

## MINIMAL SUBMANIFOLDS OF ALMOST SEMI-KÄHLER MANIFOLDS

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The purpose of this note is to study the minimality of almost Hermitian submanifolds of codimension 2 in an almost semi-Kähler manifold. Our main result is Theorem 3.4.

In § 1, we review the various classes of almost Hermitian manifolds, and prove, in passing, that all four-dimensional almost semi-Kähler manifolds are almost Kähler. In this manner, the inclusion lattice of almost Hermitian structures is greatly reduced in the four-dimensional case, (Figure 2). In § 2, we recall the configuration tensor of an immersed Riemannian submanifold and its relation to the second fundamental form of the immersion. In § 3, we study the effect of certain almost Hermitian structures on the configuration tensor, proving the minimality of codimension 2 almost Hermitian submanifolds of an almost semi-Kähler manifold.

### § 1. Almost Semi-Kähler Manifolds.

Let  $(M, g, F)$  be a smooth, *almost Hermitian manifold*. That is,  $M$  is a smooth, connected paracompact manifold;  $F$  is a smooth tensor field of type  $(1, 1)$  satisfying  $F^2 = -1$ ;  $g$  is a smooth Riemannian structure on  $M$ ; and the tensors  $F$  and  $g$  satisfy:

$$g(X, Y) = g(FX, FY),$$

for all smooth vector fields,  $X$  and  $Y$  on  $M$ . The *Kähler form* on  $M$  is the differential 2-form of bidegree  $(1, 1)$  given by:

$$\Omega(X, Y) = g(X, FY).$$

An almost Hermitian manifold is necessarily orientable and of even dimension, which we shall take to be  $2n$ .

The manifold  $(M, g, F)$  is said to be *almost semi-Kähler* if  $\Omega$  is coclosed; i. e., if its codifferential,  $\delta\Omega$ , vanishes. An almost semi-Kähler space is said to be *semi-Kähler* if it is complex, or, equivalently, if the almost complex structure,  $F$ , is integrable. Other classes of almost Hermitian manifolds are defined as follows. We say that  $(M, g, F)$  is:

$(QK)$  *Quasi-Kähler* if the components of bidegree  $(1, 2)$  and  $(2, 1)$  of the 3-form

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- $d\Omega$  vanish identically.
- ( $\mathcal{OK}$ ) *Almost Kähler* if  $d\Omega=0$ .  
(an almost Kähler space has been called an *H-space* [7]).
  - ( $\mathcal{OS}$ ) *Almost Tachibana* if  $\Omega$  is a Killing form; i. e., if  $d\Omega=\frac{1}{3}\nabla\Omega$ . (an almost Tachibana space has been called a *K-space* [7] and a nearly Kähler manifold [5]).
  - ( $\mathcal{K}$ ) *Kähler* if  $\nabla\Omega=0$ . (the Kähler condition implies that  $F$  is integrable).

There have been many studies of the differential geometry of the classes  $\mathcal{OK}$ ,  $\mathcal{OK}$ ,  $\mathcal{OS}$ , and  $\mathcal{K}$ , [3], [4], [5], [7], but not many of almost semi-Kähler spaces, [1], [7].

We cite for future reference, the following well-known facts :

- Fact 1:** *A complex quasi-Kähler manifold is Kähler.*
- Fact 2:** *A semi-Kähler space need not be Kähler, (see the example in [4]).*
- Fact 3:** *An almost Hermitian manifold which is both almost Kähler and almost Tachibana is Kähler.*
- Fact 4:** *Let  $\{E_1, \dots, E_n, FE_1, \dots, FE_n\}$  be an orthonormal  $F$ -basis for the vector fields on an open subset of the almost Hermitian manifold  $(M, g, F)$ . Then the codifferential of the Kähler form of  $M$  can be locally expressed as*

$$\delta\Omega(X) = - \sum_{i=1}^n \{ \nabla_{E_i} \Omega(E_i, X) + \nabla_{FE_i} \Omega(FE_i, X) \} .$$

when  $\delta\Omega$  acts on the vector field,  $X$ , of  $M$ .

