I. LAHIRI AND N. MANDAL
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MEROMORPHIC FUNCTIONS SHARING A SINGLE VALUE WITH UNIT WEIGHT

Indrajit Lahiri and Nintu Mandal

Abstract

We prove a uniqueness theorem for meromorphic functions sharing a single value with unit weight which improves a recent result of A. H. H. Al-Khaladi.

1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbf{C}$. For $a \in \mathbf{C} \cup \{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if the a-points of f and g coincide in locations and multiplicities. If we do not consider the multiplicities, we say that f and g share the value a IM (ignoring multiplicites). Though for the standard definitions and notations of the value distribution theory we refer to [3], some definitions and notations are given in the paper.

Definition 1.1 [6]. *Let m be a positive integer. We denote by $N(r,a;f \mid\leq m)$ $(N(r,a;f \mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.*

In a like manner we define $N(r,a;f \mid< m)$ and $N(r,a;f \mid> m)$.

Also $\overline{N}(r,a;f \mid\leq m)$, $\overline{N}(r,a;f \mid\geq m)$, $\overline{N}(r,a;f \mid< m)$ and $\overline{N}(r,a;f \mid> m)$ are defined similarly where in counting the a-points of f we ignore the multiplicities.

Further we agree to take $\overline{N}(r,a;f \mid\leq \infty) = \overline{N}(r,a;f)$ and $N(r,a;f \mid\leq \infty) = N(r,a;f)$.

Finally we define $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2)$.

In [8] R. Nevanlinna proved the following theorem.

Theorem A [8]. *Let f and g be two nonconstant entire functions satisfying $N(r,0;f) \equiv N(r,0;g) \equiv 0$. If f and g share the value 1 CM then either $f \equiv g$ or $fg \equiv 1$.*

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Recently Al-khaladi [1] improved Theorem A and proved the following result.

Theorem B [1]. *Let f and g be two nonconstant meromorphic functions satisfying* $\overline{N}(r,0;g)+\overline{N}(r,\infty;g)=S(r,g)$ *and* $\overline{N}(r,\infty;f)=S(r,f)$. *If f and g share the value* 1 *CM then f and g satisfy one of the following*:

(i) $f-1\equiv c(g-1)$, *where c is a nonzero constant. In particular, if* $c=1$ *then* $f\equiv g$;

(ii) $(f-b)g\equiv 1-b$, *where* $(b\neq 1)$ *is a constant. In particular, if* $b=0$ *then* $fg\equiv 1$;

(iii) $T(r,f)=N(r,0;f\,|\leq 2)+S(r,f)$ *and* $T(r,g)=N(r,0;f'\,|\leq 1)+S(r,f)$.

R. Brück [2] proved the following result involving a nonconstant entire function and its derivative.

Theorem C [2]. *Let f be a nonconstant entire function satisfying* $N(r,0;f')=S(r,f)$. *If f and f' share the value* 1 *CM then* $f-1\equiv c(g-1)$, *where c is a nonzero constant.*

As a consequence of Theorem B Al-khaladi [1] improved Theorem C and proved the following result.

Theorem D [1]. *Let f be a nonconstant meromorphic function satisfying* $\overline{N}(r,0;f')+\overline{N}(r,\infty;f)=S(r,f)$. *If f and* $f^{(k)}$ $(k\geq 1)$ *share the value* 1 *CM then* $f-1\equiv c(f^{(k)}-1)$, *where c is a nonzero constant.*

However a better result than Theorem D is proved in [7]. Considering $f(z)=(e^z-1)(e^z+1)^2+1$ and $g(z)=e^z$ Al-khaladi pointed out that in Theorem B the CM sharing of the value 1 cannot be replaced by the sharing of simple 1-points only. Following example shows that in Theorem B it is not even possible to replace the CM sharing of the value 1 by IM sharing.

Example 1.1. *Let* $f(z)=2e^z-e^{2z}$ *and* $g(z)=e^z$. *Then* $\overline{N}(r,0;g)=\overline{N}(r,\infty;g)=S(r,g)$, $\overline{N}(r,\infty;f)=S(r,f)$ *and f, g share* 1 *IM. Also we see that none of the possibilities of Theorem B occurs.*

So it is a natural query to explore the possibility of relaxing the nature of sharing the value 1 in Theorem B. The notion of weighted sharing of values renders a useful tool for this purpose. In the following definition we explain this idea, which measures how close a shared value is to being shared IM or to being shared CM.

