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$PSL(2,5)$ SEXTIC FIELDS WITH A POWER BASIS

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Abstract

It is shown that there exist infinitely many $PSL(2,5)$ sextic fields with a power basis.

1. Introduction

Let K be an algebraic number field of degree n. We denote the ring of integers of K by O_K. The field K is said to possess a power basis if there exists an element $\theta \in O_K$ such that $O_K = \mathbf{Z} + \mathbf{Z}\theta + \cdots + \mathbf{Z}\theta^{n-1}$. A field having a power basis is called monogenic. Every quadratic field is monogenic. Dedekind [**5**] gave an example of a cubic field which is not monogenic. If K is a cyclic cubic field, Gras [**9**], [**10**] and Archinard [**3**] have given necessary and sufficient conditions for K to be monogenic. Dummit and Kisilevsky [**6**] have shown that there exist infinitely many cyclic cubic fields which are monogenic. The same has been shown for non-cyclic cubic fields, pure quartic fields, bicyclic quartic fields, dihedral quartic fields by Spearman and Williams [**17**], Funakura [**8**], Nakahara [**16**], Huard, Spearman and Williams [**13**] respectively. It is not known if there are infinitely many monogenic cyclic quartic fields. If K is a cyclic field of prime degree $p \geq 5$ then Gras [**11**] has proved that K is monogenic if and only if K is the maximal real subfield of a cyclotomic field. In particular there is only one monogenic cyclic quintic field. Lavallee, Spearman, Williams and Yang [**15**] have shown that there exist infinitely many dihedral quintic fields with a power basis.

In this paper we exhibit infinitely many monogenic $PSL(2,5)$ sextic fields.

Theorem. *There are infinitely many integers t such that the $PSL(2,5)$ sextic fields*

$$\mathbf{Q}(\theta), \quad \theta^6 - 4\theta^5 + 2\theta^4 - 3t\theta^3 + \theta^2 + 2\theta + 1 = 0,$$

are distinct and monogenic.

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2. A parametric family of $PSL(2,5)$ sextics

Let $b, c, d \in \mathbf{Q}$. The parametric family of sextics

$$f(x; b, c, d) = x^6 + 2cx^5 + Ax^4 + Bx^3 + Cx^2 + (3b + 2c + 6)x + (b + 1),$$

where

$$A = -bd + c^2 + 2c + 2,$$
$$B = b - 2bd + 2c^2 + 2c - 4d + 2,$$
$$C = 3b - bd + c^2 + 4c + 5,$$

was studied by Anai and Kondo [**2**]. It is known that the Galois group of f is usually $PSL(2,5)(\cong A_5)$ but not always. A good source of families of polynomials with given Galois group is [**14**]. For our purposes we set

$$F_t(x) = f(x; 0, -2, (3t+6)/4), \quad t \in \mathbf{Z}.$$

We have

$$F_t(x) = x^6 - 4x^5 + 2x^4 - 3tx^3 + x^2 + 2x + 1.$$

Lemma 2.1. *$F_t(x)$ is irreducible in $\mathbf{Z}[x]$ for all $t(\neq 1) \in \mathbf{Z}$.*

Proof. Let $t(\neq 1) \in \mathbf{Z}$. We have

$$F_t(x) = (x+2)(x^5 + 2x^3 + 2x^2 + 2) \pmod 3, \tag{2.1}$$

where $x^5 + 2x^3 + 2x^2 + 2$ is irreducible (mod 3). Thus if $F_t(x)$ is reducible in $\mathbf{Z}[x]$, it must have the factorization

$$F_t(x) = l(x)m(x),$$

where $l(x)$ is a monic linear polynomial and $m(x)$ is a monic irreducible quintic polynomial. Since $F_t(0) = 1$ we must have $l(x) = x \pm 1$. If $l(x) = x - 1$, then $0 = F_t(1) = 3 - 3t$, contradicting $t \neq 1$. If $l(x) = x + 1$ then $0 = F_t(-1) = 7 + 3t$, contradicting $t \in \mathbf{Z}$. This completes the proof of the irreducibility of $F_t(x)$. □

We remark that $F_1(x) = x^6 - 4x^5 + 2x^4 - 3x^3 + x^2 + 2x + 1$ is reducible as it is divisible by $x - 1$. Using MAPLE we find

Lemma 2.2. *For $t \in \mathbf{Z}$, $\operatorname{disc}(F_t(x)) = (729t^3 + 522t^2 + 1788t + 2648)^2$.*