From these, and other well-known facts, we may construct an inclusion lattice for the principal classes of almost Hermitian structures, in which each of the inclusion relations is strict.

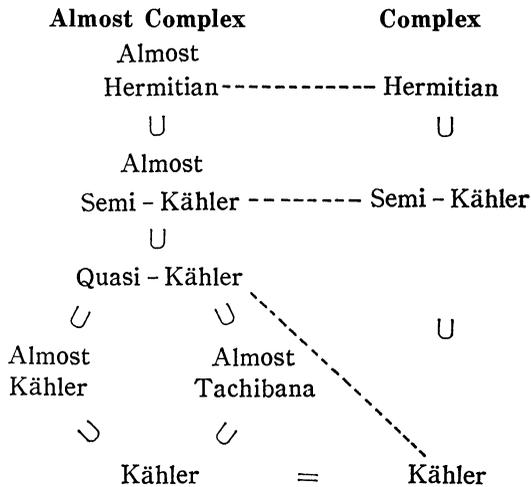


Figure 1: Lattice of Almost Hermitian Structures.

In the case of an almost Hermitian manifold of dimension 4, we can substantially reduce the inclusion lattice of Figure 1. For this, we prove the following theorem on almost semi-Kähler spaces.

**THEOREM 1.1:** *A four-dimensional almost semi-Kähler manifold is almost Kähler.*

*Proof:* Let  $*$  denote the Hodge star operator induced on the differential forms of  $M$  by  $g$  and the orientation of the almost Hermitian manifold of arbitrary dimension  $2n$ . Then it is sufficient to remark that:

$$*\Omega = \frac{1}{(n-1)!} \Omega \wedge \cdots \wedge \Omega \quad (n-1 \text{ times}).$$

Therefore, in the 4-dimensional case, we have:

$$\delta\Omega = -*d*\Omega = -*d\Omega.$$

Since  $*$  is an isomorphism,  $\delta\Omega=0$  if and only if  $d\Omega=0$ .

**COROLLARY 1.1.1:** *A four-dimensional semi-Kähler manifold is Kähler.*

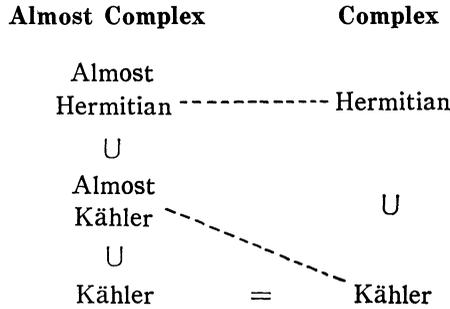
*Proof:* A complex almost Kähler space is Kähler.

**REMARK:** Theorem 1.1 is a generalization of a lemma of Gray [5] to the effect that a four-dimensional almost Tachibana manifold is Kähler. Gray's Lemma is immediate from Theorem 1.1 and Fact 3.

Tachibana and Okumura [8] have shown that the tangent bundle of a non-flat Riemannian manifold possesses an almost Hermitian structure which is almost Kähler without being Kähler. Yano and Ishihara [10] have shown the same result for tangent bundles, and Bhatia and Prakash [2] have constructed essentially the same structure in the cotangent bundle of a non-flat Riemannian manifold. In each case, the integrability condition on the almost complex structure is the flatness of the Riemannian curvature. For instance,  $T(S^2)$  is a non-Kähler, almost Kähler manifold. It is not known if there exists a compact, non-Kähler, almost Kähler manifold.

**REMARK:** The four-dimensional Hopf space,  $S^1 \times S^3$ , is a Hermitian manifold [6], but it cannot be Kähler, because compact Kähler manifolds have positive second Betti numbers.