Definition 1.2 [4, 5]. *Let k be a nonnegative integer or infinity. For* $a\in\mathbf{C}\cup\{\infty\}$ *we denote by* $E_k(a;f)$ *the set of all a-points of f where an a-point of multiplicity m is counted m times if* $m\leq k$ *and* $k+1$ *times if* $m>k$. *If* $E_k(a;f)=E_k(a;g)$, *we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f-a$ with multiplicity $m(\le k)$ if and only if it is a zero of $g-a$ with multiplicity $m(\le k)$ and z_o is a zero of $f-a$ with multiplicity $m(>k)$ if and only if it is a zero of $g-a$ with multiplicity $n(>k)$ where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integers p, $0\le p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Following theorem is the main result of the paper.

THEOREM 1.1. *Let f and g be two nonconstant meromorphic functions such that $\overline{N}(r,0;g)+\overline{N}(r,\infty;g)=S(r,g)$, $N(r,0;f)+N_2(r,\infty;f)\le T(r,f)+S(r,f)$ and $\overline{N}(r,\infty;f)\le \lambda T(r,f)+S(r,f)$ for a constant λ $(0<\lambda<1)$. If f, g share $(1,1)$ then f and g satisfy one of the following*:

(i) $f-1\equiv c(g-1)$, *where c is a nonzero constant. In particular, if $c=1$ then $f\equiv g$*;

(ii) $(f-b)g\equiv 1-b$, *where $(b\ne 1)$ is a constant. In particular, if $b=0$ then $fg\equiv 1$*;

(iii) $T(r,f)=N(r,0;f\mid\le 2)+N_2(r,\infty;f)+S(r,f)$, $N(r,0;f'\mid\le 1)\le T(r,g)+\overline{N}(r,\infty;f)+S(r,f)$ *and* $T(r,g)\le N(r,0;f'\mid\le 1)+\overline{N}(r,\infty;f\mid\ge 2)+S(r,f)$.

Following example shows that the condition $N(r,0;f)+N_2(r,\infty;f)\le T(r,f)+S(r,f)$ is necessary for Theorem 1.1.

Example 1.2. *Let* $f(z)=\dfrac{2-e^{2z}}{2-e^{z}}$ *and* $g(z)=e^{z}$. *Then f, g share $(1,\infty)$ and* $\overline{N}(r,0;g)+\overline{N}(r,\infty;g)=S(r,g)$. *Also* $\overline{N}(r,\infty;f)=N_2(r,\infty;f)=N(r,2;e^{z})=\frac{1}{2}T(r,f)+S(r,f)$ *and* $N(r,0;f\mid\le 2)=N(r,0;f)=N(r,2;e^{2z})=T(r,f)+S(r,f)$. *Further we see that none of the possibilities of Theorem* 1.1 *holds.*

Following example shows that for Theorem 1.1 the condition $\overline{N}(r,\infty;f)\le \lambda T(r,f)+S(r,f)$ is necessary, where $0<\lambda<1$.

Example 1.3. *Let* $f(z)=\dfrac{2}{1+e^{z}}$ *and* $g(z)=e^{z}$. *Then f, g share $(1,\infty)$ and* $\overline{N}(r,0;g)+\overline{N}(r,\infty;g)=S(r,g)$. *Also* $N(r,0;f)=S(r,f)$, $\overline{N}(r,\infty;f)=N_2(r,\infty;f)=T(r,f)+S(r,f)$ *and* $\overline{N}(r,\infty;f\mid\ge 2)\equiv 0$. *Further none of the possibilities of Theorem* 1.1 *occurs.*

Following example shows that the condition $\overline{N}(r,0;g)=S(r,g)$ is necessary for Theorem 1.1.

