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ON H_p CLASSIFICATION OF PLANE DOMAINS

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1. Introduction.

Throughout the present paper, it will always be understood that $0<p<\infty$, unless the contrary is explicitly stated. Let W be an open Riemann surface. Let $A(W)$ and $M(W)$ denote the families of single-valued analytic and meromorphic functions on W, respectively. We shall consider the following classes of functions:

(i) the Hardy classes $H_p(W)=\{f\in A(W): |f|^p$ admits a harmonic majorant on $W\}$;
(ii) $AB(W)=\{f\in A(W): f$ is bounded on $W\}$;
(iii) $MB^*(W)=\{f\in M(W): \log^+|f|$ admits a superharmonic majorant on $W\}$;
(iv) $AB^*(W)=A(W)\cap MB^*(W)$.

Let O_p, O_{AB}, O_{MB^*}, O_{AB^*} denote the classes of W such that $H_p(W)$, $AB(W)$, $MB^*(W)$, $AB^*(W)$, respectively, reduces to the constants. $W\in O_G$ means that W is parabolic. Finally we set $O_p^-=\cup\{O_q: 0<q<p\}$ and $O_p^+=\cap\{O_q: p<q<\infty\}$. Let S denote the Riemann sphere and Cap(E) the logarithmic capacity of the set E.

Heins [1] showed the following classification scheme:

$$O_G<O_{MB^*}<O_{AB^*}<\bigcap_{q>0}O_q<O_p^-<O_p<O_p^+<\bigcup_{q<\infty}O_q<O_{AB}, \tag{1}$$

where $<$ means a strict inclusion relation.

Suppose from now on that W is restricted to be a plane domain, and we denote by the same symbols the corresponding classes of plane domains. Then it is known that $O_G=O_{MB^*}=O_{AB^*}$ ([8, p. 280]) and $O_G<O_1$ ([1, p. 50]). But it is left unsolved to determine what parts of the classification scheme corresponding to (1) hold or not for plane domains. Recently Hejhal [3], [4] obtained some results about this problem, that is, showed the following classification scheme for plane domains:

$$O_G\leqq O_1^-<O_1\leqq O_{3/2}^-<O_{3/2}\leqq O_2^-<O_2\leqq O_{5/2}^-<O_{5/2}\leqq O_3^-<O_3\cdots<\bigcup_{q<\infty}O_q<O_{AB}. \tag{2}$$

We shall treat a decomposition of H_p functions in Section 2, and give some improvements of Hejhal's results (2) in Section 3. The idea of this paper was

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incited by that of Hejhal's [4].

2. A decomposition of H_p functions.

In this section we shall prove

THEOREM 1. *Let D_j $(j=1, 2)$ be subdomains of S such that $D_1^c \cap D_2^c = \emptyset$. Suppose that $f \in H_p(D_1 \cap D_2)$, then f can be represented in the form*

$$f(z)=f_1(z)+f_2(z) \qquad (z \in D_1 \cap D_2) \tag{3}$$

where $f_j \in H_p(D_j)$ for $j=1, 2$.

We need a lemma which was first proved by Parreau [6, p. 182]. Other proofs can be found in [4, pp. 7-9] or [9, p. 67].

LEMMA 1. *Let E be a compact subset of S. If* $\mathrm{Cap}(E)=0$, *then E is removable for every H_p function, i.e. $H_p(V-E)=H_p(V)$ for any subdomain V of S which contains E.*

Proof of Theorem 1. By Lemma 1, we may assume $\mathrm{Cap}(D_j^c)>0$ for $j=1, 2$. Let $\{D_j^{(\nu)}\}_{\nu=0}^{\infty}$ be a regular exaustion of D_j. Without loss of generality, we may assume that $(D_1^{(0)})^c \cap (D_2^{(0)})^c = \emptyset$. Define

$$f_j(z)=\lim_{\nu\to\infty} \frac{1}{2\pi i} \int_{\partial D_j^{(\nu)}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (z \in D_j)\,, \tag{4}$$

then (3) is an easy consequence of Cauchy's integral formula. Therefore $f_j \in H_p(D_j)$ is all we must prove.