The above discussion allows us, then, to construct the following strict inclusion lattice for almost Hermitian manifolds of dimension 4:



**Figure 2:** Lattice of Almost Hermitian Structures in Dimension Four.

It is easy to see that a 2-dimensional almost Hermitian manifold is Kähler, so :



**Figure 3:** Lattice of Almost Hermitian Structures in Dimension Two.

**§ 2. Immersed Submanifolds.**

We begin by recalling certain results on the configuration tensor of an immersed Riemannian manifold, without considering, as yet, any almost Hermitian structures.

Let  $M'$  be an  $m'$ -dimensional immersed submanifold of the Riemannian manifold,  $(M, g)$  of dimension  $m$ . Give  $M'$  the Riemannian structure  $g'$  induced by the immersion map into  $M$ . We let  $\overline{\mathcal{D}}(M')$  denote the restrictions of the smooth vector fields of  $M$  onto  $M'$ . Let  $\mathcal{D}(M')$  denote the subspace of such vector fields which are tangent to the submanifold  $M'$ , and let  $\mathcal{D}(M')^\perp$  be the subspace of restricted vector fields which are orthogonal to  $M'$  with respect to  $g$ . Then,  $\overline{\mathcal{D}}(M')$  is decomposed as a direct sum :

$$\overline{\mathcal{D}}(M') = \mathcal{D}(M') \oplus \mathcal{D}(M')^\perp.$$

We shall denote the tensors and tensor operators associated to the immersed submanifold,  $M'$ , by a caret,  $\hat{\phantom{x}}$ . For instance,  $\hat{R}$  will denote the Riemannian curvature tensor on  $M'$  associated to the induced Riemannian connection,  $\hat{\nabla}$ , arising from the Riemannian structure,  $g'$ , on  $M'$ .

Following [3], we define the *configuration tensor* of the immersed submanifold,  $M' \subset M$ , as the tensor  $\hat{T} : \mathcal{D}(M') \times \overline{\mathcal{D}}(M') \rightarrow \overline{\mathcal{D}}(M')$  given by :

(CT1)  $\hat{T}_X Y = \nabla_X Y - \hat{\nabla}_X Y, \quad \text{for } X, Y \in \mathcal{D}(M'),$

(CT2)  $\hat{T}_X Z = \pi(\nabla_X Z), \quad \text{for } X \in \mathcal{D}(M')$

and  $Z \in \mathcal{D}(M')^\perp$ , where  $\pi : \overline{\mathcal{D}}(M') \rightarrow \mathcal{D}(M')$  is orthogonal projection.

The most important properties of the configuration tensor,  $\hat{T}$ , are contained in the following proposition:

PROPOSITION 2.1: *Let  $M'$  be an immersed submanifold of the Riemannian manifold,  $M$ . For  $X, Y \in \mathcal{D}(M')$ , we have:*

(i)  $\hat{T}_X$  is a skew-symmetric linear operator which reverses the subspaces,  $\mathcal{D}(M')$  and  $\mathcal{D}(M')^\perp$ .

(ii)  $\hat{T}_X Y = \hat{T}_Y X$ .

(iii)  $\pi(R(X, Y)) = \hat{R}(X, Y) - [\hat{T}_X, \hat{T}_Y]$ . (the Gauss equation).

*Proof:* [3].

From condition CT1, we see that the configuration tensor,  $\hat{T}$ , is essentially the second fundamental form of the immersion. In fact, we say that  $M'$  is a totally geodesic submanifold of  $M$ , whenever  $\hat{T}$  is identically zero.

Let  $\{E_1, \dots, E_{m'}\}$  be an orthonormal local basis for  $\mathcal{D}(M')$ . From proposition 2.1 (i), we see that the vector field  $\hat{T}_{E_i}(E_i)$  is orthogonal to  $\mathcal{D}(M')$  for every  $i=1, \dots, m'$ . Thus, the mean curvature vector field,  $H$ , of  $M'$  in  $M$ , defined by:

$$H = \sum_{i=1}^{m'} \hat{T}_{E_i}(E_i)$$

is a vector field in the orthogonal subspace,  $\mathcal{D}(M')^\perp$ . We say that  $M'$  is a minimal submanifold of  $M$  when the mean curvature vector field is identically zero. Clearly, totally geodesic implies minimal. There are many examples, however, of submanifolds of Euclidean  $n$ -space, for instance, which are minimally immersed, but which are not totally geodesic.