Example 1.4. *Let* $f(z) = e^z - 1$ *and* $g(z) = (e^z - 1)^2$. *Then* f, g *share* $(1, \infty)$, $\overline{N}(r, \infty; f) = S(r, f)$, $\overline{N}(r, \infty; g) = S(r, g)$ *and* $\overline{N}(r, 0; g) \neq S(r, g)$. *Also we see that none of the possibilities of Theorem* 1.1 *holds.*

Following example shows that the condition $\overline{N}(r, \infty; g) = S(r, g)$ is necessary for Theorem 1.1.

Example 1.5. *Let* $f(z) = 1 + e^{2z}$ *and* $g(z) = \dfrac{1}{1 - e^z}$. *Then* f, g *share* $(1, \infty)$, $\overline{N}(r, \infty; f) = S(r, f)$, $\overline{N}(r, 0; g) = S(r, g)$ *and* $\overline{N}(r, \infty; g) \neq S(r, g)$. *Also none of the possibilities of Theorem* 1.1 *occurs.*

Also Example 1.1 shows that in Theorem 1.1 it is not possible to relax the nature sharing from $(1, 1)$ to $(1, 0)$.

Finally following three examples show that all the three possibilities of Theorem 1.1 can actually occur.

Example 1.6. *Let* $f(z) = 3e^z - 2$ *and* $g(z) = e^z$. *Then* f, g *share* $(1, \infty)$, $\overline{N}(r, 0; g) + \overline{N}(r, \infty; g) = S(r, g)$ *and* $\overline{N}(r, \infty; f) = S(r, f)$. *Also* $f - 1 \equiv 3(g - 1)$, *which is the possibility* (i) *of Theorem* 1.1.

Example 1.7. *Let* $f(z) = 2 - \dfrac{1}{e^z}$ *and* $g(z) = e^z$. *Then* f, g *share* $(1, \infty)$, $\overline{N}(r, 0; g) + \overline{N}(r, \infty; g) = S(r, g)$ *and* $\overline{N}(r, \infty; f) = S(r, f)$. *Also* $(f - 2)g \equiv 1 - 2$, *which is the possibility* (ii) *of Theorem* 1.1.

Example 1.8. *Let* $f(z) = \dfrac{e^z(1 + e^z)}{e^z - 1}$ *and* $g(z) = -e^{2z}$. *Then* f, g *share* $(1, \infty)$, $T(r, f) = T(r, g) + O(1)$, $N(r, 0; f) = N(r, 0; f \mid\leq 2) = N(r, -1; e^z) = \frac{1}{2}T(r, f) + S(r, f)$, $\overline{N}(r, \infty; f) = N_2(r, \infty; f) = N(r, 1; e^z) = \frac{1}{2}T(r, f) + S(r, f)$, $\overline{N}(r, \infty; f \mid\geq 2) \equiv 0$ *and* $\overline{N}(r, 0; g) + \overline{N}(r, \infty; g) = S(r, g)$. *Since*

$$\begin{aligned} N(r, 0; f' \mid\leq 1) &= N(r, 1 + \sqrt{2}; e^z) + N(r, 1 - \sqrt{2}; e^z) \\ &= 2T(r, e^z) + S(r, e^z) \\ &= T(r, g) + S(r, f), \end{aligned}$$

it follows that $T(r, f) = N(r, 0; f \mid\leq 2) + N_2(r, \infty; f) + S(r, f)$, $N(r, 0; f' \mid\leq 1) \leq T(r, g) + \overline{N}(r, \infty; f) + S(r, f)$ *and* $T(r, g) \leq N(r, 0; f' \mid\leq 1) + \overline{N}(r, \infty; f \mid\geq 2) + S(r, f)$, *which is the possibility* (iii) *of Theorem* 1.1.

Following result is a direct consequence of Theorem 1.1 and improves Theorem B.

Corollary 1.1. *Let f and g be two nonconstant meromorphic functions satisfying $\overline{N}(r,0;g)+\overline{N}(r,\infty;g)=S(r,g)$ and $\overline{N}(r,\infty;f)=S(r,f)$. If f and g share $(1,1)$ then f and g satisfy one of the following*:

(i) $f-1\equiv c(g-1)$, *where c is a nonzero constant. In particular, if $c=1$ then $f\equiv g$*;

(ii) $(f-b)g\equiv 1-b$, *where $(b\neq 1)$ is a constant. In particular, if $b=0$ then $fg\equiv 1$*;

(iii) $T(r,f)=N(r,0;f\,|\leq 2)+S(r,f)$ *and* $T(r,g)=N(r,0;f'\,|\leq 1)+S(r,f)$.

