked Riemann surface $\tau_t = [S_{f_t}, \iota_{f_t}]$ varies continuously in $T(S)$, and
(iii) $\tau_t \neq \tau_0$ for $t \neq 0$.

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2. Example

In this section we give an example of triple (S, E, φ) which satisfies the assumptions of Theorem 1.6.

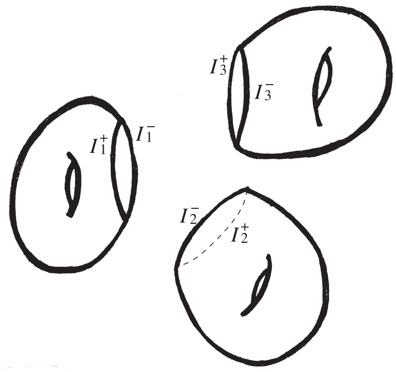

FIGURE 1.

First take three copies M_1, M_2, M_3 of the rectangle

$$M = \{z = x + iy \in \mathbf{C} \,|\, |x| \leq 2, |y| \leq 1\},$$

and let z_j be the coordinate corresponding to z on each M_j. Next on each M_j, identify the two pairs of parallel sides under the translations

$$z_j \to z_j + 4, \quad z_j \to z_j + 2i.$$

Then we obtain three copies T_1, T_2, T_3 of a torus T. The quadratic differential dz^2 on M induces a holomorphic quadratic differential φ_0 on T.

Cut T_j along the segment

$$I_j = \{z_j = x_j + iy_j \,|\, -1 \leq x_j \leq 0, y_j = 0\},$$

and glue them cyclically. More precisely, we paste the upper edge I_1^+ of the slit I_1 to the lower edge I_2^- of the slit I_2, the upper edge I_2^+ of the slit I_2 to the lower edge I_3^- of the slit I_3, and the upper edge I_3^+ of the slit I_3 to the lower edge I_1^- of the slit I_1. Then we obtain a compact Riemann surface S of genus three (see Figures 1 and 2).

Let Π be the natural projection of S onto the torus T, and φ be the pull-back of φ_0 by Π. Finally, let E be the subset of S consisting of ℓ_1 and ℓ_2, where ℓ_i is the arc on T_i corresponding to $\{z \,|\, 0 \leq x \leq 1, y = 0\}$.

We now consider the obstacle problem for (S, E, φ). Then the obstacle E is a horizontal slit for φ. Hence we know that the identity mapping of S gives an extremal slit map associated with the extremal problem for this triple. Moreover, we can easily see that the point $p_0 = \Pi^{-1}(0)$ in S is a refolding point of order 4 for (S, E, φ).

Thus the assumptions in Theorem 1.6 are satisfied and, as a consequence, the points in $T(S)$ which are induced by the extremals for (S, E, φ) are not uniquely determined.

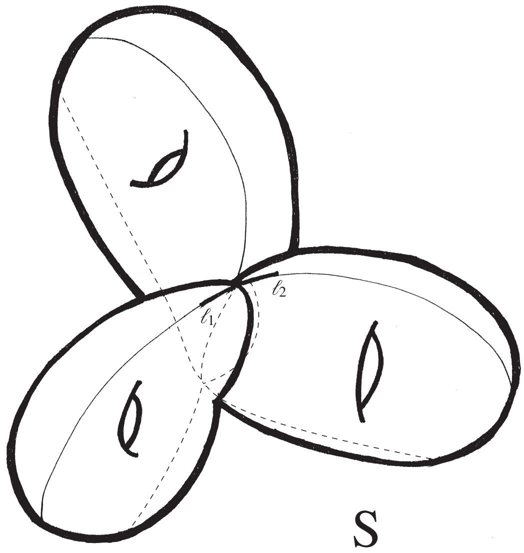

FIGURE 2.

3. Proof of Theorem 1.6

Assume that a component J of $g(E)$ contains a refolding point p_0 of order $m \geq 3$ for $(S_g, g(E), \varphi_g)$ and horizontal arcs ℓ_1 and ℓ_2 with common end point p_0 and that an angle formed by ℓ_1 and ℓ_2 at p_0 is

$$\frac{2k\pi}{m+2} \quad \left(2 \leq k \leq \frac{m+2}{2}\right).$$

Note that the arcs ℓ_1, ℓ_2 are segments on the real axis with endpoint at the origin with respect to the natural parameter

$$\zeta = \int_{z_0}^{z} \sqrt{\varphi_g(z)}\, dz,$$

where z is a local chart near p_0 and $z_0 = z(p_0)$.

We take closed subarcs $\kappa_j \subset \ell_j$, $j = 1, 2$, with the same φ_g-length such that p_0 is an endpoint of each κ_j and that φ_g has no zeros on $\kappa_j \setminus \{p_0\}$. Let p_j be the other endpoint of κ_j for each j. Also set $K = \kappa_1 \cup \kappa_2$.

Now, cut S_g along κ_1 and κ_2. For each j, let κ_j^+ and κ_j^- be the right-side and the left-side edges of the slit κ_j, respectively, with respect to the orientation which corresponds to the move along the slit from p_0 to p_j. Assume that κ_1^- and κ_2^+, κ_1^+ and κ_2^- form the angles

$$\frac{2k\pi}{m+2} \quad \text{and} \quad \frac{2\pi(m+2-k)}{m+2}.$$

at p_0, respectively.

Paste κ_1^- and κ_2^+ so that points having the same absolute value with respect to ζ are identified. In the same way, paste κ_1^+ and κ_2^-. Let $\tilde{K}$ be the union of the pasted segments. Then we obtain a new analytically finite Riemann surface R and the natural conformal embedding $u : S_g \backslash K \to R$. Set $f = u \circ g$ and $S_f = R$. Then the pair (f, S_f) is an element of the family $\mathfrak{F}(S, E)$. (Figures 2 exhibits the case when $m = 4$ and $k = 2$.)