Lemma 2.3. *If $t(\neq 1) \in \mathbf{Z}$ then $\operatorname{Gal}(F_t(x)) \cong PSL(2,5)$.*

Proof. Let $t(\neq 1) \in \mathbf{Z}$. By Lemma 2.1 $F_t(x)$ is irreducible in $\mathbf{Z}[x]$. From (2.1) we see that $\operatorname{Gal}(F_t(x))$ contains a 5-cycle so that 5 divides the order of $\operatorname{Gal}(F_t(x))$. The only possible Galois groups of an irreducible sextic polynomial $\in \mathbf{Z}[x]$ up to conjugation are the fifteen groups

$$C_6, \quad S_3, \quad D_6, \quad A_4, \quad C_3^2 \rtimes C_2 \cong C_3 \times D_3,$$
$$A_4 \times C_2, \quad S_4, \quad C_3^2 \rtimes C_2^2 \cong D_3 \times D_3, \quad C_3^2 \rtimes C_4, \quad S_4 \times C_2,$$
$$PSL(2,5) \cong A_5, \quad C_3^2 \rtimes D_4, \quad PGL(2,5) \cong S_5, \quad A_6, \quad S_6,$$

see [**4**, pp. 329–331], whose orders are respectively 6, 6, 12, 12, 18, 24, 24, 36, 36, 48, 60, 72, 120, 360, 720. The only groups having order divisible by 5 are

$$PSL(2,5), \quad PGL(2,5), \quad A_6, \quad S_6.$$

By Lemma 2.2 the discriminant of $F_t(x)$ is a perfect square so $\mathrm{Gal}(F_t(x)) \ncong PGL(2,5)$, S_6. Hence $\mathrm{Gal}(F_t(x)) \cong PSL(2,5)$ or A_6.

Let ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, ϕ_6 be the six complex roots of $F_t(x)$. Let $f_{15}(x)$ be the polynomial in $\mathbf{Z}[x]$ of degree 15 whose roots are

$$\begin{array}{lll}
\phi_1\phi_2 + \phi_3\phi_4 + \phi_5\phi_6, & \phi_1\phi_2 + \phi_3\phi_5 + \phi_4\phi_6, & \phi_1\phi_2 + \phi_3\phi_6 + \phi_4\phi_5, \\
\phi_1\phi_3 + \phi_2\phi_4 + \phi_5\phi_6, & \phi_1\phi_3 + \phi_2\phi_5 + \phi_4\phi_6, & \phi_1\phi_3 + \phi_2\phi_6 + \phi_4\phi_5, \\
\phi_1\phi_4 + \phi_2\phi_3 + \phi_5\phi_6, & \phi_1\phi_4 + \phi_2\phi_5 + \phi_3\phi_6, & \phi_1\phi_4 + \phi_2\phi_6 + \phi_3\phi_5, \\
\phi_1\phi_5 + \phi_2\phi_3 + \phi_4\phi_6, & \phi_1\phi_5 + \phi_2\phi_4 + \phi_3\phi_6, & \phi_1\phi_5 + \phi_2\phi_6 + \phi_3\phi_4, \\
\phi_1\phi_6 + \phi_2\phi_3 + \phi_4\phi_5, & \phi_1\phi_6 + \phi_2\phi_4 + \phi_3\phi_5, & \phi_1\phi_6 + \phi_2\phi_5 + \phi_3\phi_4.
\end{array}$$

Using MAPLE we find

$$\begin{aligned}
f_{15}(x) = {} & x^{15} - 6x^{14} + (36t+6)x^{13} + (-27t^2 - 144t - 114)x^{12} \\
& + (540t^2 - 30t + 255)x^{11} + (-648t^3 - 864t^2 - 2976t + 612)x^{10} \\
& + (243t^4 + 3024t^3 + 513t^2 + 1626t + 5473)x^9 \\
& + (-4374t^4 + 1350t^3 - 25677t^2 + 10566t + 5726)x^8 \\
& + (2916t^5 - 486t^4 + 30456t^3 - 23544t^2 + 109836t - 9589)x^7 \\
& + (-729t^6 - 14013t^4 - 24894t^3 - 83466t^2 + 234012t - 115054)x^6 \\
& + (1944t^5 + 79056t^4 + 63558t^3 + 160884t^2 + 298728t - 443860)x^5 \\
& + (-69012t^5 - 57834t^4 - 183276t^3 + 624681t^2 + 98616t - 957110)x^4 \\
& + (19683t^6 + 17982t^5 + 83106t^4 - 660204t^3 + 162576t^2 \\
& \quad - 1250718t - 1735331)x^3 \\
& + (-26244t^5 + 470691t^4 + 210006t^3 - 177102t^2 - 3543564t - 3305952)x^2 \\
& + (-131220t^5 + 59454t^4 + 723006t^3 + 1004607t^2 - 2921238t - 4182059)x \\
& + (-52488t^5 - 156492t^4 + 16686t^3 + 469719t^2 - 285438t - 2317862).
\end{aligned}$$