We now explain some more notations.

Definition 1.3 [5]. *Let f and g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the counting function of those a-points of f whose multiplicities are not equal to the multriplicities of the corresponding a-points of g, where each a-point is counted only once.*

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$.

Definition 1.4. *We denote by $N_0(r,0;f^{(k)})$ $(\overline{N}_0(r,0;f^{(k)}))$ the counting function (reduced counting function) of those zeros of $f^{(k)}$ which are not the zeros of f.*

Definition 1.5. *We denote by $N_{\otimes}(r,0;f^{(k)})$ $(\overline{N}_{\otimes}(r,0;f^{(k)}))$ the counting function (reduced counting function) of those zeros of $f^{(k)}$ which are not the zeros of $f(f-1)$.*

Definition 1.6. *We denote by $N_{\oplus}(r,0;f^{(k)})$ $(\overline{N}_{\oplus}(r,0;f^{(k)}))$ the counting function (reduced counting function) of those zeros of $f^{(k)}$ which are not the zeros of $f-1$.*

Throughout the paper we mean by f, g two nonconstant meromorphic functions defined in the open complex plane $\mathbf{C}$.

2. Lemmas

In this section we present some necessary lemmas. Henceforth we denote by H the function defined by

$$H=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right).$$

Lemma 2.1 [5]. *If f, g share $(1,1)$ and $H\not\equiv 0$ then*

(i) $N(r,1;f\,|\le 1) \le N(r,H) + S(r,f) + S(r,g)$,

(ii) $N(r,1;g\,|\le 1) \le N(r,H) + S(r,f) + S(r,g)$.

LEMMA 2.2 [5]. *Let* f, g *share* $(1,0)$ *and* $H \not\equiv 0$. *Then*

$$N(r,H) \le \overline{N}(r,\infty;f\,|\ge 2) + \overline{N}(r,0;f\,|\ge 2) + \overline{N}(r,\infty;g\,|\ge 2) + \overline{N}(r,0;g\,|\ge 2) + \overline{N}_*(r,1;f,g) + \overline{N}_\otimes(r,0;f') + \overline{N}_\otimes(r,0;g').$$

LEMMA 2.3 [6]. *If* k *is a positive integer then*

$$N_0(r,0;f^{(k)}) \le k\overline{N}(r,\infty;f) + N(r,0;f\,|< k) + k\overline{N}(r,0;f\,|\ge k) + S(r,f).$$

LEMMA 2.4. *If* f, g *share* $(1,1)$ *then*

$$\overline{N}_0(r,0;g') + \overline{N}(r,1;g\,|\ge 2) + \overline{N}_*(r,1;f,g) \le 3\overline{N}(r,0;g) + 3\overline{N}(r,\infty;g) + S(r,g).$$

Proof. Since f, g share $(1,1)$, we get by Lemma 2.3 for $k=1$

$$\begin{aligned}&\overline{N}_0(r,0;g') + \overline{N}(r,1;g\,|\ge 2) + \overline{N}_*(r,1;f,g)\\ &\quad\le \overline{N}_0(r,0;g') + 2\overline{N}(r,1;g\,|\ge 2)\\ &\quad\le 3N_0(r,0;g')\\ &\quad\le 3\overline{N}(r,0;g) + 3\overline{N}(r,\infty;g) + S(r,g).\end{aligned}$$

This proves the lemma. □

3. Proof of the main result

Proof of Theorem 1.1. We consider the following two cases.

CASE I. Let $H \equiv 0$. Then on integration we get

$$f-1 \equiv \frac{g-1}{A-B(g-1)}, \tag{3.1}$$

where $A(\ne 0)$ and B are constants.

If $B=0$ then from (3.1) we get

$$f-1 \equiv c(g-1),$$

where $c = \dfrac{1}{A}$ is a nonzero constant. This is possibility (i) of the theorem.