Minimal submanifolds have great intrinsic importance since they arise in the study of Plateau's Problem, and other classical problems of partial differential equations. Thus, existence or non-existence theorems concerning minimal submanifolds have substantial interest. We include two facts relating to minimal submanifolds.

**Fact 5:** *Canonically embedded subspheres of a sphere are totally geodesically, and therefore, minimally immersed.*

**Fact 6:** *There do not exist any compact minimal submanifolds of Euclidean  $n$ -space.*

### § 3. Submanifolds of Almost Hermitian Manifolds.

We now consider almost complex submanifolds of an almost Hermitian manifold in order to see how much this additional structure restricts the possibilities for the configuration tensor,  $\hat{T}$ .

If  $(M', g', F')$  is an almost Hermitian immersed submanifold of the almost Hermitian manifold  $(M, g, F)$  such that  $g'$  is the inherited Riemannian structure on  $M'$  from  $g$ , and such that  $F'$  is identical to the restriction of  $F$  onto  $M$ ,

then we say that  $(M', g', F')$  is an *almost Hermitian submanifold* of  $(M, g, F)$ . Since the almost complex structure  $F'$  on the submanifold is essentially the restriction of  $F$  to  $M'$ , it should cause no confusion to denote both almost complex structures by  $F$ .

Using the facts that the restriction of the Kähler form  $\Omega$  of  $M$  to  $M'$  is  $\Omega'$ , and that the immersion mapping commutes with the  $d$ -operator on differential forms, the proof of the following proposition is straightforward [3].

**PROPOSITION 3.1:** *Let  $M'$  be an almost Hermitian submanifold of the almost Hermitian manifold,  $M$ . Then if  $M$  is Kähler (resp., almost Kähler; resp., almost Tachibana; resp., quasi-Kähler; resp., Hermitian), then  $M'$  is Kähler (resp., almost Kähler; resp., almost Tachibana; resp., quasi-Kähler; resp., Hermitian).*

**REMARK:** It is not known if Proposition 3.1 obtains for the more general classes of almost semi-Kähler and semi-Kähler manifolds.

A classic result in the folk-lore of minimal submanifolds states that any holomorphically immersed submanifold of a Kähler manifold is minimal. In [3], Gray extended this statement to apply to any almost Hermitian submanifold of a quasi-Kähler manifold. Thus,

**PROPOSITION 3.2:** *An almost Hermitian submanifold of a quasi-Kähler manifold is minimal.*

*Proof:* Let  $M'$  be an almost Hermitian submanifold of the quasi-Kähler manifold,  $M$ . Then,  $M'$  is quasi-Kähler by Proposition 3.1. It is easily checked that the quasi-Kähler condition  $(Q\mathcal{K})$  is equivalent to:

$$\nabla_X Y + F\nabla_X(FY) + \nabla_{FX}(FY) - F\nabla_{FX}(Y) = 0,$$

for all tangent vector fields,  $X$  and  $Y$ . In particular, for  $X$ , tangent to  $M'$ , we have:

$$\nabla_X X + F\nabla_X(FX) + \nabla_{FX}(FX) - F\nabla_{FX}(X) = 0,$$

and

$$-\hat{\nabla}_X X - F\hat{\nabla}_X(FX) - \hat{\nabla}_{FX}(FX) + F\hat{\nabla}_{FX}(X) = 0.$$