As $PSL(2,5) \cong A_5$ and A_6 are non-solvable groups, $F_t(x)$ is a non-solvable, irreducible sextic, which has a square discriminant (by Lemma 2.2), and we can apply [**12**, Prop. 5, p. 717] to deduce that

$$\mathrm{Gal}(F_t(x)) \cong PSL(2,5) \quad \text{if } f_{15}(x) \text{ is reducible in } \mathbf{Z}[x]$$

and

$$\mathrm{Gal}(F_t(x)) \cong A_6 \quad \text{if } f_{15}(x) \text{ is irreducible in } \mathbf{Z}[x].$$

It follows from [**2**] that $f_{15}(x)$ factors into a product of two polynomials in $\mathbf{Z}[x]$, one of degree 5, and the other of degree 10. The two factors are

$$x^5 - 2x^4 + (12t-2)x^3 + (-9t^2+6t-72)x^2 + (-27t^2+12t-107)x + (-81t^2-282t-446)$$

and

$$\begin{aligned} x^{10} &- 4x^9 + 24tx^8 + (-18t^2-54t-50)x^7 + (207t^2-78t-26)x^6 \\ &+ (-216t^3-9t^2-582t+478)x^5 + (81t^4+378t^3-189t^2-1530t+993)x^4 \\ &+ (-243t^4+540t^3+189t^2+672t+1446)x^3 \\ &+ (-216t^3-2961t^2+5214t+4623)x^2 \\ &+ (1404t^3-5418t^2+2184t+8130)x + (648t^3-324t^2-2646t+5197). \end{aligned}$$

Thus $\mathrm{Gal}(F_t(x)) \cong PSL(2,5)$ for $t(\neq 1)$ in $\mathbf{Z}$. □

3. Field discriminant calculations

Let $t(\neq 1) \in \mathbf{Z}$. Let θ_t be a root of $F_t(x)$, so that $\theta_t \in \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$. Set $K_t = \mathbf{Q}(\theta_t)$ so that, by Lemma 2.3, K_t is a sextic field with Galois group $PSL(2,5)$. Set $g(t) := 729t^3 + 522t^2 + 1788t + 2648$. In this section we determine the discriminant $d(K_t)$ of K_t when either $g(t)$ or $g(t)/8$ is squarefree and odd. The former possibility can only occur when $t \equiv 1 \pmod 2$ and the latter when $t \equiv 0 \pmod 4$. First we prove a general lemma.

LEMMA 3.1. *Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbf{Z}[x]$ be irreducible. Suppose that θ is a root of $f(x)$ and $K = \mathbf{Q}(\theta)$. If p is a prime number such that $p \| a_0$ and $p | a_1$ then the ideal $\langle p \rangle$ ramifies in K.*

Proof. Suppose that $\langle p \rangle$ does not ramify in K. Then there exist distinct prime ideals $\wp_1, \ldots, \wp_r$ of O_K such that

$$\langle p \rangle = \wp_1 \cdots \wp_r.$$

As $p \| a_0$ we have $\langle a_0 \rangle = \wp_1 \cdots \wp_r \langle b \rangle$ for some $b \in \mathbf{Z}$ with $p \nmid b$. Thus $\wp_i \nmid \langle b \rangle$ for $i = 1, \ldots, r$. Since $N(\theta) = \pm a_0 \equiv 0 \pmod p$ the ideal $\langle \theta \rangle$ must be divisible by at least one $\wp_i$, say $\wp$. As $\wp | a_1$ we have

$$a_0 = a_0 - f(\theta) = -a_1\theta - \cdots - a_{n-1}\theta^{n-1} - \theta^n \equiv 0 \pmod{\wp^2},$$

contradicting $p \| a_0$. Thus $\langle p \rangle$ ramifies in K. □

Returning to the sextic fields defined at the beginning of this section, we prove the following result.