Let $B \ne 0$. If $A+B \ne 0$ then from (3.1) we get by the second fundamental theorem

$$\begin{aligned} T(r,g) &\le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}\left(r,\frac{A+B}{B};g\right) + S(r,g) \\ &= \overline{N}(r,\infty;f) + S(r,g) \\ &\le \lambda T(r,f) + S(r,g) \\ &= \lambda T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction as $0 < \lambda < 1$.

Therefore $A + B = 0$ and so from (3.1) we get

$$\left(f - \frac{B-1}{B}\right)g \equiv \frac{1}{B}.$$

If we put $b = \dfrac{B-1}{B}$ then $b \neq 1$ and from above we get

$$(f-b)g \equiv 1-b,$$

which is possibility (ii) of the theorem.

Case II. Let $H \not\equiv 0$. Since f, g share $(1,1)$, by the second fundamental theorem we get

$$\begin{aligned} T(r,g) &\le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}(r,1;g) + S(r,g) \\ &= \overline{N}(r,1;g) + S(r,g) \\ &\le T(r,f) + S(r,g). \end{aligned}$$

This shows that every $S(r,g)$ is replacable by $S(r,f)$. Let $h = (f-1)/(g-1)$. Since f, g share $(1,1)$ we get by Lemma 2.4

$$\begin{aligned} \overline{N}(r,0;h) &\le \overline{N}_*(r,1;f,g) + \overline{N}(r,\infty;g) \\ &\le 3\overline{N}(r,0;g) + 4\overline{N}(r,\infty;g) \\ &= S(r,g) \\ &= S(r,f) \end{aligned}$$

and

$$\begin{aligned} \overline{N}(r,\infty;h) &\le \overline{N}_*(r,1;f,g) + \overline{N}(r,\infty;f) \\ &\le 3\overline{N}(r,0;g) + 3\overline{N}(r,\infty;g) + \overline{N}(r,\infty;f) \\ &= \overline{N}(r,\infty;f) + S(r,g) \\ &= \overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

Since

$$f' = h(g-1)\left(\frac{h'}{h} + \frac{g'}{g-1}\right),$$

we see that possible zeros of f' occur from the following sources: (i) zeros of h, (ii) zeros of $g-1$ and (ii) zeros of $\frac{h'}{h}+\frac{g'}{g-1}$.

Let z_0 be a simple zero of $g-1$. Since f, g share $(1,1)$, z_0 is neither a zero nor a pole of h. On the other hand z_0 is a simple pole of $\frac{h'}{h}+\frac{g'}{g-1}$. Hence z_0 is not a zero of f'. Therefore by Lemma 2.4 we get

$$
\begin{aligned}
(3.2)\quad & \overline{N}(r,0;f') \\
&\le \overline{N}(r,0;h)+\overline{N}(r,1;g\,|\ge 2)+T\left(r,\frac{h'}{h}+\frac{g'}{g-1}\right) \\
&\le 3\overline{N}(r,0;g)+3\overline{N}(r,\infty;g)+N\left(r,\frac{h'}{h}\right)+N\left(r,\frac{g'}{g-1}\right)+S(r,f) \\
&\le \overline{N}(r,0;h)+\overline{N}(r,\infty;h)+\overline{N}(r,1;g)+\overline{N}(r,\infty;g)+S(r,f) \\
&\le N(r,1;g\,|\le 1)+\overline{N}(r,1;g\,|\ge 2)+\overline{N}(r,\infty;f)+S(r,f) \\
&\le N(r,1;g\,|\le 1)+3\overline{N}(r,0;g)+3\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+S(r,f) \\
&= N(r,1;g\,|\le 1)+\overline{N}(r,\infty;f)+S(r,f).
\end{aligned}
$$

Again since f, g share $(1,1)$, by Lemma 2.1, Lemma 2.2 and Lemma 2.3 we get

$$
\begin{aligned}
(3.3)\quad N(r,1;g\,|\le 1) &\le \overline{N}(r,0;f\,|\ge 2)+\overline{N}(r,0;g\,|\ge 2) \\
&\quad +\overline{N}_*(r,1;f,g)+\overline{N}(r,\infty;g\,|\ge 2) \\
&\quad +\overline{N}_\otimes(r,0;f')+\overline{N}_\otimes(r,0;g')+\overline{N}(r,\infty;f\,|\ge 2) \\
&\le \overline{N}(r,0;f\,|\ge 2)+\overline{N}(r,1;f\,|\ge 2)+\overline{N}_\otimes(r,0;f') \\
&\quad +N_0(r,0;g')+\overline{N}(r,\infty;f\,|\ge 2)+S(r,g) \\
&\le \overline{N}(r,0;f')+\overline{N}(r,\infty;f\,|\ge 2)+S(r,f).
\end{aligned}
$$