Let $\gamma = \partial D_1^{(0)}$ and χ be the least harmonic majorant of $|f|^p$ in $D_1 \cap D_2$. Since f_2 is bounded on $(D_1^{(0)})^c$, we see

$$\begin{aligned} |f_1(z)|^p &= |f(z)-f_2(z)|^p \\ &\leqq 2^p(|f(z)|^p + |f_2(z)|^p) \\ &\leqq 2^p(\chi(z)+M) \qquad (z \in D_1 \cap (D_1^{(0)})^c) \end{aligned} \tag{5}$$

for a large M. Let $t \in D_1^{(0)} \cap D_2^{(0)}$ be fixed and $g(z\,;t\,;D)$ denote Green's function for a domain D with pole at t. Since $g(z\,;t\,;D_1^{(\nu)} \cap D_2^{(\nu)})$ and $g(z\,;t\,;D_1^{(\nu)})$ converge uniformly on γ to $g(z\,;t\,;D_1 \cap D_2)$ and $g(z\,;t\,;D_1)$, respectively, it is easily shown that

$$\min_{z\in\gamma} \frac{g(z\,;t\,;D_1^{(\nu)} \cap D_2^{(\nu)})}{g(z\,;t\,;D_1^{(\nu)})} \longrightarrow \min_{z\in\gamma} \frac{g(z\,;t\,;D_1 \cap D_2)}{g(z\,;t\,;D_1)} > 0 \tag{6}$$

as $\nu \to \infty$. Here note that we have assumed $\mathrm{Cap}(D_1^c)>0$. Using (6) we can choose $\varepsilon>0$ independently on ν so that

$$g(z\,;t\,;D_1^{(\nu)} \cap D_2^{(\nu)}) - \varepsilon g(z\,;t\,;D_1^{(\nu)}) > 0 \tag{7}$$

for $z\in\gamma$, and hence for $z\in D_1^{(\nu)}\cap(D_1^{(0)})^c$ by maximum principle. Therefore we see

$$\frac{\partial g}{\partial n_z}(z\,;\,t\,;\,D_1^{(\nu)}\cap D_2^{(\nu)})\geqq\varepsilon\frac{\partial g}{\partial n_z}(z\,;\,t\,;\,D_1^{(\nu)}) \tag{8}$$

for $z\in\partial D_1^{(\nu)}$, where the derivative is taken along the inner normal. Combining (5) and (8), we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial D_j^{(\nu)}}|f_1(z)|^p\frac{\partial g}{\partial n_z}(z\,;\,t\,;\,D_1^{(\nu)})|dz|\\ &\leqq\frac{1}{2\pi}\int_{\partial D_j^{(\nu)}}2^p(\chi(z)+M)\frac{1}{\varepsilon}\frac{\partial g}{\partial n_z}(z\,;\,t\,;\,D_1^{(\nu)}\cap D_2^{(\nu)})|dz|\\ &\leqq\frac{2^p}{\varepsilon}(\chi(t)+M)\,,\end{aligned}$$

and hence $f_1\in H_p(D_1)$. By symmetry we have $f_2\in H_p(D_2)$.

REMARK 1. Theorem 1 was proved by Rudin [7, pp. 56-57] in case the boundary ∂D_j of D_j is an analytic Jordan curve for $j=1, 2$.

REMARK 2. By using Theorem 1, we can considerably shorten the proofs of Theorem 5, $\cdots$, 8 of Hejhal's paper [4, pp. 9-13].

3. H_p classification.

In this section we shall prove

THEOREM 2. *Let k be any integer not less than 2. Then*

$$O_{k/2}<O_p \tag{9}$$

for any real number p with $p>k/2$.