So,

$$\hat{T}_X X + F\hat{T}_X(FX) + \hat{T}_{FX}(FX) - F\hat{T}_{FX}(X) = 0.$$

Since  $\hat{T}_X(FX) = \hat{T}_{FX}(X)$ , by Proposition 2.1 (ii), we have,

$$\hat{T}_X X + \hat{T}_{FX}(FX) = 0.$$

Now consider an orthonormal  $F$ -basis for  $\mathcal{D}(M')$ ,  $\{E_1, \dots, E_{m'}, FE_1, \dots, FE_{m'}\}$ . The last identity then implies

$$H = \sum_{i=1}^{m'} \{\hat{T}_{E_i}(E_i) + \hat{T}_{FE_i}(FE_i)\} = 0.$$

It was the attempt to extend Proposition 3.2 to the weaker class of almost

semi-Kähler manifolds which motivated this study. While we have not been able to prove such an extension (we conjecture that Proposition 3.2 is false for almost semi-Kähler spaces), we have been able to prove the minimality of certain submanifolds of almost semi-Kähler manifolds.

Before stating our result, we define the partial codifferential of the Kähler from  $\Omega$  of  $M$ , and derive its relationship to the minimality of  $M'$ .

Let  $M'$  be an almost Hermitian submanifold of the almost Hermitian manifold,  $(M, g, F)$ , and let  $\{E_1, \dots, E_{m'}, FE_1, \dots, FE_{m'}\}$  be an orthonormal  $F$ -basis for  $\mathcal{D}(M')$ . We define, as in [3], the *partial codifferential of  $\Omega$*  via :

$$\begin{aligned} \delta\Omega(X) = & - \sum_{i=1}^{m'} \{ \nabla_{E_i}(\Omega)(E_i, X) + \nabla_{FE_i}(\Omega)(FE_i, X) \}, \\ & \text{for all } X \in \overline{\mathcal{D}}(M'). \end{aligned}$$

Note that the summation contains the covariant differentiation operator of  $M$ , but only sums to the dimension of  $M'$ . The usefulness of  $\delta\Omega$  is derived from the following :

**PROPOSITION 3.3:** *An almost Hermitian submanifold,  $M'$ , of the almost Hermitian manifold,  $(M, g, F)$ , is minimal if and only if  $\delta\Omega(Z)=0$ , for every  $Z \in \mathcal{D}(M')^\perp$ .*

*Proof:* Now  $\nabla_X(\Omega)(Y, Z) = g(\nabla_X(FY) - F\nabla_X Y, Z)$ . So,

$$\begin{aligned} \nabla_{E_i}(\Omega)(E_i, Z) &= g(\nabla_{E_i}(FE_i), Z) - g(F\nabla_{E_i}(E_i), Z) \\ &= g(\hat{T}_{E_i}(FE_i), Z) - g(F\hat{T}_{E_i}(E_i), Z) \\ &\quad + g(\hat{V}_{E_i}(FE_i), Z) + g(F\hat{V}_{E_i}(E_i), Z) \\ &= g(\hat{T}_{E_i}(FE_i), Z) - g(F\hat{T}_{E_i}(E_i), Z), \end{aligned}$$

because the vector field  $Z$  is orthogonal to  $\mathcal{D}(M')$ , and so to  $\nabla_{E_i}(FE_i)$  and  $F\nabla_{E_i}(E_i)$ . Therefore,

$$\begin{aligned} \delta\Omega(Z) &= - \sum_{i=1}^{m'} g(\hat{T}_{E_i}(FE_i) - F\hat{T}_{E_i}(E_i) - \hat{T}_{FE_i}(E_i) - F\hat{T}_{FE_i}(FE_i), Z) \\ &= g(FH, Z). \end{aligned}$$

Thus,  $\delta\Omega=0$  if and only if  $H=0$ .

Using Proposition 3.3, we are able to prove the minimality of certain almost Hermitian submanifolds of an almost semi-Kähler manifold. Notice that we do know if these submanifolds inherit an almost semi-Kähler structure from the ambient almost semi-Kähler space.