Moreover, from the construction we can extend $(u^{-1})^* \varphi_g$ naturally to a holomorphic quadratic differential $\psi \in A(S_f)$ satisfying $\|\psi\|_{L^1(S_f)} = \|\varphi_g\|_{L^1(S_g)}$. The obstacle $f(E)$ is a horizontal slit for ψ.

The following proposition is crucial in the proof of Lemma 3.2 and Lemma 3.3.

Proposition 3.1 (Second Minimal Norm Property [2, p. 54]). *Assume S is an analytically finite Riemann surface. Let $\varphi \in A(S)$ and let ψ be a quadratic differential, continuous except possibly at the punctures of S. Suppose* $\operatorname{height}_\varphi[\gamma] \leq \operatorname{height}_\psi[\gamma]$ *for every* $[\gamma] \in \mathfrak{S}[S]$. *Then*

$$\|\varphi\| \leq \iint_S |\sqrt{\varphi}\sqrt{\psi}| \, dxdy \leq \|\varphi\|^{1/2} \|\psi\|^{1/2}$$

and $\|\varphi\| = \|\psi\|$ *if only if* $\varphi = \psi$.

Lemma 3.2. $\psi = \varphi_f$.

Proof. If $\operatorname{height}_\psi[\gamma] \leq \operatorname{height}_{\varphi_f}[\gamma]$ for every $[\gamma] \in \mathfrak{S}[S_f]$, then by Proposition 3.1 we can see $\|\psi\|_{L^1(S_f)} \leq \|\varphi_f\|_{L^1(S_f)}$. On the other hand, since (g, S_g) is extremal for (S, E, φ) and $\|\psi\|_{L^1(S_f)} = \|\varphi_g\|_{L^1(S_g)}$, we obtain $\|\varphi_f\|_{L^1(S_f)} \leq \|\psi\|_{L^1(S_f)}$. Hence, $\|\psi\|_{L^1(S_f)} = \|\varphi_f\|_{L^1(S_f)}$. Proposition 3.1 implies $\psi = \varphi_f$ on S_f. So we have only to show that $\operatorname{height}_\psi[\gamma] \leq \operatorname{height}_{\varphi_f}[\gamma]$ for every $[\gamma] \in \mathfrak{S}[S_f]$.

We say that a curve β on S_g is a φ_g-polygonal curve, if β is the union of finitely many horizontal arcs and vertical arcs of φ_g. Note that for every $[\gamma] \in \mathfrak{S}[S_g]$

$$\operatorname{height}_{\varphi_g}[\gamma] = \inf_\beta \operatorname{height}_{\varphi_g}(\beta),$$

where the infimum is taken over all φ_g-polygonal curves β freely homotopic to γ in S_g.

Let $[\gamma] \in \mathfrak{S}[S]$ and β be a φ_g-polygonal curve in S_g with $[\beta] = \iota_g[\gamma]$ in $\mathfrak{S}[S_g]$. We can add horizontal segments contained in $\tilde{K}$ to the (possibly broken) curve $u(\beta \backslash K)$ so that the resulting set $\tilde{\beta}$ is a ψ-polygonal (closed) curve and satisfies $[\tilde{\beta}] = \iota_f[\gamma]$ in $\mathfrak{S}[S_f]$. Then,

$$\operatorname{height}_\psi(\iota_f[\gamma]) \leq \operatorname{height}_\psi(\tilde{\beta}) = \operatorname{height}_{\varphi_g}(\beta).$$

Hence we obtain

$$\operatorname{height}_\psi(\iota_f[\gamma]) \leq \operatorname{height}_{\varphi_g}(\iota_g[\gamma]) = \operatorname{height}_{\varphi_f}(\iota_f[\gamma]).$$

Thus we have proved the assertion. $\square$

By Lemma 3.2, we see that (f, S_f) is extremal for (S, E, φ) and the obstacle $f(E)$ of S_f is a horizontal slit for φ_f.

LEMMA 3.3. *$[S_g, \iota_g] \neq [S_f, \iota_f]$ in $T(S)$.*

Proof. Suppose that $[S_g, \iota_g] = [S_f, \iota_f]$ in $T(S)$. Then there exists a conformal map $h : S_g \to S_f$ with $\iota_f = h_* \circ \iota_g$. Since $\mathrm{height}_{h^*\varphi_f}[\gamma] = \mathrm{height}_{\varphi_f}[h(\gamma)]$, we obtain

$$\mathrm{height}_{h^*\varphi_f}[\gamma] = \mathrm{height}_{\varphi_g}[\gamma]$$

for every $[\gamma] \in \mathfrak{S}[S_g]$. Hence Proposition 3.1 implies that

$$h^*\varphi_f = \varphi_g \quad \text{on } S_g.$$

In particular, the map h sends the zeros of φ_g to those of φ_f while keeping multiplicities. From the argument together with the relation $u^*\varphi_f = \varphi_g$ on $S_g \setminus K$, the number of zeros of a given order of φ_g on K is equal to that of φ_f on $\tilde{K}$. This is impossible, because from the construction the zero p_0 of φ_g of order $m \geq 3$ breaks into two zeros of φ_f of orders $k-2$ and $m-k$, respectively, where $2 \leq k \leq (m+2)/2$. Hence the number of zeros of φ_g of order m on K is less than that of φ_f on $\tilde{K}$, which is a contradiction. $\square$

Thus we have proved the assertion in Remark 1.7, and hence Theorem 1.6.

In [5], the author gave the uniqueness result under the condition that the obstacle possibly consists of infinitely many components. It is expantion of Theorem 1.4.

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