Proof. We may assume that $k/2<p\leqq k$. Let p be fixed with $k/2<p\leqq k$, then

$$\pi/k\leqq\pi/p<2\pi/k \tag{10}$$

$$\pi/2\leqq k\pi/2p<\pi\,. \tag{11}$$

In order to prove the theorem, we must construct a plane domain W for which $H_p(W)$ reduces to the constants but $H_{k/2}(W)$ contains non-constant functions. Our construction is similar to that of Hejhal's (cf. Example 1 and 2 of [4, pp. 19-20]).

Let A be a Cantor set constructed on the arc $\{z: |z|=1 \text{ and } |\arg z|\leqq\pi/4\}$. We can construct A so that (i) A is a compact totally disconnected set of linear measure 0, (ii) $Cap\,(A)>0$, (iii) A is symmetric with respect to the real axis and (iv) $1\notin A$ (see [5, p. 150]). Let E_0 be the union of the images under all the branches of the multi-valued function $h(z)=(\log z)^{-2/k}$, and define $E_1=$

$E_0 \cup \{0\}$. Then it is easily shown that E_1 is a compact totally disconnected set of linear measure 0 which lies on the k-star $K_1=\{z: \arg z=\pi/k+2m\pi/k,\ m=0, 1, \cdots, k-1\}$. Let $E_2=e^{\pi i/p}E_1$, that is, E_2 be the set obtained by rotating E_1 through the angle π/p. Then E_2 is a compact totally disconnected set of linear measure 0 which lies on the k-star $K_2=\{z: \arg z=\pi/k+\pi/p+2m\pi/k,\ m=0, 1, \cdots, k-1\}$. Finally we set $E=E_1 \cup E_2$ and $W=S-E$.

Hejhal [4, pp. 15-18] proved the following lemma.

LEMMA 2. *Let $c_1, \cdots, c_n$ be positive numbers with $c_1+\cdots+c_n=2$, $n \geqq 2$. Suppose that E is a compact totally disconnected set of linear measure 0 which lies on an n-star formed by n rays emanating from the origin to ∞, with successive angles $\pi c_1, \cdots, \pi c_n$. Let $c_0=\max\{c_j: j=1, \cdots, n\}$ and $1 \leqq p<\infty$. Suppose that $f \in H_p(S-E)$, then*

$$f(z)=\sum_{j=0}^{\infty} a_\nu z^{-\nu} \qquad (0<|z| \leqq \infty),$$

with $a_\nu=0$ for all $\nu \geqq 1/pc_0$.

Using (10) we easily see that the maximum angle of the $2k$ angles formed by the $2k$-star $K_1 \cup K_2$ on which E lies is π/p, i.e. $c_0=1/p$, where c_0 is as in Lemma 2. Applying Lemma 2, it turns out that $H_p(W)$ contains no non-constant functions, i.e.

$$W \in O_p. \tag{12}$$

Next we shall prove that $f(z)=z^{-1}$ belongs to $H_{k/2}(W)$. For this we must show that the subharmonic function $w(z)=|z|^{-k/2}$ admits a harmonic majorant in W. First we deal with the case where k is even and next the other.

Let $k=2m$, $m=1, 2, \cdots$, and we consider the following functions analytic in W:

$$f_1(z)=\exp(z^{-m}),$$

$$f_2(z)=\exp(-z^{-m}),$$

$$f_3(z)=\exp(e^{m\pi i/p}z^{-m}) \equiv f_1(e^{-\pi i/p}z),$$

$$f_4(z)=\exp(-e^{m\pi i/p}z^{-m}) \equiv f_2(e^{-\pi i/p}z).$$

We can easily check that A and the image $f_j(W)$ of W under f_j are disjoint, in other words, f_j omits in W the set A of positive capacity, for $j=1, 2, 3, 4$. Then, by Nevanlinna-Frostman theorem (see [2, p. 150] or [4, p. 18]), we see that $f_j \in AB^*(W)$, i.e. $\log^+|f_j|$ admits a harmonic majorant in W. Let χ_j be the least harmonic majorant of $\log^+|f_j|$ in W for $j=1, 2, 3, 4$. Then we easily see

$$|\operatorname{Re} z^{-m}| \leqq \chi_1(z)+\chi_2(z), \tag{13}$$

$$|\operatorname{Re} e^{i\theta}z^{-m}| \leqq \chi_3(z)+\chi_4(z), \tag{14}$$

where $\theta=m\pi/p$, since $\log^+|e^a|+\log^+|e^{-a}|=|\operatorname{Re} a|$ for any complex number a.