LEMMA 3.2. *Let $t(\neq 1) \in \mathbf{Z}$ be such that either $g(t)$ or $g(t)/8$ is squarefree and odd. Let p be an odd prime such that $p \mid g(t)$. Then $p \mid d(K_t)$.*

Proof. Let

$$\alpha_t = \theta_t^3 - 4\theta_t^2 + 5\theta_t - 3 \in O_{K_t}.$$

MAPLE shows that α_t is a root of

$$\begin{aligned} h_t(x) = x^6 &+ (6 - 9t)x^5 + (27t^2 - 63t - 26)x^4 \\ &+ (-27t^3 + 216t^2 - 60t - 240)x^3 \\ &+ (-243t^3 + 414t^2 + 192t - 416)x^2 \\ &- g(t)x - g(t). \end{aligned}$$

As $K_t = \mathbf{Q}(\theta_t)$ is a primitive sextic field, the degree of the minimal polynomial of α_t over $\mathbf{Q}$ is 6. Hence $h_t(x)$ is the minimal polynomial of α_t over $\mathbf{Q}$. Thus the conditions of Lemma 3.1 are satisfied and so $\langle p \rangle$ ramifies in K_t, proving $p \mid d(K_t)$. □

LEMMA 3.3. *Let $t(\neq 1) \in \mathbf{Z}$ be such that $g(t)/8$ is squarefree and odd. Then $2^6 \,\|\, d(K_t)$.*

Proof. Using MAPLE we show that none of the 32 elements of K_t given by

$$\frac{1}{2}(a_0 + a_1\theta_t + a_2\theta_t^2 + a_3\theta_t^3 + a_4\theta_t^4 + \theta_t^5), \quad a_i = 0, 1,$$

is an integer of K_t. We just provide the details in one typical case. We consider

$$\beta_t = \theta_t + \theta_t^3 + \theta_t^4 + \theta_t^5.$$

As $g(t)/8$ is squarefree and odd, we have $t = 4k$, $k \in \mathbf{Z}$. The minimal polynomial of β_t over $\mathbf{Q}$ is

$$\begin{aligned} q(x) = x^6 &+ (-1068k - 610)x^5 + (-39744k^3 + 66672k^2 - 20976k + 7536)x^4 \\ &+ (-248832k^5 + 82944k^4 - 198720k^3 + 70560k^2 - 38736k - 2584)x^3 \\ &+ (82944k^4 + 20736k^3 + 63504k^2 + 8736k - 2548)x^2 \\ &+ (41472k^4 + 22464k^3 + 32544k^2 + 11664k + 600)x \\ &+ (20736k^4 + 5184k^3 + 15840k^2 + 3312k + 1592). \end{aligned}$$

Suppose that $\beta_t/2$ is an integer of K_t. Then from $q(x)$ we see that

$$2^6 \mid 20736k^4 + 5184k^3 + 15840k^2 + 3312k + 1592.$$

This is clearly impossible as $2^4 \mid 20736k^4 + 5184k^3 + 15840k^2 + 3312k$ but $2^3 \| 1592$. Hence $\beta_t/2 \notin O_{K_t}$.

For $l \in \{0,1,2,3,4,5\}$ let α_l be a minimal integer in θ_t of degree l, see [**1**, Definition 7.2.1, p. 164]. Then $\alpha_0 = 1$ and $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is an integral basis for K_t [**1**, Theorem 7.2.7, p. 168]. Each α_l is of the form

$$\alpha_l = \frac{a_{l,0} + a_{l,1}\theta_t + \cdots + a_{l,l-1}\theta_t^{l-1} + \theta_t^l}{d_l},$$

where $a_{l,0}, a_{l,1}, \ldots, a_{l,l-1} \in \mathbf{Z}$ and $d_l \in \mathbf{N}$, so that $d_0 = 1$ [**1**, Theorem 7.2.6, p. 166]. Moreover $d_0 d_1 d_2 d_3 d_4 d_5 = \text{ind}\ \theta_t$ [**1**, Theorem 7.2.8, p. 169]. If $2 | d_l$ for some $l \in \{0,1,2,3,4,5\}$ then

$$\frac{d_l}{2}\theta_t^{5-l}\alpha_l = \frac{a_{l,0}\theta_t^{5-l} + \cdots + \theta_t^5}{2} \in O_{K_t},$$

which is a contradiction in view of what we proved first. Hence $2 \nmid d_l$ $(l = 0,1,2,3,4,5)$. Thus $2 \nmid \text{ind}\ \theta_t$. But $2^3 \parallel g(t)$ and