By the second fundamental theorem and Lemma 2.3 we get

$$
\begin{aligned}
T(r,g) &\le \overline{N}(r,1;g)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+S(r,g) \\
&\le N(r,1;g\,|\le 1)+N_0(r,0;g')+S(r,g) \\
&\le N(r,1;g\,|\le 1)+N_0(r,0;g')+S(r,g) \\
&\le N(r,1;g\,|\le 1)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+S(r,g) \\
&= N(r,1;g\,|\le 1)+S(r,g)
\end{aligned}
$$

so that

$$
(3.4)\qquad N(r,1;g\,|\le 1)=T(r,g)+S(r,g)=T(r,g)+S(r,f).
$$

Since f, g share $(1,1)$ by Lemma 2.3 we get

$$\begin{aligned}\overline{N}(r,1;f\,|\geq 2) &= \overline{N}(r,1;g\,|\geq 2)\\ &\leq N_0(r,0;g')\\ &\leq \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+S(r,g)\\ &= S(r,g)\\ &= S(r,f).\end{aligned}$$

Now by the second fundamental theorem we get from (3.3) and the given condition

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,\infty;f)+N(r,0;f)+\overline{N}(r,1;f)-N_{\otimes}(r,0;f')+S(r,f)\\ &= \overline{N}(r,\infty;f)+N(r,0;f)+N(r,1;g\,|\leq 1)-N_{\otimes}(r,0;f')+S(r,f)\\ &\leq N_2(r,\infty;f)+N(r,0;f)+\overline{N}(r,0;f')-N_{\otimes}(r,0;f')+S(r,f)\\ &= N_2(r,\infty;f)+N(r,0;f)+\overline{N}_{\otimes}(r,0;f')-N_{\otimes}(r,0;f')\\ &\quad +\overline{N}(r,1;f\,|\geq 2)+S(r,f)\\ &= N(r,0;f)+N_2(r,\infty;f)+\overline{N}_{\otimes}(r,0;f')-N_{\otimes}(r,0;f')+S(r,f)\\ &\leq N_2(r,\infty;f)+N(r,0;f)+S(r,f)\\ &\leq T(r,f)+S(r,f).\end{aligned}$$

This shows that

$$T(r,f)=N(r,0;f)+N_2(r,\infty;f)+S(r,f) \tag{3.5}$$

and

$$N_{\otimes}(r,0;f')-\overline{N}_{\otimes}(r,0;f')=S(r,f). \tag{3.6}$$

From (3.6) we get

$$N(r,0;f\,|\geq 3)\leq 3\{N_{\otimes}(r,0;f')-\overline{N}_{\otimes}(r,0;f')\}=S(r,f).$$

Hence from (3.5) we get

$$T(r,f)=N(r,0;f\,|\leq 2)+N_2(r,\infty;f)+S(r,f).$$

Again from (3.6) we get by Lemma 2.4

$$\begin{aligned}\overline{N}(r,0;f'\,|\geq 2) &\leq \overline{N}(r,1;f\,|\geq 3)+2\{N_{\otimes}(r,0;f')-\overline{N}_{\otimes}(r,0;f')\}\\ &\leq \overline{N}(r,1;f\,|\geq 2)+S(r,f)\\ &= S(r,f).\end{aligned}$$

So from (3.2), (3.3) and (3.4) we obtain

$$N(r,0;f'|\leq 1) \leq T(r,g) + \overline{N}(r,\infty;f) + S(r,f)$$

and

$$T(r,g) \leq N(r,0;f'|\leq 1) + \overline{N}(r,\infty;f|\geq 2) + S(r,f).$$

This proves the theorem. □

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Department of Mathematics
University of Kalyani
West Bengal 741235
India
E-mail: ilahiri@vsnl.com, ilahiri@hotpop.com