For simplicity, we write $z^{-m}=u+iv$. Then (14) is

$$|u\cos\theta-v\sin\theta|\leqq\chi_3+\chi_4\,. \tag{15}$$

Since $\sin\theta>0$ by (11), we can obtain from (13) and (15)

$$|v|\leqq\frac{1}{\sin\theta}(|u|+\chi_3+\chi_4)\leqq\frac{1}{\sin\theta}\sum_{j=1}^{4}\chi_j\,. \tag{16}$$

Combining (13) and (16), we see

$$w(z)=|z|^{-m}\leqq|u|+|v|\leqq\chi_1+\chi_2+\frac{1}{\sin\theta}\sum_{j=1}^{4}\chi_j\,, \tag{17}$$

and hence

$$f(z)=z^{-1}\in H_{k/2}(W)\,. \tag{18}$$

Next we assume that k is odd. Let R be the 2-sheeted Riemann surface associated with the function z^2, and R_0 be the Riemann surface obtained by removing from R all the points whose projections lies on E. We consider the following functions single-valued and analytic on R_0:

$$f_1(z)=\exp(z^{-k/2})\,,$$

$$f_2(z)=\exp(-z^{-k/2})\,,$$

$$f_3(z)=\exp(e^{k\pi i/2p}z^{-k/2})\,,$$

$$f_4(z)=\exp(-e^{k\pi i/2p}z^{-k/2})\,.$$

It is easy to see that f_j omits the points of A on R_0, and hence $f_j\in AB^*(R_0)$, for $j=1, 2, 3, 4$. Let χ_j be the least harmonic majorant of $\log^+|f_j|$ on R_0. Then, in the same manner as (13) and (14), we get

$$|\mathrm{Re}\, z^{-k/2}|\leqq\chi_1(z)+\chi_2(z) \tag{19}$$

$$|\mathrm{Re}\, e^{i\theta}z^{-k/2}|\leqq\chi_3(z)+\chi_4(z) \tag{20}$$

for $z\in R_0$, where $\theta=k\pi/2p$. But the both sides of (19) and (20) are defined and single-valued for $z\in W-\{\infty\}$, and the right sides are defined and single-valued for $z\in W-\{\infty\}$, and the right sides are harmonic in $W-\{\infty\}$, since $z^{-k/2}=-z^{*-k/2}$, where, for $z\in R_0$, z^* represents the other point on R_0 which is projected on the same point as z. Therefore we can show that $w(z)=|z|^{-k/2}$ admits a harmonic majorant in $W-\{\infty\}$, by the same way as we obtained (17) from (13) and (14). Hence

$$f(z)=z^{-1}\in H_{k/2}(W-\{\infty\})=H_{k/2}(W)\,, \tag{21}$$

since any isolated point is removable for every Hardy class (see Lemma 1).

By (18) and (21) we see $W\notin O_{k/2}$. Combining this with (12), we obtain $W\in O_p-O_{k/2}$, and hence $O_{k/2}<O_p$ as desired.

4. Concluding remarks.

Combining Theorem 2 and Hejhal's results (2), we have got the following H_p classification scheme for plane domains:

$$O_G \leqq O_1^- < O_1 < O_{p_2} \leqq O_{3/2}^- < O_{3/2} < O_{p_3} \leqq O_2^- < O_2 < O_{p_4} \tag{22}$$
$$\leqq O_{5/2}^- < O_{5/2} < O_{p_5} \cdots < \bigcup_{q<\infty} O_q < O_{AB} ,$$

where p_k is any real number with $k/2 < p_k < (k+1)/2$ for $k=2, 3, \cdots$.

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