$$g(t)^2 = \text{disc}(F_t(x)) = (\text{ind}\ \theta_t)^2 d(K_t)$$

so $2^6 \parallel d(K_t)$. □

4. Proof of Theorem

By a theorem of Erdös [7] there are infinitely many integers t such that $g(t)$ is squarefree. These integers are necessarily odd, as is $g(t)$. Let p be any prime number dividing $\text{disc}(F_t(x))$. By Lemma 2.2 we have $\text{disc}(F_t(x)) = g(t)^2$. Thus, as $g(t)$ is odd and squarefree, we have $p \neq 2$ and $p^2 \parallel \text{disc}(F_t(x))$. From Lemma 3.2 we deduce that $p \mid d(K_t)$. Then, from $\text{disc}(F_t(x)) = \text{ind}(\theta_t)^2 d(K_t)$, we see that $p^2 \mid d(K_t)$. Thus $\text{disc}(F_t(x)) = d(K_t)$ and $\text{ind}(\theta_t) = 1$ so $\{1, \theta_t, \theta_t^2, \theta_t^3, \theta_t^4, \theta_t^5\}$ is a power basis for O_{K_t}.

Again, by Erdös' theorem, there are infinitely many integers k such that $w(k) = 729 \cdot 2^3 k^3 + 522 \cdot 2k^2 + 894k + 331$ is squarefree and necessarily odd. Hence there are infinitely many integers $t = 4k$ such that $\frac{g(t)}{8} = \frac{g(4k)}{8} = w(k)$ is squarefree and odd. Exactly as in the previous case, the powers of any odd prime p in $\text{disc}(F_t(x))$ and $d(K_t)$ are both 2. By Lemmas 2.2 and 3.3 we have $2^6 \parallel \text{disc}(F_t(x))$ and $2^6 \parallel d(K_t)$. Hence $\text{disc}(F_t(x)) = d(K_t)$ and $\text{ind}(\theta_t) = 1$ so $\{1, \theta_t, \theta_t^2, \theta_t^3, \theta_t^4, \theta_t^5\}$ is a power basis for O_{K_t}.

Finally, as $g(t) = \pm g(u)$ has at most six solutions u for a given integer t, we can pick an infinite subsequence of the original sequence of t's for which $g(t)$ or $g(t)/8$ is squarefree and odd in such a way that the sextic fields K_t are distinct.

5. Other power bases

Let $t(\neq 1) \in \mathbf{Z}$. If $g(t)$ or $g(t)/8$ is squarefree and odd then $K_t = \mathbf{Q}(\theta_t)$ has an additional power basis, namely $\{1, \beta_t, \dots, \beta_t^5\}$, where

$$\beta_t = \theta_t - 3t\theta_t^2 + 2\theta_t^3 - 4\theta_t^4 + \theta_t^5,$$

since the minimial polynomial of β_t is

$$x^6 + 10x^5 + 41x^4 + (88 + 3t)x^3 + (106 + 18t)x^2 + (76 + 36t)x + (33 + 24t),$$

whose discriminant is $g(t)^2$.

In the case $t = -1$, we have, setting $\theta = \theta_{-1}$, additional power basis generators $\gamma_1, \dots, \gamma_8$ given by

$$\begin{aligned}
\gamma_1 &= 5\theta + 5\theta^2 - 2\theta^3 - 3\theta^4 + \theta^5,\\
\gamma_2 &= -\theta + 7\theta^2 + 10\theta^3 - 13\theta^4 + 3\theta^5,\\
\gamma_3 &= \theta - 3\theta^2 + \theta^3,\\
\gamma_4 &= \theta - 4\theta^2 - 2\theta^3 + 4\theta^4 - \theta^5,\\
\gamma_5 &= \theta^2 - 4\theta^3 + 7\theta^4 - 2\theta^5,\\
\gamma_6 &= -6\theta^3 + 5\theta^4 - \theta^5,\\
\gamma_7 &= 2\theta^3 - \theta^4,\\
\gamma_8 &= \theta + \theta^2 + 2\theta^3 + 2\theta^4 - \theta^5.
\end{aligned}$$

If $t = -2$ we have, setting $\theta = \theta_{-2}$, an additional power basis generator γ given by

$$\gamma = 2\theta - \theta^2 + 7\theta^3 - 5\theta^4 + \theta^5.$$

We do not know if there are any further power bases.

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