**THEOREM 3.4:** *Let  $M^{2n+2}$  be an almost semi-Kähler manifold and  $M'$  be an almost Hermitian submanifold of  $M$  of codimension 2. Then  $M'$  is a minimal submanifold of  $M$ .*

*Proof:* Choose an orthonormal  $F$ -basis,  $\{E_1, \dots, E_n, FE_1, \dots, FE_n, Z, FZ\}$ , for  $\overline{\mathcal{D}}(M)$  such that  $\{Z, FZ\}$  is a basis for  $\mathcal{D}(M')^\perp$  and  $\{E_1, \dots, E_n, FE_1, \dots, FE_n\}$  is a basis for  $\mathcal{D}(M')$ . According to Proposition 3.3, it is enough to show that  $\tilde{\delta}\Omega(Z)=0=\tilde{\delta}\Omega(FZ)$ . We know that the usual codifferential,  $\delta\Omega$ , of  $\Omega$ , vanishes on all vector fields of  $M$ , because  $M$  is an almost semi-Kähler manifold. So,

$$0=\delta\Omega(Z)=-\sum_{i=1}^n \{\nabla_{E_i}(\Omega)(E_i, Z)+\nabla_{FE_i}(\Omega)(FE_i, Z)\} \\ -\nabla_Z(\Omega)(Z, Z)-\nabla_{FZ}(\Omega)(FZ, Z).$$

Thus,

$$\tilde{\delta}\Omega(Z)=\nabla_Z(\Omega)(Z, Z)+\nabla_{FZ}(\Omega)(FZ, Z).$$

But,

$$\nabla_X(\Omega)(Y, W)=-\nabla_X(\Omega)(W, Y),$$

and,

$$\nabla_X(\Omega)(FY, W)=\nabla_X(\Omega)(Y, FW).$$

Therefore,

$$\tilde{\delta}\Omega(Z)=0,$$

and similarly,

$$\tilde{\delta}\Omega(FZ)=0.$$

REMARK: It is of interest to note the similarity of the two statements:

(a) Proposition 3.2 is true for quasi-Kähler manifolds, but is difficult (or false ?) for almost semi-Kähler manifolds.

(b) The complexification of a quasi-Kähler space is Kähler, but the corresponding statement for almost semi-Kähler spaces is false. (See Figure 1).

REMARK: In [9], we proved a series of results on almost complex Riemannian submersions between almost Hermitian manifolds. In particular, we showed that the base space inherits the same structure from the total space, when the total space is quasi-Kähler or stronger. However, when the total space is almost semi-Kähler, that result no longer obtains. We found, instead, that the base space is almost semi-Kähler if and only if the fibres of the Riemannian fibration are minimally immersed. Perhaps this indicates a relationship between the existence of an almost semi-Kähler structure on the submanifolds and their minimality.

In this direction, we now prove the orthogonal complement to Proposition 3.3. This is, Proposition 3.3 says that  $\tilde{\delta}\Omega$ , when restricted to  $\mathcal{D}(M')^\perp$ , gives information on the minimality of  $M'$ . We now show that the same partial codifferential,  $\tilde{\delta}\Omega$ , when restricted to  $\mathcal{D}(M')$ , gives information on whether  $M'$  is almost semi-Kähler or not. Notice that the ambient manifold,  $M$ , in the following theorem and its corollary, is only supposed to be almost Hermitian.

**THEOREM 3.5:** *An almost Hermitian submanifold,  $M'$ , of the almost Hermitian manifold,  $(M, g, F)$ , is almost semi-Kähler if and only if  $\tilde{\delta}\Omega(W)=0$  for every  $W \in \mathcal{D}(M')$ .*

*Proof:* It is sufficient to show that  $\tilde{\delta}\Omega(W)=\hat{\delta}\Omega(W)$  for every  $W\in\mathcal{D}(M')$ . But, clearly, as in the proof of Proposition 3.3, we have  $\tilde{\delta}\Omega(W)-\hat{\delta}\Omega(W)=-g(FH, W)$ . Since  $FH$  is orthogonal to  $W$ , the theorem obtains.

**COROLLARY 3.5.1:** *If  $\tilde{\delta}\Omega=0$ , then  $M'$  is a minimal, almost semi-Kähler submanifold of the almost Hermitian manifold,  $M$ .*

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